Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

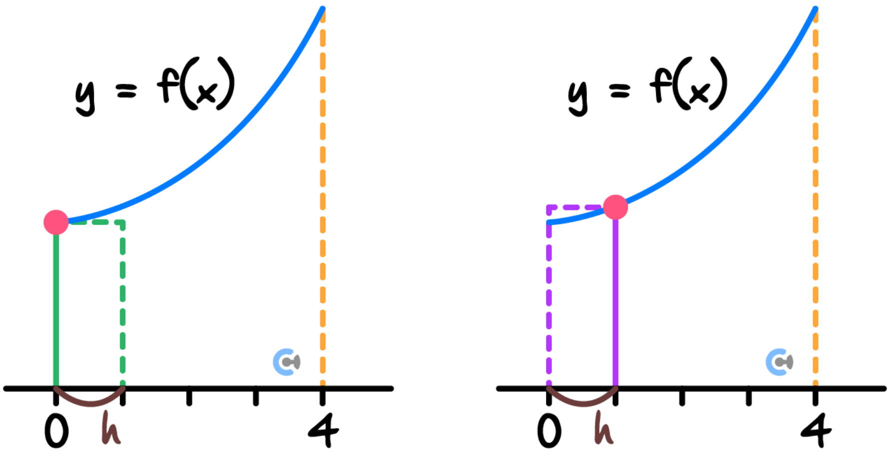
VCE Mathematical Methods ¾
Integration II [4.3]
Workbook

Outline:

Section A: Introduction

Let's quickly review last week's material.

Learning Objective: [4.2.1] – Approximate areas under the function

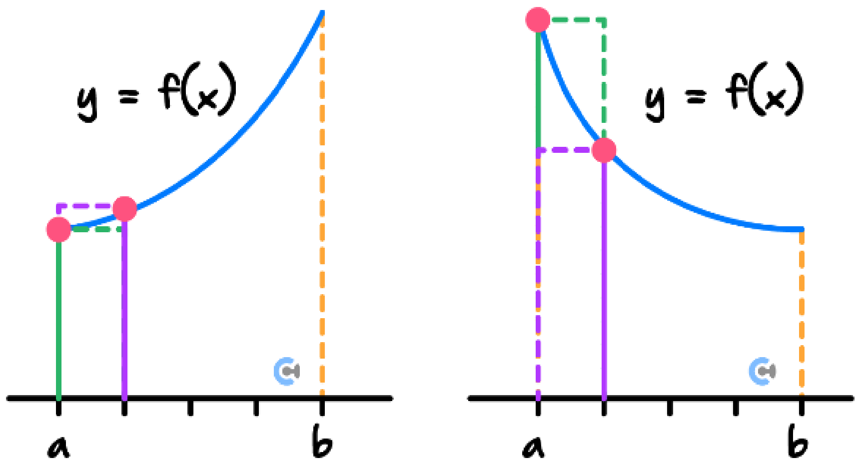


- **Riemann Sum:** Approximates the area under a curve using rectangles.
- **Height of the rectangles:** Determined by either the left or right y value of the function.
- **Step size (h):** Thickness of the rectangle.

$$h = \frac{b-a}{n} \text{ where } n = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Space for Personal Notes

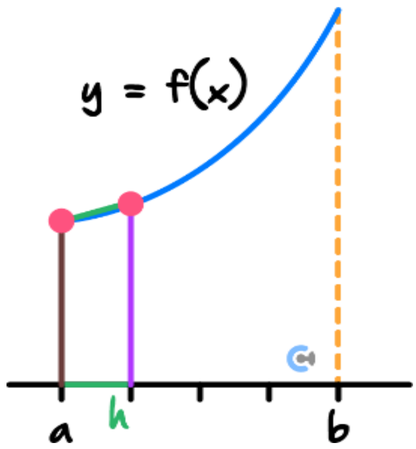
Overestimation and Underestimation of Riemann Sum



- Left endpoint approximation:
  - Increasing function: ______________________
  - Decreasing function: ______________________
- Right endpoint approximation:
  - Increasing function: ______________________
  - Decreasing function: ______________________

Space for Personal Notes

Trapezoidal Approximation



- **Trapezoidal Approximations:** Approximates the area under a curve using trapeziums.
- **Formula for each trapezium:** Approximates the area under a curve using trapeziums.

$$Area = \frac{h}{2}\left(f(a) + f(a+h)\right)$$

- **Step size (h):** Thickness of the trapezium.

$$h = \frac{b-a}{n} \text{ where } n = number\ of\ rectangles$$

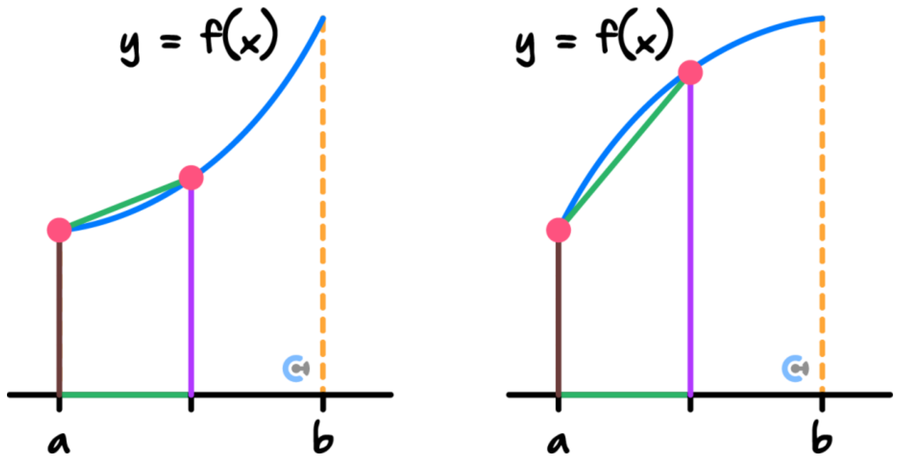
- **Formula:** Sum of all the trapeziums

$$Approximation = \frac{b-a}{2n}\left(f(a) + 2f(a+h) + 2f(a+2h) \cdots + f(b)\right)$$

$$VCAA\ Version = \frac{x_n - x_0}{2n}\left(f(x_0) + 2f(x_1) + 2f(x_2) \cdots + f(x_n)\right)$$

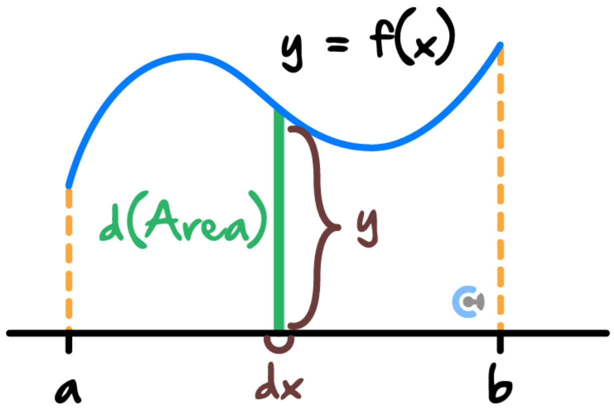
Space for Personal Notes

Overestimation and Underestimation of Trapezoidal Approximation



- **Trapezoidal approximation:**
  - Concave Up: ____________________________
  - Concave Down: ____________________________

Learning Objective: [4.2.2] – Find signed and total areas



- The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$Area = \_\_\_\_\_\_\_\_\_\_\_\_$$

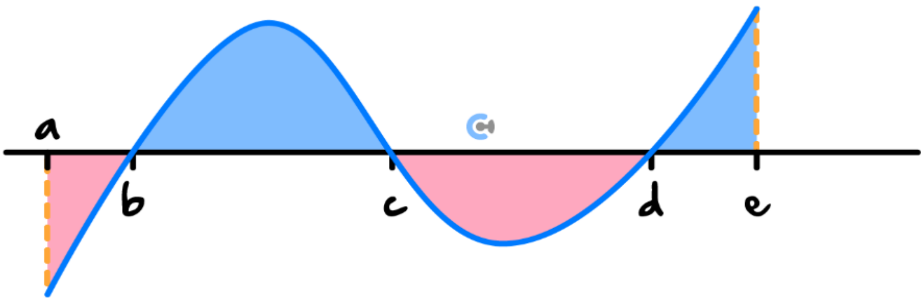
Space for Personal Notes

Signed Area

$$\textit{Signed Area} = \int_a^b y\, dx$$

- **Signed Area**: Areas with signs due to the sign of the y-value of the function.
- Positive Signed Area: ______________
- Negative Signed Area: ______________
- Positive and negative areas will cancel out.

Total Area



- **Total Area:** Total ______________________ area.
- Require ____________________ integrals for areas with _______________________________ y-value.

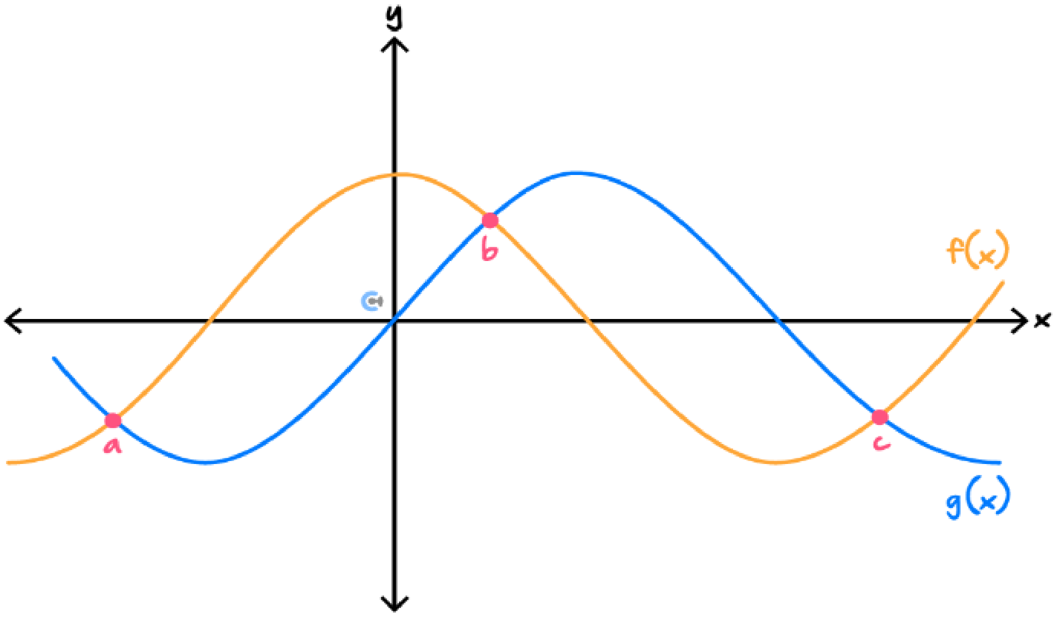
$$\textit{Total Area} = -\int_a^b y\, dx + \int_b^c y\, dx - \int_c^d y\, dx + \int_d^e y\, dx$$

- **Steps:**

1. __________________ the integrals for ______________________ and ____________________ y-values.
2. Turn the negative signed areas to positive by putting a _________________________ in front!

Space for Personal Notes

Learning Objective: [4.2.3] - Find the area between two functions



- **Area between two functions**:

Integrand (inside the integral) is the ________ function minus the ________________ function.

$$Area = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

What are we learning today?

Context: **Integration I Learning Objective**

- [4.3.1] - Find Average Value of the Function.

We will learn how to find an average y value of a function.

- [4.3.2] - Apply Integration by Recognition.

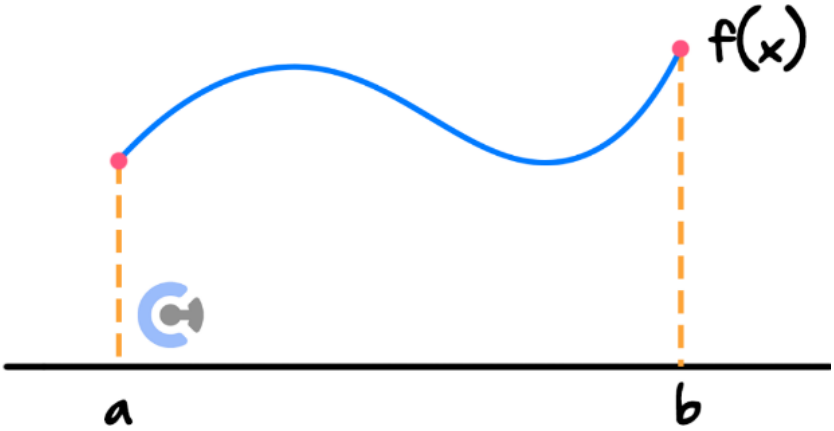
Learn how to integrate something impossible (usually).

- [4.3.3] - Advanced Areas

Trickier integration problems where we use inverse function properties.

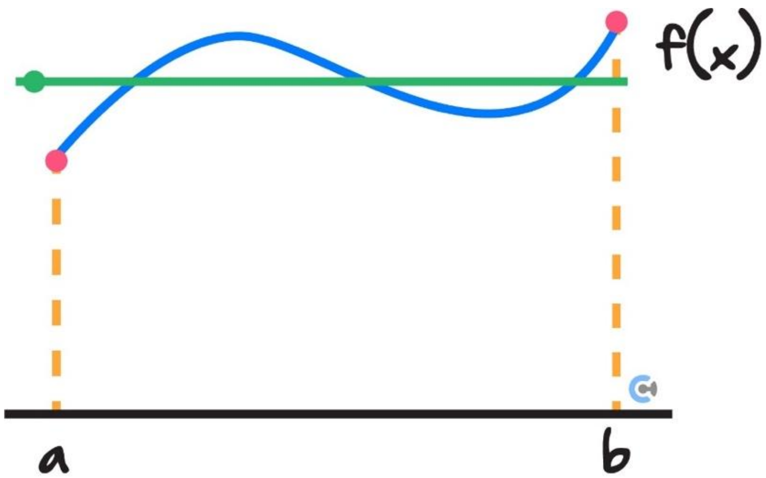
Section B: Average Value of the Function

Context: Average Value of the Function



- The value of the function represents the ______________________________ of the graph.
- Hence, the average value of the function is the average height of the function.

Average Value of the Function = Average height of the function



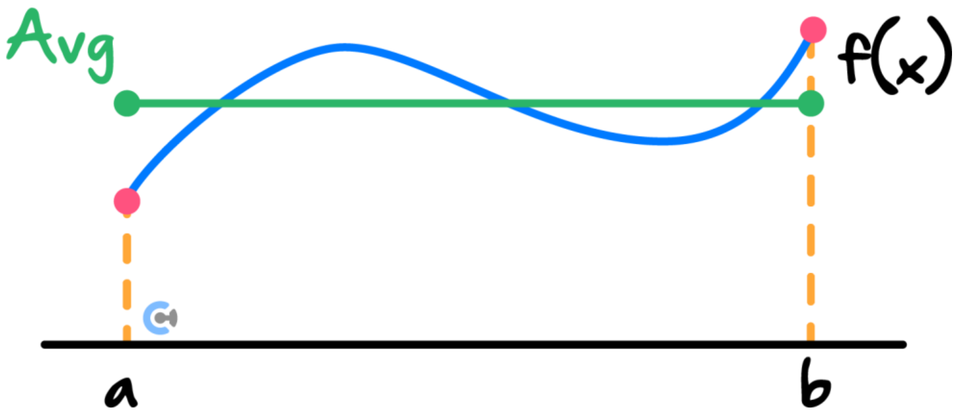
- How can we find the average height of the function?

Space for Personal Notes

Sub-Section: Average Value of the Function

Exploration: Average Value of the Function

- Consider the graph below.



- Visualise the average value of the function above.
  - Find the area under a function from $x = a$ to $x = b$.

$$Area = \_\_\_\_\_\_\_\_\_\_$$

  - Find the area under the average value.

$$Area = \_\_\_\_\_\_\_\_\_\_$$

  - What do you notice about the two areas above?

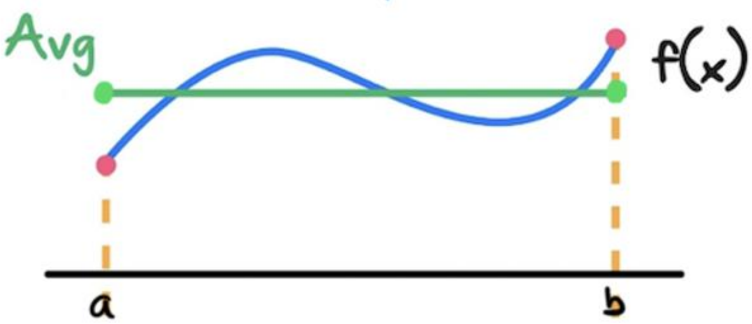
$$\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_$$

- Hence, what is the average value of $f(x)$ from $x = a$ to $x = b$?

$$Avg = \_\_\_\_\_\_\_\_\_\_$$

Space for Personal Notes

Average Value of a Function



- **Average value of the function**: Find the average y value of a function from $x = a$ and $x = b$.
- Average value of the function from $x = a$ to $x = b$ is given by:

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Discussion: How many integrals are we using? Hence, is the average value the signed area divided by the width, or the total area divided by the width?

Question 1 Walkthrough.

Find the average value of $f(x) = \frac{\cos(x)}{3}$ on the interval $\left[0, \frac{\pi}{2}\right]$.

TIP: Checking your average value.

- You can quickly check your answer by checking if it's between the minimum y value and maximum y value.
- Average MUST ALWAYS be in between.

Question 2

Find the average value of $f(x) = x^2 + 4$ on the interval $[0, 5]$.

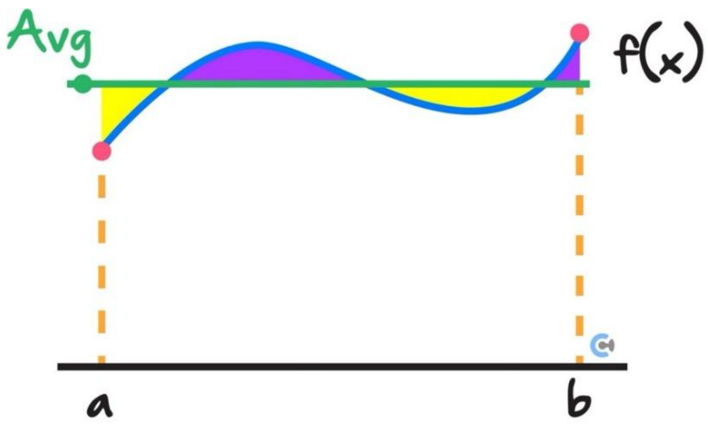
Question 3

Find the average value of $f(x) = \frac{1}{x} + 1$ on the interval $[2, 4]$.

Sub-Section: Visualising Average Value of the Function

Discussion: What should the area above the average value and the area under the average value be?

Visualisation of Average Value



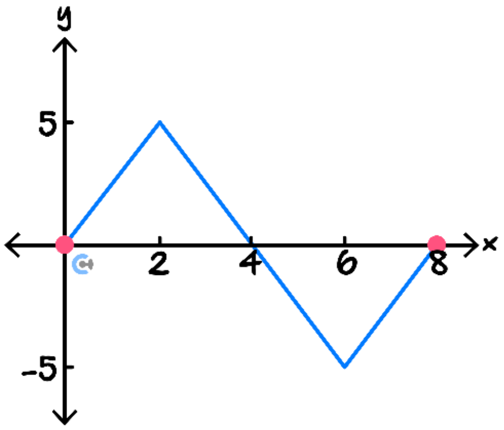
Area Above = Area Below the Avg Value

Space for Personal Notes

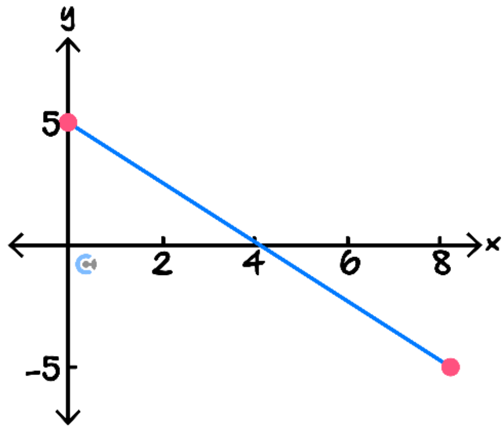
Question 4 (1 mark)

Which one of the following functions has an average value of 5 over the interval $[0, 8]$?

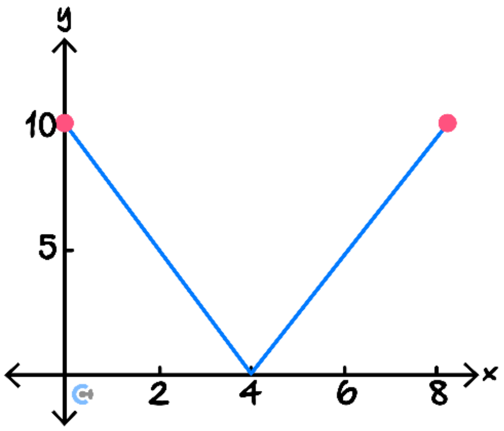
A.



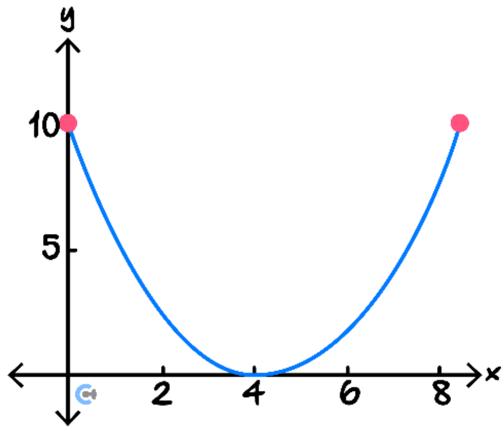
B.



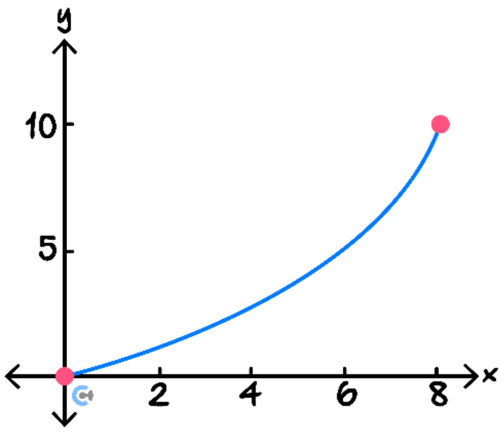
C.



D.



E.



Section C: Integration by Recognition

Sub-Section: Introduction to Integration by Recognition

Context: Limitation of MM Integration

- Consider the following integrals,

$$\int x \sin(x)\, dx, \ \int \log_e(x)\, dx$$

- What do they have in common?

- As we can see, MM34 have some integrals we cannot find.

- What can we do instead for the above integrals?

Space for Personal Notes

Sub-Section: Integration by Recognition

Let's take a look at an example of integration by recognition.

Question 5 Walkthrough.

a. Find $\frac{d}{dx}\left(x \log_e(x)\right)$.

b. Hence, how can we find $\int \log_e(x)\, dx$?

Discussion: What is the purpose of integration by recognition?

Integration by Recognition

- **Integration by Recognition:** Finding impossible antiderivatives using prior information.
- Questions follow the structure:
  - **a)** Find the derivative of ...
  - **b)** Hence, find the integral ...

Question 6

a. Find $f'(x)$ where $f(x) = 4x \log_e(2x)$.

b. Hence, find $\int \log_e(2x)\, dx$.

Space for Personal Notes

Question 7

a. Find $\frac{d}{dx}(xe^{3x})$.

b. Hence, find $\int xe^{3x}\,dx$.

Space for Personal Notes

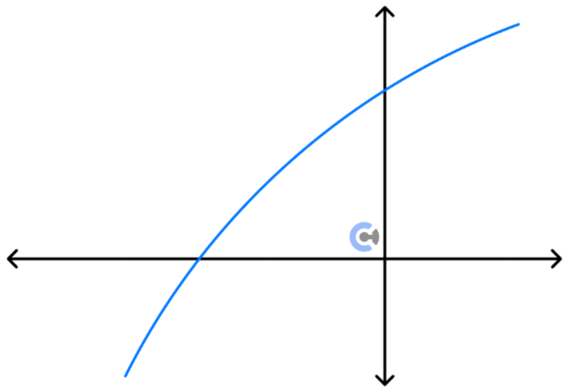
Section D: Advanced Area

Sub-Section: Horizontal Strips

Discussion: We have been solving for areas using vertical strips ($y\,dx$). Is it possible to find areas using horizontal strips instead?

Exploration: Horizontal Strips

- Consider the area bounded by the graph below and its axes intercepts.



- Cut the area into thin horizontal strips and define the area.

Area of the thin strips = ________________

- Hence, define the integral that will help evaluate the area using horizontal strips.

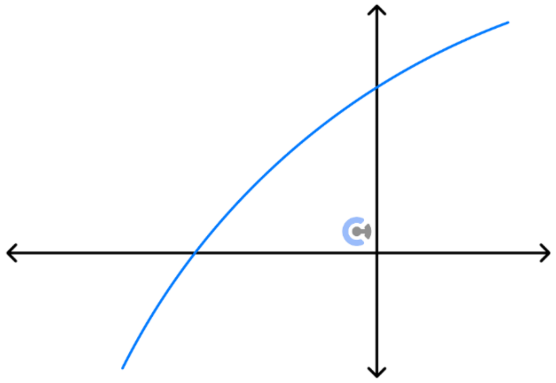
Area = ____________

  - The terminals must be your y values.

Space for Personal Notes

Definition

Horizontal Strips



- **Horizontal Strip:** Cut the area into thin horizontal strips.
- Useful when the function is hard to antiderive but the inverse is easy.

EG: $\log_e(x)$

Area of the thin strips $= x\,dy$

$$Area = \int_a^b x\,dy$$

The terminals must be y values.

Question 8 Walkthrough.

Let $f(x) = \log_e(x)$. Find the area bounded by the graph of $y = f(x)$, the y-axis, the x-axis, and the line $y = 2$.

Question 9

Let $f(x) = \log_e(2x)$. Find the area bounded by the graph of $y = f(x)$, the y-axis, the x-axis, and the line $y = 3$.

Space for Personal Notes

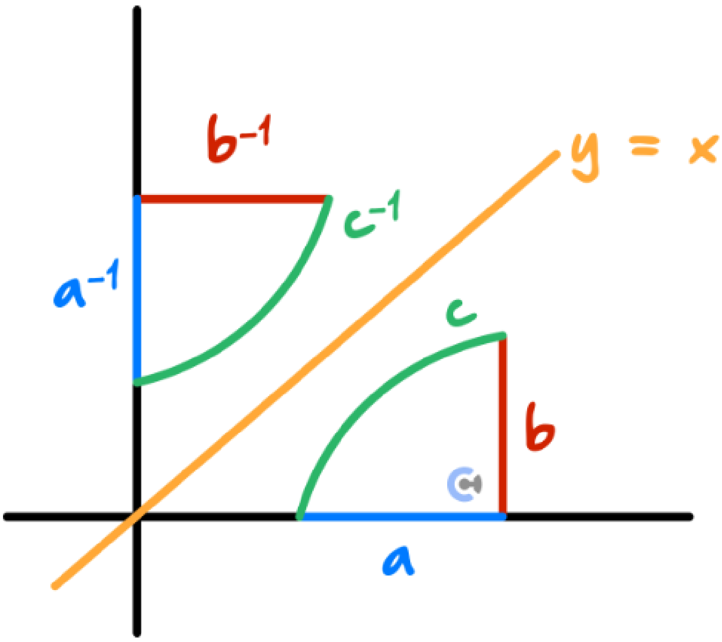
Sub-Section: Inverse Boundaries

Discussion: What would happen to the size of the area if you reflect the area around $y = x$?

Discussion: Hence, area bounded by a, b, c will be the same as area bounded by which boundaries?

Definition

Using Inverse Boundaries



- **Idea:** Reflecting the area around the $y = x$ results in the same area.

$$\text{Area bounded by } a, b, c = \text{Area bounded by } a^{-1}, b^{-1}, c^{-1}$$

Question 10 Walkthrough. Tech-Active.

Let $f: [3, \infty) \to f(x) = x^3 - 3x^2 - 9x + 5$.

Find the area bounded by $y = f^{-1}(x), y = 6, y = 8$ and the y-axis.

NOTE: Inverse function is impossible to find here!

Question 11 Tech-Active.

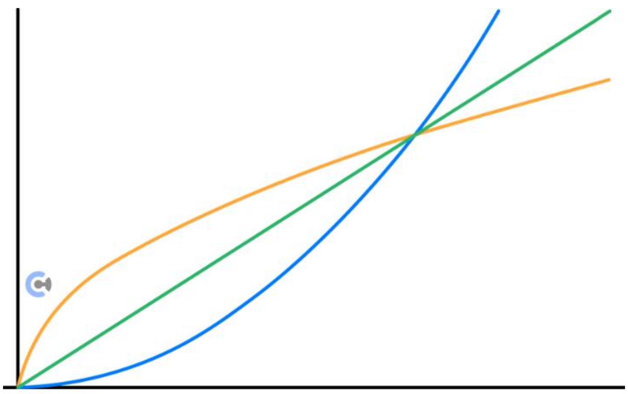
Let $g: [2, \infty) \to g(x) = x^3 + 4x^2 - 5x + 10$.

Find the area bounded by $y = g^{-1}(x)$, $y = 5$, $y = 8$ and the y-axis.

Sub-Section: Area Between Inverses

Discussion: Given that inverses are symmetrical around $y = x$, how can we find the area between two inverses?

Area Between Inverses



- **Area between inverses:** Can be cut up by the line of symmetry: $y = x$.

$$\int_a^b f(x) - f^{-1}(x)\,dx = 2\int_a^b f(x) - x\ dx = 2\int_a^b x - f^{-1}(x)\ dx$$

Space for Personal Notes

Question 12 Walkthrough. Tech-Active.

Let $f(x) = \log_e(3x) + 10$.

Find the area bounded by $y = f(x)$ and $y = f^{-1}(x)$. Give your answer correct to two decimal places.

Question 13 (2 marks) **Tech-Active.**

Let $f(x) = e^x - 2$.

Find the area bounded by $y = f(x)$ and $y = f^{-1}(x)$. Give your answer correct to two decimal places.

Sub-Section: Area and Family of Functions

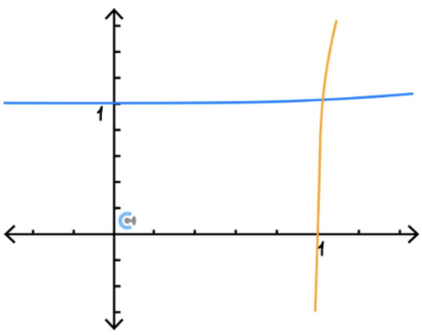
Question 14 Walkthrough.

Let $f(x) = e^{2(x-k)} + 2$. The area bounded by f, f^{-1} and the coordinate axes in the first quadrant is $A(k)$.

Find $\lim_{k \to \infty} A(k)$.

Area and Family of Functions

Definition



$$\lim_{k \to l} Area = Area\ of\ a\ simple\ shape$$

Steps:

1. Use Manipulate / Sliders.
2. Identify the "simple shape" that the area approaches to.

NOTE: This was in the VCAA exam a lot!

Question 15

Let $g(x) = e^{3(x-k)} + 4$. The area bounded by g, g^{-1} and the coordinate axes in the first quadrant is $A(k)$.

Find $\lim_{k \to \infty} A(k)$.

Space for Personal Notes

Website: contoureducation.com.au | Phone: 1800 888 300 | Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make the Most of These (Free) Services!

- Experienced Contour tutors ($45+$ raw scores, $99+$ ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
➤ Book via bit.ly/contour-methods-consult-2025 (or QR code below). ➤ One active booking at a time (must attend before booking the next).	➤ Message +61 440 138 726 with questions. ➤ Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726