Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Integration Exam Skills [4.4]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2- Pg 19
Supplementary Questions	Pg 20 - Pg 37

Section A: Compulsory Questions

Sub-Section: Exam 1 (Tech-Free)

Question 1

a. Find an anti-derivative of $3x^4 - \frac{2}{x^2}$ with respect to x.

$$\int \left(3\cdot x^4 - \frac{2}{x^2}\right) dx \quad \frac{3\cdot x^5}{5} + \frac{2}{x}$$

Since, it's 'an' anti-derivative, no need to $+c$.

b. Find $\int (4-2x)^{-5} dx$.

$$\int (4-2x)^{-5}\, dx$$
$$= -\frac{1}{4}(4-2x)^{-4} \times \left(-\frac{1}{2}\right) + C$$
$$= \frac{1}{8}(4-2x)^{-4} + C \quad \text{(1A)}$$

c. The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$. Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

$$\int \sin(2\pi x)\, dx$$
$$= -\frac{\cos(2\pi x)}{2\pi} + C \quad \text{(1M)}$$

sub in $\left(1, \frac{1}{\pi}\right)$:

$$\frac{1}{\pi} = -\frac{\cos(2\pi)}{2\pi} + c \Rightarrow C = \frac{3}{2\pi}.$$

$$g(x) = \frac{-\cos(2\pi x)}{2\pi} + \frac{3}{2\pi}. \quad \text{(1A)}$$

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Question 2

The gradient of a curve is given by $2\sin(2x) - 4e^{-2x}$. The curve passes through the origin. What is the equation of the curve?

$\int 2\sin(2x) - 4e^{-2x}\, dx$

$= 2e^{-2x} - \cos(2x) + c$ (1M)

sub in $(0, 0)$, solve for c:

$0 = 2e^{-2(0)} - \cos(2(0)) + c$

$c = -1$

Equation of the curve:

$y = 2e^{-2x} - \cos(2x) - 1$ (1A)

Question 3

a. Find the derivative of $x\sin(x)$.

$\frac{d}{dx}(x\sin(x)) = x\cos(x) + \sin(x)$ (1A)

b. Hence, find an antiderivative of $x\cos(x)$.

$x\cos(x) = \frac{d}{dx}(x\sin(x)) - \sin(x)$

$\int x\cos(x)\, dx = x\sin(x) - \int \sin(x)\, dx$ (1M)

$= x\sin(x) + \cos(x)$ (1A)

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Question 4

Find the average value of $y = -x^2 + 8x + 12$ over the interval $[1, 4]$.

$$\frac{1}{4-1}\int_1^4 -x^2+8x+12\,dx$$

$$= \frac{1}{3}\left[\frac{-x^3}{3}+4x^2+12x\right]_1^4 \quad \text{(1M)}$$

$$= \frac{1}{3}\left(\left(-\frac{4^3}{3}+4(4^2)+12(4)\right)-\left(-\frac{1}{3}+4+12\right)\right)$$

$$= \frac{1}{3}\times 75$$

$$= 25 \quad \text{(1A)}$$

Question 5

Find $3\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$, given $\int_0^k (g(x))dx = 3k$, where function g is continuous for $x \in R$ and given $g(x) \geq 0$ for $x \in [0, k]$.

$$3\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$$

$$= 3\int_0^{3k} g\left(\frac{x}{3}\right)dx - 3\int_0^{3k} 1\,dx$$

$$= 3\times 3(3k) - 3[x]_0^{3k} \quad \text{(1M)}$$

$$= 27k - 9k$$

$$= 18k \quad \text{(1A)}$$

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Question 6

Let $g: R \to R, g(x) = (a - x)^2$, where a is a real constant.

The average value of a on the interval $[-1, 1]$ is $\frac{31}{12}$. Find the value(s) of a.

$\frac{31}{12} = \frac{1}{1-(-1)} \int_{-1}^{1} (a-x)^2 \, dx$ (1M)

since $(a-x)^2 = a^2 - 2ax + x^2$,

$\int_{-1}^{1} (a-x)^2 \, dx = \int_{-1}^{1} a^2 - 2ax + x^2 \, dx$

$= \int_{-1}^{1} a^2 \, dx - \int_{-1}^{1} 2ax \, dx + \int_{-1}^{1} x^2 \, dx$

$= 2a^2 + \frac{2}{3}$ (1M)

$\frac{31}{12} = \frac{1}{2}\left(2a^2 + \frac{2}{3}\right) \Rightarrow a^2 = \frac{9}{4}$ (1A)

$a = \pm \frac{3}{2}$

Question 7

If $y = \frac{\tan(x)}{4}$, find $\frac{dy}{dt}$, given $\frac{dx}{dt} = \frac{2}{\sqrt{t}}$ and $x = 4$ when $t = 1$.

$x = \int \frac{dx}{dt} \, dt = \int \frac{2}{\sqrt{t}} \, dt = 4\sqrt{t} + c.$

Sub in $x = 4$ when $t = 1$:

$4 = 4\sqrt{1} + c \rightarrow c = 0.$ (1M)

Hence, $x = 4\sqrt{t}$.

$y = \frac{\tan(4\sqrt{t})}{4}$ (1M)

$\frac{dy}{dt} = \frac{1}{\sqrt{t} \times 2\cos^2(4\sqrt{t})}$ (1A)

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Sub-Section: Exam 2 (Tech-Active)

Question 8

If $\int ae^{bx}dx = -2e^{2x} + c$, then:

A. $a = 4$ and $b = -2$.

B. $a = -2$ and $b = 2$.

C. $a = -1$ and $b = 2$.

D. $a = -4$ and $b = 2$.

Question 9

The gradient of a curve is given by $2\cos\left(\frac{x}{2}\right)$. If the x-intercept is $x = \frac{5\pi}{3}$ then, the y-intercept will be at $y =$

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. -2

D. $\frac{\sqrt{3}}{2}$

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Question 10

Let $f(x) = px^r$ and $g(x) = qx^s$, where a, b, m and n are positive integers. The domain of f = domain of $g = R$. If $f'(x)$ is an anti-derivative of $g(x)$, then which one of the following must be true?

A. $\frac{r}{s}$ is an integer.

B. $\frac{s}{r}$ is an integer.

C. $\frac{p}{q}$ is an integer.

D. $\frac{q}{p}$ is an integer.

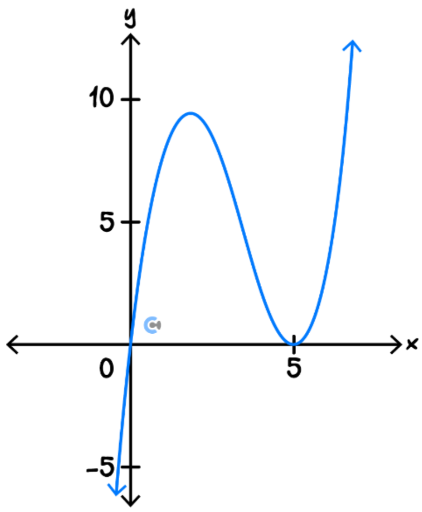
$f'(x) = prx^{r-1} = \int g(x)\,dx$

$= \frac{q}{s+1}x^{s+1}$

$prx^{r-1} = \frac{q}{s+1}x^{s+1}$

$\frac{q}{p} = r(s+1) \Rightarrow$ integer.

Question 11



The graph of $y = f'(x)$ is shown above. Which of the following statements is true for the graph of $y = f(x)$?

A. The graph has a local maximum at $x = 0$ and a stationary point of inflection at $x = 5$.

B. The graph has a local minimum at $x = 0$ and a stationary point of inflection at $x = 5$.

C. The graph has a local maximum at $x = 5$ and a stationary point of inflection at $x = 0$.

D. The graph has a local minimum at $x = 5$ and a stationary point of inflection at $x = 0$.

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Question 12

If $\int_2^6 f(x)dx = 8$, then $\int_0^2 f(2x+2)dx$ is equal to:

A. 4

B. 6

C. 8

D. 10

Question 13

If $\int_0^3 g(x)dx = 18$ and $\int_0^3 (2g(x) + ax)dx = 72$, then the value of a is:

A. 2

B. 4

C. 6

D. 8

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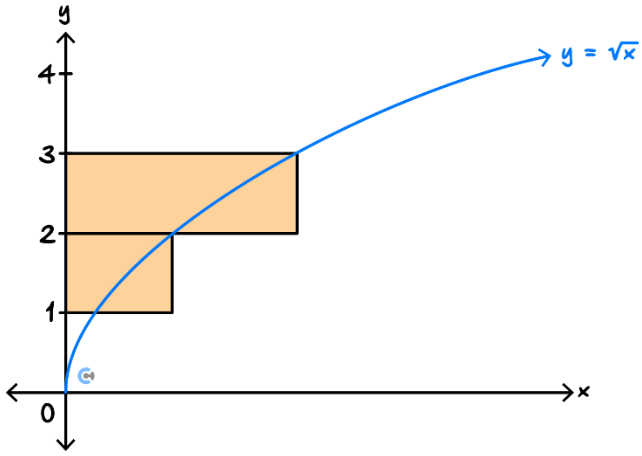
Question 14

Lily and Max are calculating the area between the graph of $y = \sqrt{x}$ and the y-axis between $y = 1$ and $y = 3$. Jake uses a partitioning, shown in the diagram below, while Anita uses a definite integral to find the exact area.



The difference between the results obtained by Jake and Anita is:

A. 0

B. $\frac{22}{3}$

C. $\frac{26}{3}$

D. $\frac{11}{3}$

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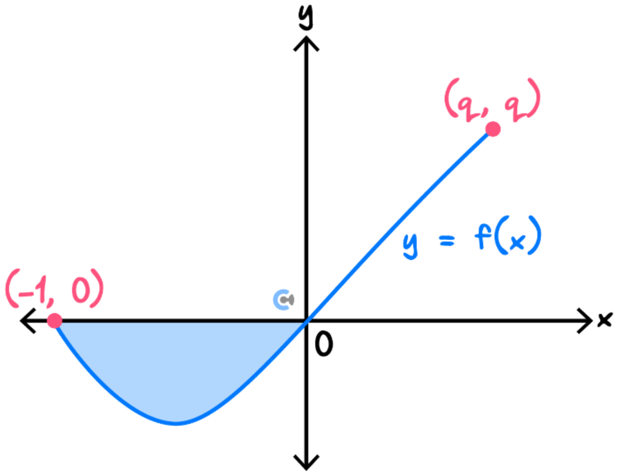
Question 15

The graph of a function $\boldsymbol{f}: [-\mathbf{1}, \boldsymbol{q}] \to \boldsymbol{R}$ is shown below.



The average value of $\boldsymbol{f}$ over the interval $[-\mathbf{1}, \boldsymbol{q}]$ is zero. The area of the shaded region is $\frac{9}{2}$.

If the graph is a straight line, for $\mathbf{0} \leq \boldsymbol{x} \leq \boldsymbol{q}$, then the value of q is:

A. 2

B. 5

C. $\frac{5}{2}$

D. 3

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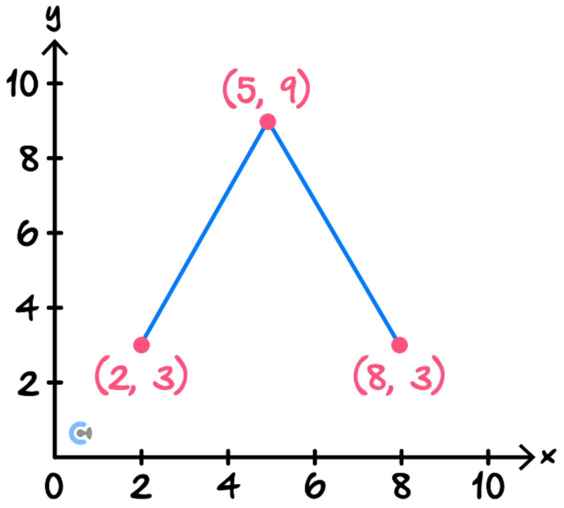
Question 16

The graph of a function, h, is shown below.



The average value of h is:

A. 3

B. 5

C. 6

D. 7

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Question 17

The algorithm below, described in pseudocode, estimates the value of a definite integral using the trapezium rule.

Inputs:

- $f(x)$, the function to integrate.
- a, the lower terminal of integration.
- b, the upper terminal of integration.
- n, the number of trapeziums to use.

Define trapezium $(f(x), a, b, n)$
    $h \leftarrow (b - a) \div n$
    $sum \leftarrow f(a) + f(b)$
    $x \leftarrow a + h$
    $i \leftarrow 1$
    While $i < n$ Do
        $sum \leftarrow sum + 2 \times f(x)$
        $x \leftarrow x + h$
        $i \leftarrow i + 1$
    EndWhile
    $area \leftarrow (h \div 2) \times sum$
    Return area

Consider the algorithm implemented with the following inputs:

$$trapezium\ (\ln(x), 1, 5, 4)$$

What is the value of sum **after the 3rd iteration** of the loop?

A. $2\ln(1) + 2\ln(2) + 2\ln(3) + 2\ln(4) + 2\ln(5)$

B. $\ln(1) + 2\ln(2) + 2\ln(3) + \ln(5)$

C. $2\ln(2) + 2\ln(3) + 2\ln(4) + \ln(5)$

D. $2\ln(2) + 2\ln(3) + 2\ln(4)$

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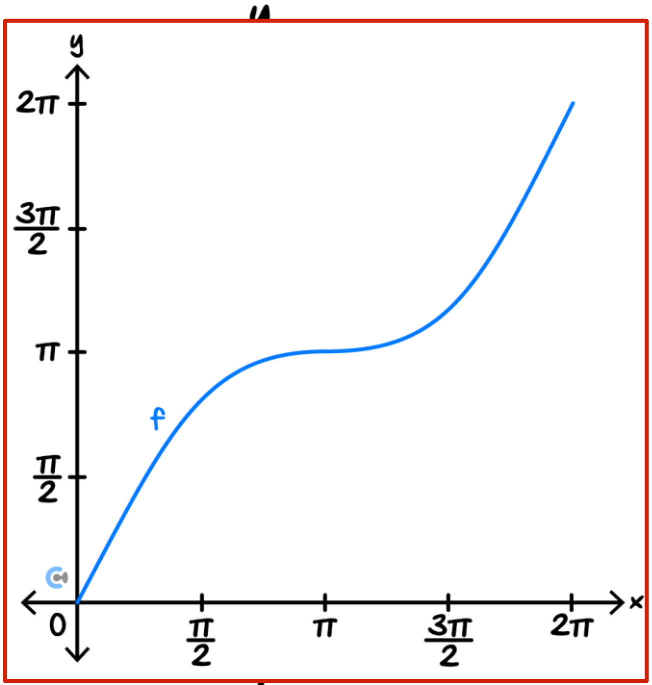
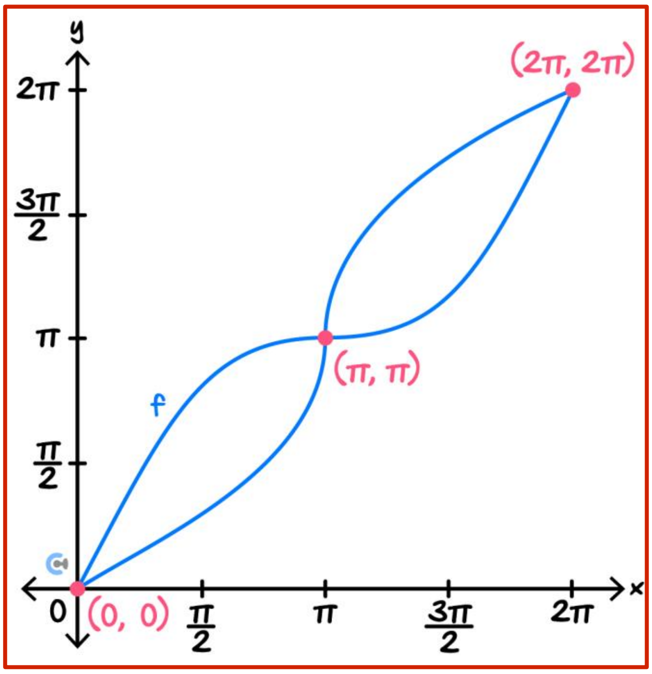
Question 18

A data science researcher is studying a nonlinear transformation model described by the function:
$f(x) = x + \sin(x), x \in [0, 2\pi]$.

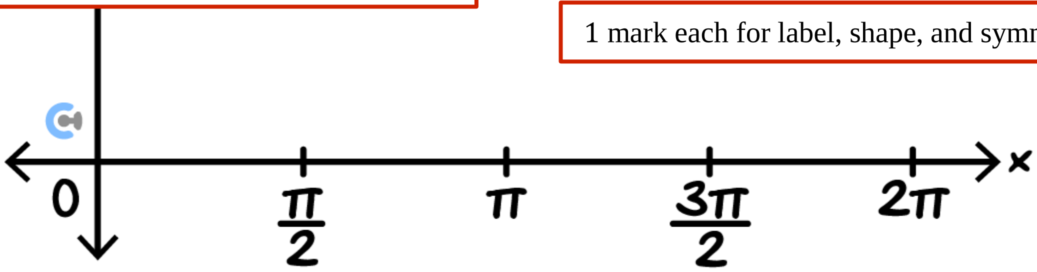
a. **Sketch the function** $f(x) = x + \sin(x)$ over the interval $[0, 2\pi]$. Label the endpoints clearly.





Label $(0, 0)$ and $(2\pi, 2\pi)$. (1 mark), shape (1 mark)

1 mark each for label, shape, and symmetry.



b. Given that $f(x)$ is strictly increasing on $[0, 2\pi]$, and thus invertible, **state the domain of** $f^{-1}(x)$.

$[0, 2\pi]$

c. **Sketch** $f^{-1}(x)$ on the same axis and **identify the coordinates of any intersections** between the graphs of $f(x)$ and $f^{-1}(x)$.

d. **Find the area between** $f(x)$ and $f^{-1}(x)$ on their domain of overlap.

$$\text{Area} = 4\int_0^{\pi} f(x) - x \, dx \quad \text{(1M)}$$

$$= 8 \quad \text{(1A)}$$

e. **Calculate the area** under the curve $f^{-1}(x)$ from $x = 0$ to $x = 4$, bounded by the x-axis. Round your answer to **2 decimal places**.

solve$(x+\sin(x)=4,x)$ $\quad x=4.96761$

$$4\cdot 4.96761-\int_0^{4.96761}(\sin(x)+x)\,dx \quad 6.78432$$

2 marks for method, 1 mark for solution.

f. **Verify** that the point $A\left(\frac{5\pi}{6}+\frac{1}{2},\frac{5\pi}{6}\right)$ lies on the graph of $f^{-1}(x)$.

$$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \sin\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{1}{2} \Rightarrow f^{-1}\left(\frac{5\pi}{6} + \frac{1}{2}\right) = \frac{5\pi}{6}$$

g. **Find the area bounded** by the tangent to $f^{-1}(x)$ at point A and the curve $f^{-1}(x)$. Give your answer correct to **2 decimal places**.

2 marks for method, 1 mark for solution.

$$\text{tangentLine}\left(\sin(x)+x,x,\frac{5\cdot\pi}{6}\right) \qquad \frac{5\cdot\pi\cdot\sqrt{3}+6}{12}-\frac{(\sqrt{3}-2)\cdot x}{2}$$

$$\text{solve}\left(\frac{5\cdot\pi\cdot\sqrt{3}+6}{12}-\frac{(\sqrt{3}-2)\cdot x}{2}=\sin(x)+x,x\right) \qquad x=4.20402$$

$$\int_{\frac{5\cdot\pi}{6}}^{4.20402}\left(\frac{5\cdot\pi\cdot\sqrt{3}+6}{12}-\frac{(\sqrt{3}-2)\cdot x}{2}-(\sin(x)+x)\right)dx \qquad 0.083051$$

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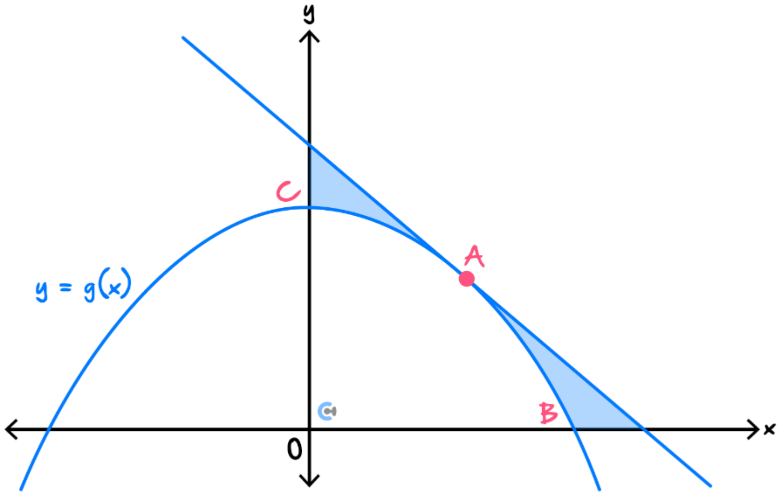
Question 19

Part of the graph of a function $R \to R, g(x) = 12 - 2x^2$ is shown below.



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the line perpendicular to the graph of g at the point A.

$C(0, 12), B(\sqrt{6}, 0)$ gradient of $BC = \frac{12-0}{0-\sqrt{6}} = \frac{-12}{\sqrt{6}}$

$\text{solve}\left(\frac{d}{dx}\left(12-2\cdot x^2\right)=\frac{-12}{6^{\frac{1}{2}}},x\right) \quad x=\frac{\sqrt{6}}{2}$

$\text{normalLine}\left(12-2\cdot x^2,x,\frac{\sqrt{6}}{2}\right) \quad \frac{\sqrt{6}\cdot x}{12}+\frac{35}{4}$

1 mark for method, 1 mark for solution.

ii. Find the average rate of change of $f(x)$ between $x = 0$ and the x-coordinate of point A.

$\text{Define } f(x)=12-2\cdot x^2 \quad \textit{Done}$

$\frac{f\left(\frac{\sqrt{6}}{2}\right)-f(0)}{\frac{\sqrt{6}}{2}-0} \quad -\sqrt{6}$

1 mark for method, 1 mark for solution.

iii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

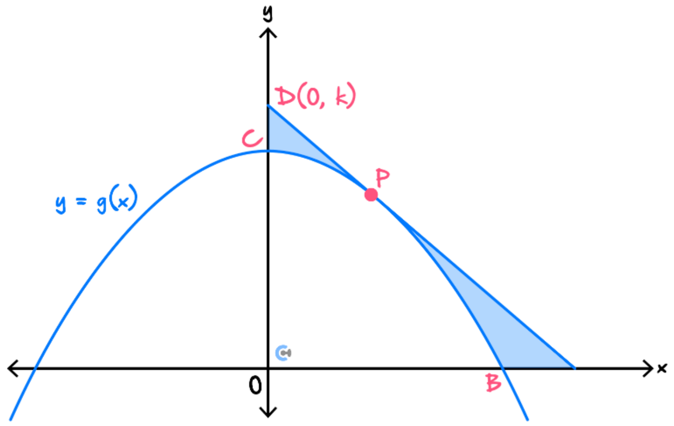
Calculate the area of this shaded region.

Area = Area of Triangle − Area under parabola

$\text{tangentLine}\left(f(x),x,\frac{\sqrt{6}}{2}\right)$	$15-2\cdot\sqrt{6}\cdot x$
$15-2\cdot\sqrt{6}\cdot x\vert x=0$	15
$\text{solve}(15-2\cdot\sqrt{6}\cdot x=0,x)$	$x=\frac{5\cdot\sqrt{6}}{4}$
$\frac{15\cdot\frac{5\cdot\sqrt{6}}{4}\cdot 1}{2}-\int_0^{\sqrt{6}} f(x)\,dx$	$\frac{11\cdot\sqrt{6}}{8}$

2 marks for method, 1 mark for solution.

b. The tangent to the graph of g at a point p has a negative gradient and intersects the y-axis at point $D(0, k)$, where $14 \leq k \leq 20$.



i. Find the equation of the tangent line at point p in terms of k.

tangentLine$(f(x),x,k)$ $\quad 2\cdot(k^2+6)-4\cdot k\cdot x$

1 mark for method, 1 mark for solution.

ii. Find the rule $A(k)$ for the function of k that gives the area of the shaded region.

tangentLine$(f(x),x,k)$ $\quad 2\cdot(k^2+6)-4\cdot k\cdot x$

$2\cdot(k^2+6)-4\cdot k\cdot x|x=0$ $\quad 2\cdot(k^2+6)$

solve$(2\cdot(k^2+6)-4\cdot k\cdot x=0,x)$ $\quad x=\frac{k^2+6}{2\cdot k}$

$$2\cdot(k^2+6)\cdot\frac{k^2+6}{2\cdot k}\cdot\frac{1}{2}-\int_0^{\sqrt{6}} f(x)\,dx \quad \frac{k^4+12\cdot k^2-16\cdot k\cdot\sqrt{6}+36}{2\cdot k}$$

1 mark for method, 1 mark for solution.

iii. Find the maximum area of the shaded region and the value of k for which this occurs, give to 2 decimal places.

$$\text{fMax}\left(\frac{k^4+12\cdot k^2-16\cdot k\cdot\sqrt{6}+36}{2\cdot k},k,14,20\right) \quad k=20$$

$$\frac{k^4+12\cdot k^2-16\cdot k\cdot\sqrt{6}+36}{2\cdot k}|k=20 \quad \frac{-(80\cdot\sqrt{6}-41209)}{10}$$

ANS: 4101.30
2 marks for answer.

iv. If $12 \leq k \leq 24$, find the minimum area of the shaded region and the value of k for which this occurs, give to 2 decimal places.

$$\text{fMin}\left(\frac{k^4+12\cdot k^2-16\cdot k\cdot\sqrt{6}+36}{2\cdot k},k,12,24\right) \quad k=12$$

$$\frac{k^4+12\cdot k^2-16\cdot k\cdot\sqrt{6}+36}{2\cdot k}|k=12 \quad \frac{-(16\cdot\sqrt{6}-1875)}{2}$$

ANS: 917.90
2 marks for answer.

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Section B: Supplementary Questions

Sub-Section: Exam 1 (Tech-Free)

Question 20

a. Evaluate $\int_1^5 \left(\frac{1}{\sqrt{x}}\right) dx$.

$$\int_1^5 \left(\frac{1}{\sqrt{x}}\right) dx$$

$$= \left[2\sqrt{x}\right]_1^5 \quad \text{(1M)}$$

$$= 2\sqrt{5} - 2 \quad \text{(1A)}$$

b. If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

$$f(x) = \int 2\cos(x) - \sin(2x)\, dx$$

$$= \frac{\cos(2x)}{2} + 2\sin(x) + c \quad \text{(1M)}$$

Sub in $\left(\frac{\pi}{2}, \frac{1}{2}\right)$: $c = -1$

$$f(x) = \frac{1}{2}\cos(2x) + 2\sin(x) - 1 \quad \text{(1A)}$$

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Question 21

a. Find $\int_1^2 3x^2 - 4x + \frac{5}{x} + \sin(x)$.

$$\int \left(3x^2 - 4x + \frac{5}{x} + \sin(x)\right) dx = x^3 - 2x^2 + 5\ln|x| - \cos(x) + C$$ 1M

$$[5\ln(2) - \cos(2)] - [-1 - \cos(1)] = 5\ln(2) - \cos(2) + 1 + \cos(1)$$ 1A

b. If $f(x) = x\log_e(x)$,

i. Find $f'(x)$.

$$f'(x) = \log_e(x) + 1$$

ii. Hence, find $\int \log_e(x)dx$.

$$\int \log_e(x)\,dx = x\log_e(x) - \int 1\,dx$$ (1M)

$$= x\log_e(x) - x + c$$ (1A)

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Question 22

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, find $2\int_0^{7a} \left(f\left(\frac{x}{7}\right) + 2\right)dx$.

$$2\int_0^{7a} \left(f\left(\frac{x}{7}\right) + 2\right) dx$$

$$= 2\int_0^{7a} f\left(\frac{x}{7}\right) dx + 2\int_0^{7a} 2\, dx$$

$$= 2 \cdot (7a + 14a)$$

$$= 42a.$$

Question 23

Find the value(s) of $\boldsymbol{k}$ for which the average value of $\boldsymbol{y = \sin(kx)}$ over the interval $[\boldsymbol{0, \pi}]$ is equal to the average value of $\boldsymbol{y = \cos(x)}$ over the same interval.

$$\frac{1}{\pi}\int_0^{\pi} \cos(x)\, dx = \frac{1}{\pi}\left[\sin(x)\right]_0^{\pi} = 0$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(kx)\, dx = 0$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(kx)\, dx = \frac{1}{\pi}\left[-\frac{\cos(kx)}{k}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{-1}{k}\left(\cos(k\pi) - \cos(0)\right)$$

$$= \frac{-1}{\pi k}\left(\cos(k\pi) - 1\right)$$

$\cos(k\pi) - 1 = 0 \Rightarrow \cos(k\pi) = 1 \Rightarrow k = 2n,\ n \in \mathbb{Z}.$

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Question 24

Let $f: R \to R, f(x) = x^2 e^{-kx}$, where k is a negative real constant.

a. Show that $f'(x) = xe^{-kx}(-kx + 2)$.

$$f(x) = x^2 \cdot e^{-kx} \Rightarrow f'(x) = \frac{d}{dx}(x^2) \cdot e^{-kx} + x^2 \cdot \frac{d}{dx}(e^{-kx})$$

$$= 2x \cdot e^{-kx} + x^2 \cdot (-ke^{-kx}) = e^{-kx}(2x - kx^2) = xe^{-kx}(2 - kx) \Rightarrow \boxed{f'(x) = xe^{-kx}(-kx + 2)}$$

b. Find the value(s) of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection.

$f(x) = f'(x)$

$x^2 = (-kx + 2)x$ since $e^{-kx} \neq 0$

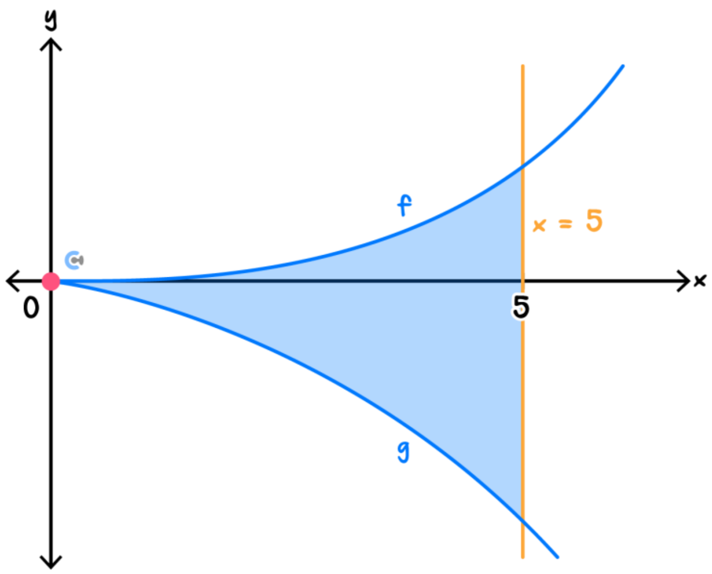
$x(x + kx - 2) = 0$

① $x = 0$

② $(k+1)x = 2 \Rightarrow x = \frac{2}{k+1}$

Hence $k = -1$

Let $g(x) = \frac{2xe^{-kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.



Let A be the area of the region bounded by the curves $y = f(x), y = g(x)$ and the line $x = 5$.

c. Write down a definite integral that gives the value of A.

$$\int_0^5 f(x) - g(x)\, dx$$

$$= \int_0^5 x^2 e^{-kx} - \frac{2xe^{-kx}}{k}\, dx$$

d. Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{10}{-k}$.

$$A = \int_0^5 xe^{-kx}\left(x - \frac{2}{k}\right) dx$$

$$= -\frac{1}{k}\int_0^5 f'(x)\, dx$$

$$= -\frac{1}{k}\left[x^2 e^{-kx}\right]_0^5$$

$$= -\frac{1}{k}\left(25e^{-5k}\right) = \frac{10}{-k}$$

$$25e^{-5k} = 10$$

$$e^{-5k} = \frac{2}{5}$$

$$-5k = \log_e\left(\frac{2}{5}\right)$$

$$k = -\frac{1}{5}\log_e\left(\frac{2}{5}\right)$$

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Sub-Section: Exam 2 (Tech-Active)

Question 25

If $f(x) = \int_0^x \left(\sqrt{t^2+4}\right)dt$, then $f'(-2)$ is equal to:

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

Question 26

Which one of the following options is an anti-derivative of $\frac{1}{x^2} - \frac{1}{\cos^2\left(\frac{x}{2}\right)}$?

A. $-\frac{1}{x} - 2\tan\left(\frac{x}{2}\right)$

B. $-\frac{2}{x^3} - \frac{(2)}{\cos^3\left(\frac{x}{2}\right)}$

C. $\frac{1}{x} - \frac{1}{2}\tan\left(\frac{x}{2}\right)$

D. $\log_e(x^2) - \tan\left(\frac{x}{2}\right)$

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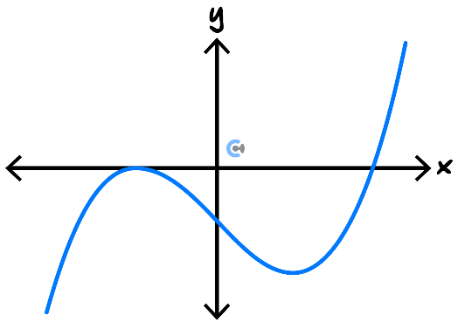
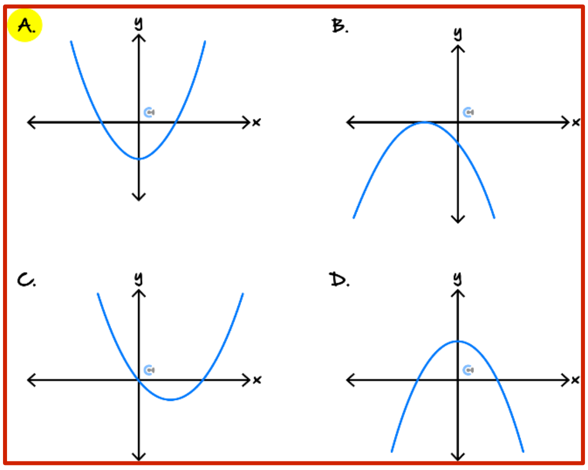
The following information applies to the two questions that follow.

For Questions 27 and 28, refer to the graph of $y = f(x)$ shown below.



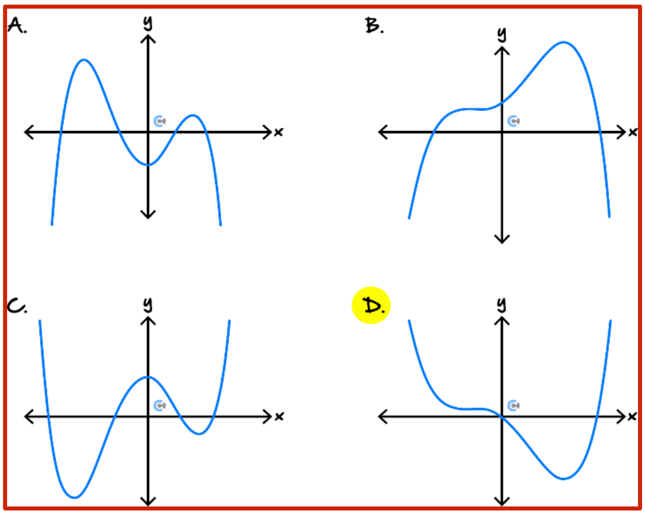
Question 27

The corresponding part of the derivative graph of $y = f(x)$ is best represented by:



Question 28

The corresponding part of the antiderivative graph of $y = f(x)$ is best represented by:



Question 29

Given that $\int_1^5 (f(x))dx = 4$, $\int_5^1 (f(x) - 2)dx$ is equal to:

A. 0

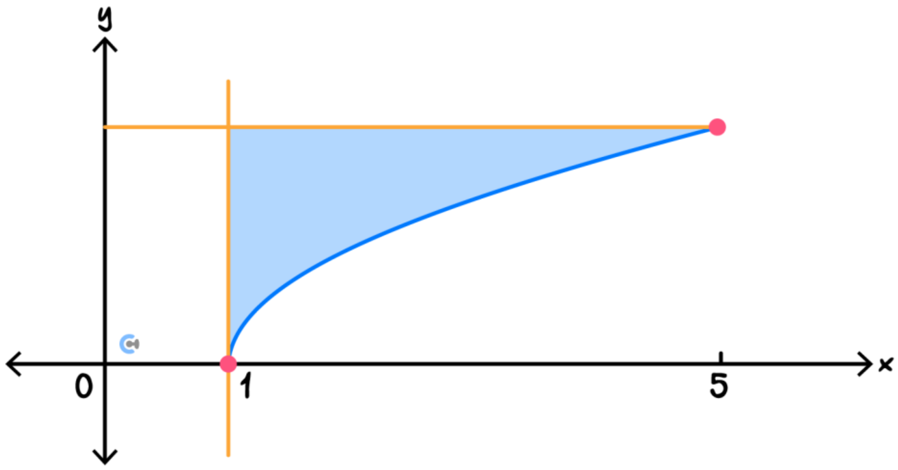
B. 1

C. 4

D. 7

Question 30

The graph of $\boldsymbol{g}: [\mathbf{1}, \mathbf{5}] \to \boldsymbol{R}, \boldsymbol{g}(\boldsymbol{x}) = 2\sqrt{\boldsymbol{x} - \mathbf{1}}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 (2\sqrt{x - 1})dx$

B. $\int_0^4 \left(\frac{x^2}{4}\right) dx$

C. $\int_0^4 (4 - 2\sqrt{x - 1})dx$

D. $\int_0^4 \left(\frac{x^2}{4} + 1\right) dx$

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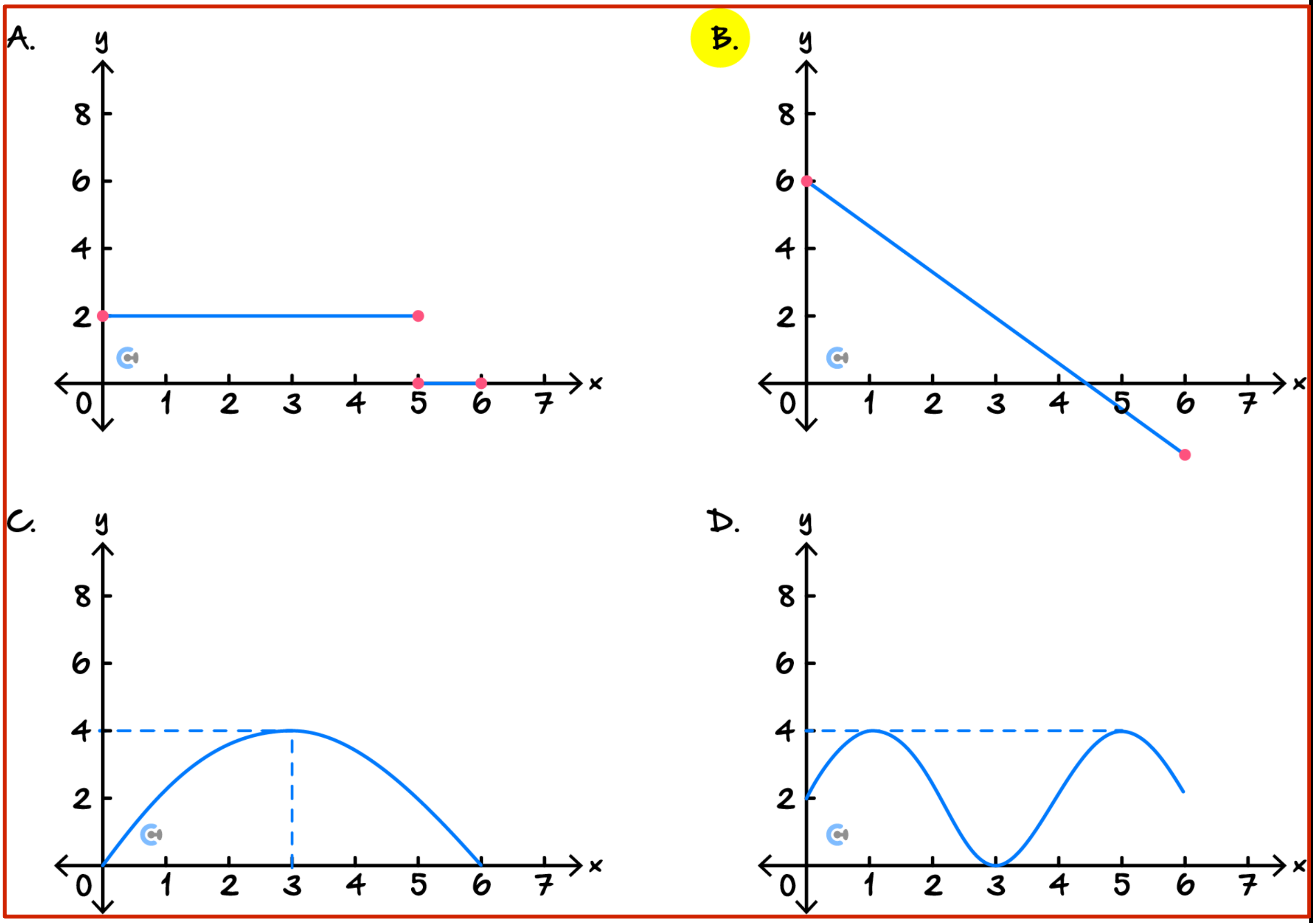
Question 31

Let g be a function with an average value of 2 over the interval $[0, 6]$.

The graph of g over this interval could be:



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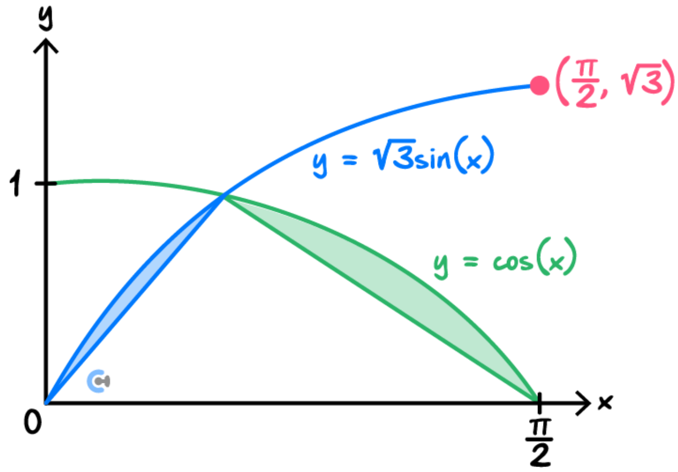
Question 32

The area of the shaded region would be:



A. $\sqrt{3} - 1 - \frac{\sqrt{3}\pi}{8}$

B. $\sqrt{3} - 1 - \frac{\sqrt{3}\pi}{4}$

C. $\frac{\sqrt{3}\pi}{8}$

D. $\frac{\pi}{2}(\sqrt{3} - 1)$

E. $\frac{3\pi}{8} - \sqrt{3}$

Question 33

The temperature $T°$ over a time period of a day is given by the function $T(t) = 17 - 6\sin\left(\frac{\pi t}{12}\right)$, where t is the time in hours. Using the given function, the average temperature over the first 12 hours is equal to:

A. 17

B. $204 - \frac{12}{\pi}$

C. $17 + \frac{12}{\pi}$

D. $17 - \frac{12}{\pi}$

E. $\frac{12}{\pi}$

Question 34

The following pseudocode is intended to estimate the value of a definite integral using the trapezium rule. However, one line in the loop is missing.

Inputs:

- $f(x)$, the function to integrate.
- a, the lower terminal of integration.
- b, the upper terminal of integration.
- n, the number of trapeziums to use.

Define trapezium $(f(x), a, b, n)$
    $h \leftarrow (b - a) \div n$
    $sum \leftarrow f(a) + f(b)$
    $x \leftarrow a + h$
    $i \leftarrow 1$
    While $i < n$ Do
        ____
        $x \leftarrow x + h$
        $i \leftarrow i + 1$
    EndWhile
    $area \leftarrow (h \div 2) \times sum$
    Return area

A. $sum \leftarrow f(x) + 2$

B. $sum \leftarrow sum + f(x)$

C. $sum \leftarrow sum + 2 \times f(x)$

D. $sum \leftarrow sum \times 2 \times f(x)$

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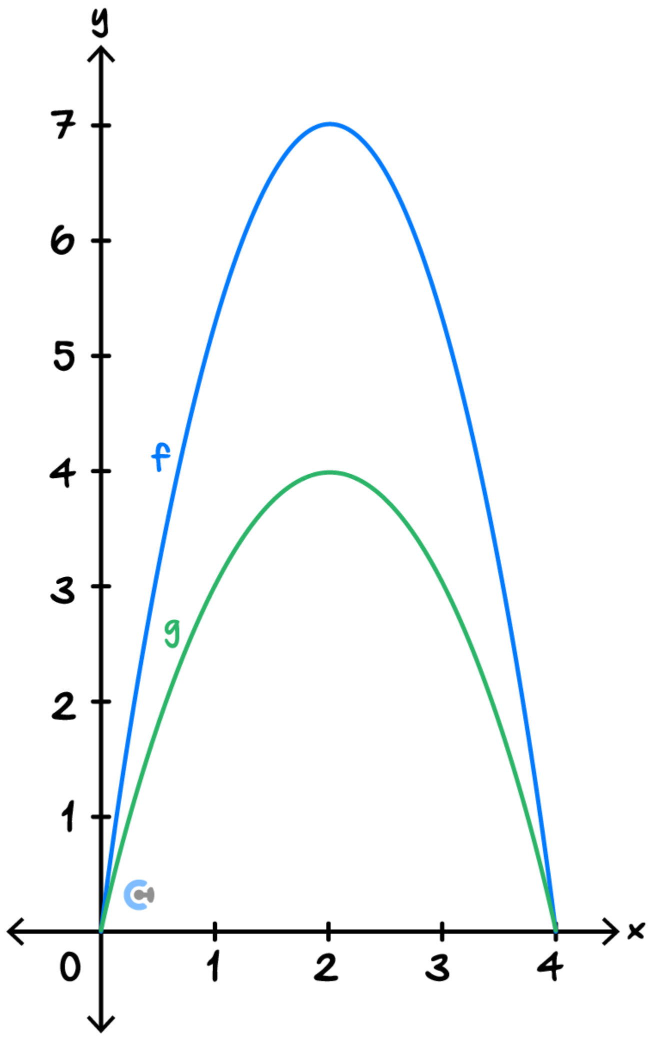
Question 35

Assume that $f(x) = \frac{7x}{4}(4 - x)$ and $g(x) = 4x - x^2$, $0 \leq x \leq 4$.



a. Find the angle between the tangents drawn to f and g when $x = 0$, in degrees, correct to 2 decimal places.

Define $f(x)=\frac{(4-x)\cdot 7\cdot x}{4}$	*Done*
Define $g(x)=4\cdot x-x^2$	*Done*
$\tan^{-1}\left(\frac{d}{dx}(f(x))\right)-\tan^{-1}\left(\frac{d}{dx}(g(x))\right)\|x=0$	$\tan^{-1}\left(\frac{1}{4}\right)-\tan^{-1}\left(\frac{1}{7}\right)$
$\tan^{-1}\left(\frac{d}{dx}(f(x))\right)-\tan^{-1}\left(\frac{d}{dx}(g(x))\right)\|x=0$	0.103082

$\frac{0.103082\cdot 180}{\pi}$	5.90616

1 mark for method, 1 mark for answer: 5.91

b. Find the average value of the function $y = f(x) - g(x)$ on the interval $[0, 4]$.

$$\frac{1}{4}\cdot\int_0^4 (f(x)-g(x))\,dx \qquad 2$$

1 mark for method, 1 mark for answer.

c. Let $(k, g(k))$ be a random point on graph of g, find the value of area bounded by the tangent of $g(x)$ at $x = k$, $g(x)$, and $f(x)$.

Area required = Area bounded by f and g – area bounded by f and tangent line

$$\text{tangentLine}(g(x),x,k) \qquad k^2-2\cdot(k-2)\cdot x$$

$$\text{solve}(f(x)=k^2-2\cdot(k-2)\cdot x,x)$$

$$x=\frac{-2\cdot\left(\sqrt{-3\cdot(k^2-4\cdot k-3)}-2\cdot k-3\right)}{7} \text{ or } x=\frac{2\cdot\left(\sqrt{-3\cdot(k^2-4\cdot k-3)}+2\cdot k+3\right)}{7}$$

$$\int_{\frac{-2\cdot\left(\sqrt{-3\cdot(k^2-4\cdot k-3)}-2\cdot k-3\right)}{7}}^{\frac{2\cdot\left(\sqrt{-3\cdot(k^2-4\cdot k-3)}+2\cdot k+3\right)}{7}} \left(f(x)-(k^2-2\cdot(k-2)\cdot x)\right)dx \qquad \frac{8\cdot(-k^2+4\cdot k+3)^{\frac{3}{2}}\cdot\sqrt{3}}{49}$$

1 mark for method, 1 mark for answer.

$$\int_0^4 (f(x)-g(x))\,dx-\frac{8\cdot(-k^2+4\cdot k+3)^{\frac{3}{2}}\cdot\sqrt{3}}{49} \qquad 8-\frac{8\cdot(-k^2+4\cdot k+3)^{\frac{3}{2}}\cdot\sqrt{3}}{49}$$

d. Find the value of k such that the bounded area is minimum, and state the minimum value.

$$\text{fMin}\left(8-\frac{8\cdot(-k^2+4\cdot k+3)^{\frac{3}{2}}\cdot\sqrt{3}}{49},k,0,4\right) \qquad k=2$$

$$8-\frac{8\cdot(-k^2+4\cdot k+3)^{\frac{3}{2}}\cdot\sqrt{3}}{49}\Big|k=2 \qquad 8-\frac{8\cdot\sqrt{21}}{7}$$

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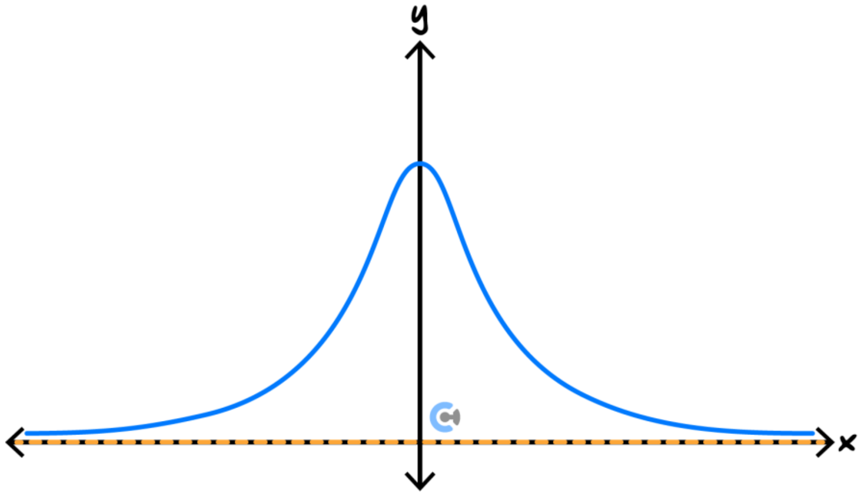
Question 36

An engineer is exploring the safety of the jumps that have been built along the track. A typical jump follows the rule $h(x) = \frac{1}{x^2+1}$ as shown in the diagram.



a. Find an approximation, for the area under the curve from $x = -3$ to $x = 3$ using intervals of width one unit and right endpoint rectangles.

$$1 \times \big(h(-2) + h(-1) + h(0) + h(1) + h(2) + h(3)\big) = \frac{5}{2}$$

1 mark for method, 1 mark for answer.

b. Find the exact area under $h(x)$ from $x = -3$ to $x = 3$.

$\int_{-3}^{3} h(x)\,dx$

$= \pi - 2\tan^{-1}\left(\frac{1}{3}\right)$ sq. units (1A)

c. Hence, show your working to find $\int_{-1}^{1} h(3x)dx - \frac{\pi}{6}$.

$\int_{-1}^{1} h(3x)\,dx - \frac{\pi}{6}$

$= \frac{1}{3}\left[\pi - 2\tan^{-1}\left(\frac{1}{3}\right)\right] - \frac{\pi}{6}$ (1M)

$= \frac{\pi}{6} - \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$ (1A)

Let $g: [a, \infty] \to R, g(x) = \frac{1}{x^2+1}$ where a is the least possible value such that the inverse function $g^{-1}(x)$ exists.

d. State the value of a.

$a = 0$

e. Find the integral(s) required to find the area defined by the regions bounded by the graphs of $g(x), g^{-1}(x)$ and the lines $x = \frac{7}{10}$ and $x = 2$. You do not need to evaluate the integrals.

Area $= \int_{\frac{7}{10}}^{1} g(x) - g^{-1}(x)\, dx + \int_{1}^{2} g(x)\, dx$

or: $\int_{\frac{7}{10}}^{2} g(x)\, dx - \int_{\frac{7}{10}}^{1} g^{-1}(x)\, dx$

1A

The graph of g undergoes the listed transformations below to become the graph of p:

- Dilated by a factor of 3 from x-axis.
- Dilated by a factor of $\frac{1}{2}$ from y-axis.
- Reflected in the y-axis.

f. Find the rule for p^{-1} and state the domain.

$p(x) = \frac{3}{4x^2+1}, \quad x \in (-\infty, 0)$ (1M)

Swap x and y ... (1M)

$p^{-1}(x) = -\sqrt{\frac{3-x}{4x}}, \quad x \in (0, 3]$ (1A)

g. Find the average value of p^{-1} in the interval $[1, 2]$, correct to 2 decimal places.

$$\frac{1}{2-1}\cdot\int_1^2 -\left(\frac{3-x}{4}\cdot x\right)^{\frac{1}{2}}\,dx \qquad \frac{-\left(9\cdot\cos^{-1}\left(\frac{\sqrt{3}}{3}\right)-9\cdot\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)+2\cdot\sqrt{2}\right)}{8}$$

$$\frac{1}{2-1}\cdot\int_1^2 -\left(\frac{3-x}{4}\cdot x\right)^{\frac{1}{2}}\,dx \qquad -0.73587$$

1 mark for method, 1 mark for answer.

h. The area between $y = kp^{-1}(x)$, the lines $x = 1$, $x = 2$ and the x-axis is found to be at least 6 square units. Find the possible values of k correct to 1 decimal place.

$$\text{Area} = k_1\int_2^1 p^{-1}(x)\,dx \geq 6 \Rightarrow k \geq 11.8$$

or

$$\text{Area} = k_2\int_1^2 p^{-1}(x)\,dx \geq 6 \Rightarrow k \leq -11.8$$

$$k \in (-\infty, -11.8] \cup [11.8, \infty)$$

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Question 37

Let $g: [0, \infty) \to R, g(x) = \frac{3}{4x^2+1}$.

a. Using the fact that $g\left(g^{-1}(x)\right) = x$ or otherwise, find the rule for g^{-1} and its domain.

(a) $\frac{3}{4(g^{-1}(x))^2+1} = x$

$4(g^{-1}(x))^2+1 = \frac{3}{x}$ (1M)

$g^{-1}(x) = \pm\sqrt{\frac{3-x}{4x}}$

As domain of g is $[0, \infty)$

$g^{-1}(x) = \sqrt{\frac{3-x}{4x}}$. (1A)

b. Find the area bounded between $g(x)$ and $g^{-1}(x)$, correct to 2 decimal places.

$\text{solve}\left(\frac{3}{4\cdot x^2+1}=\left(\frac{3-x}{4\cdot x}\right)^{\frac{1}{2}}, x\right)$ $x=0.085786$ or $x=0.817183$ or $x=2.91421$

$\int_{0.085786}^{0.817183}\left(\frac{3}{4\cdot x^2+1}-\left(\frac{3-x}{4\cdot x}\right)^{\frac{1}{2}}\right)dx+\int_{0.817183}^{2.91421}\left(\left(\frac{3-x}{4\cdot x}\right)^{\frac{1}{2}}-\frac{3}{4\cdot x^2+1}\right)dx$

0.582254

1 mark for method, 1 mark for answer.

c. Find the value of a so that $\int_0^{0.5} g(x)dx = 1.5 - \int_a^3 0.5 - g^{-1}(x)dx$.

$\frac{3}{4\cdot x^2+1}|x=0.5$ 1.5

Using property of symmetry,

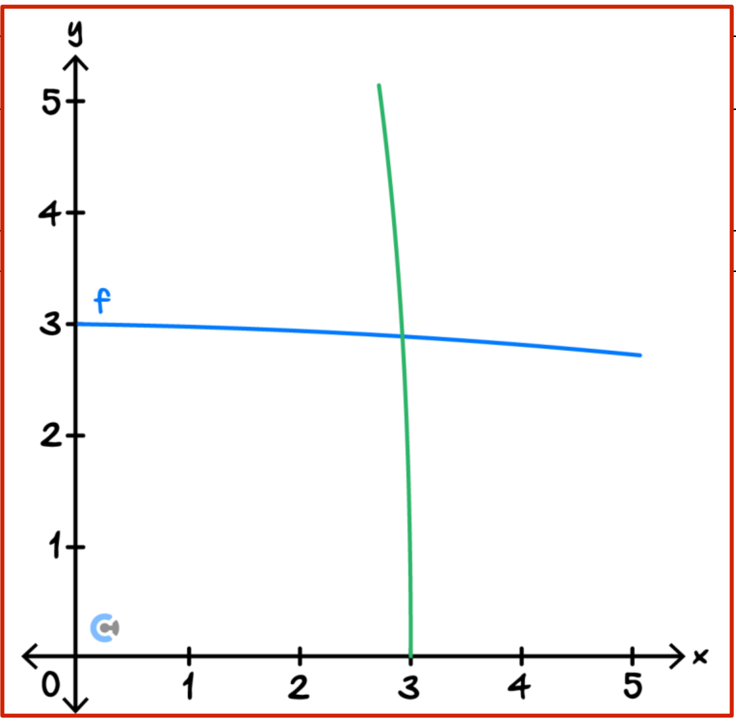
1 mark for method, 1 mark for answer.

Now, consider $f: [0, \infty) \to R, f(x) = \frac{3}{kx^2+1}, k > 0$.

d. State possible number of intersection points between $f(x)$ and $f^{-1}(x)$.

1 or 3 intersections (observed from the graph).

e. Consider the set of k that there is only one intersection point between $f(x)$ and $f^{-1}(x)$, find the largest possible area that is bounded by $f(x)$, $f^{-1}(x)$, x-axis and y-axis in terms of k.



Set a slider for k, then the largest possible area would be $3 \times 3 = 9$.

Spa

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