CONTOUR EDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Integration I [4.2]
Workbook

Outline:

Section A: Introduction

Let's quickly review last week's material.

Learning Objective: [4.1.1] – Find antiderivative functions

- **Indefinite Integrals:**
  - The indefinite integral of $f(x)$ gives us the **Antiderivative Function.**

$$\int \frac{dy}{dx}\, dx = y + c$$

  - We call $dy\,/\,dx$ the ____________________ and y the ________________________ function.
  - c is an arbitrary real constant.

- **Table of Standard Integrals with Reverse Chain Rule:**

$f(x)$	$\int f(x)dx$
$(ax+b)^n$	
$\sin(ax+b)$	
$\cos(ax+b)$	
$\sec^2(ax+b)$	
e^{ax+b}	
$\frac{1}{ax+b}$	

Learning Objective: [4.1.2] - Solve definite integrals

- **Definite Integrals:**

$$\int_a^b f(x)\,dx = [F(x)]_a^b = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Definite Integrals give us the _______________ in the antiderivative as x changes from ______________.

Learning Objective: [4.1.3] - Apply integral properties to tackle integration questions

- **Integral Properties**

- The following rules also hold for definite integrals:

$$\int_b^a f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\int_a^a f(x)\,dx = \_\_\_\_\_\_\_\_$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\int_a^b c \cdot f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Space for Personal Notes

What are we learning today?

Context: Integration I Learning Objective

We will learn to approximate areas under a function.

- [4.2.1] - Approximate areas under the function

And move away from approximation to find exact areas!

There are going to be two types of exact areas: Signed and Total. We will learn together.

- [4.2.2] - Find signed and total areas

Finally, we will learn how to find areas between two functions!

- [4.2.3] - Find the area between two functions

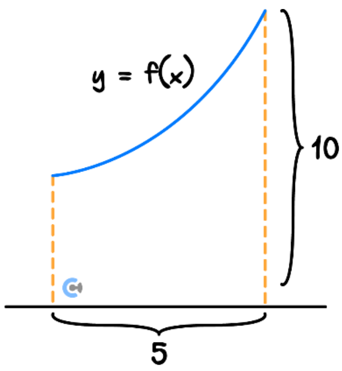
Space for Personal Notes

Section B: Approximating the Area

Let's start with an easy example.

Approximating an Area

- Consider the area under the function $f(x)$.



- What is the area? $50, 25$, excuse me?

The area under the function is not in a simple shape!

- Instead, let's fit simple shapes inside the area!
- **Fit any simple shapes (rectangle, trapezium...) on the diagram above!**
- How can we approximate the area using these simple shapes?
- We can find the ________________of the ____ instead.

This is an approximation of the area!

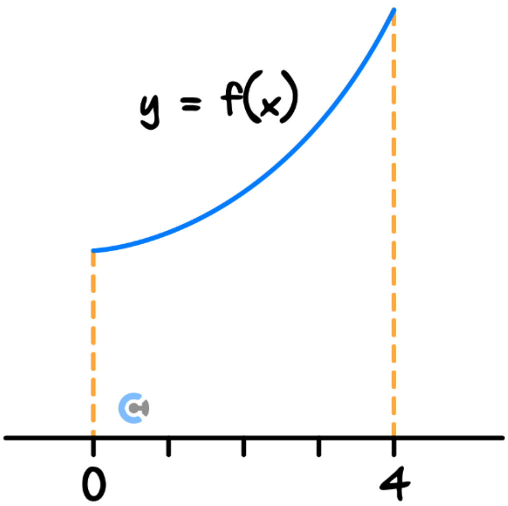
Space for Personal Notes

Sub-Section: Riemann Sum

Let's rectangles to approximate the area!

Exploration: Number of Rectangles and Step Size

- Take a look at the area under $y = f(x)$ from $x = 0$ to $x = 4$.



- Firstly, each rectangle needs to have the ______________________________!
- To fit 4 rectangles, how thick should each rectangle be? $[4, 1]$
- There is a special mathematical term ______________________

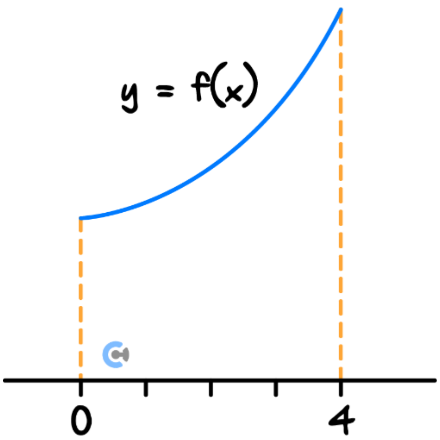
NOTE: Step size is the thickness of each rectangle.

Space for Personal Notes

Now, how tall should these rectangles be!

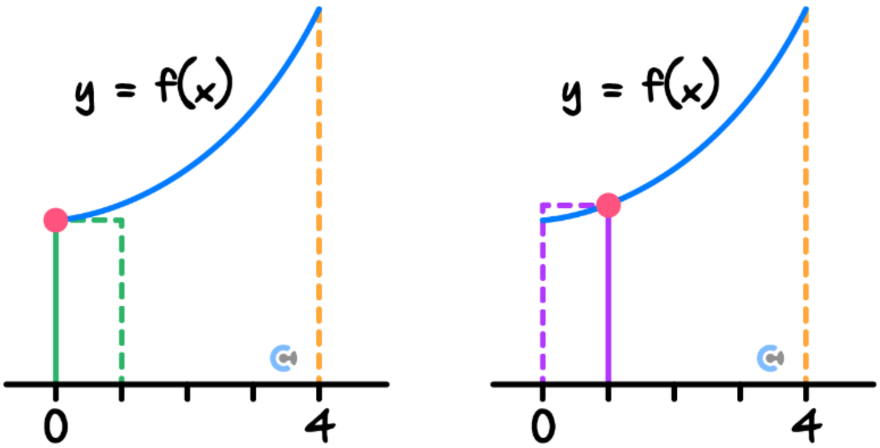
<u>Exploration</u>: Rectangle's Height and Left / Right End Point

- Let's approximate the area using 4 rectangles with a step size of 1.



- For the first rectangle from $x = 0$ to $x = 1$, where should the height end?

  [Left diagram / Right diagram]



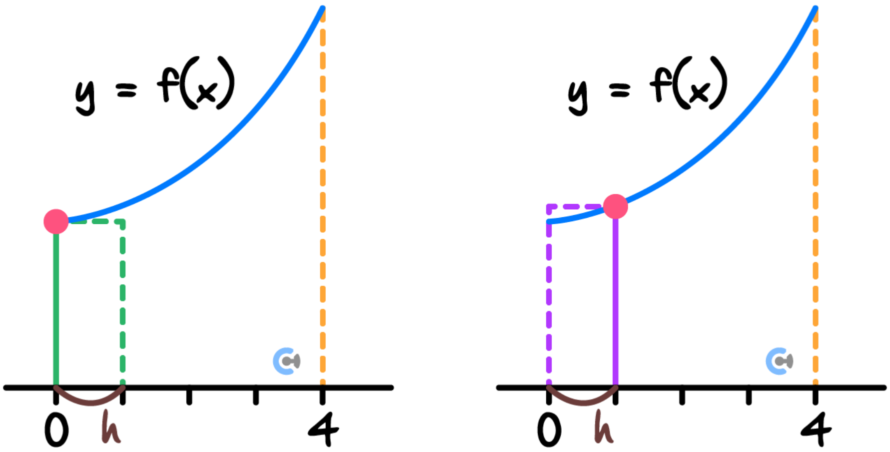
- The red rectangle's height is determined by the left intersection with the function.

This is called the left endpoint approximation!

- The blue rectangle's height is determined by the right intersection with the function.

This is called the right endpoint approximation!

Riemann Sum



- **Riemann Sum:** Approximates the area under a curve using rectangles.
- **Height of the rectangles:** Determined by either the left or right y value of the function.
- **Step size (h):** Thickness of the rectangle.

$$h = \frac{b-a}{n} \text{ where } n = number\ of\ rectangles$$

Space for Personal Notes

Let's try a question!

Question 1 Walkthrough.

Approximate the area under $y = \frac{1}{2}x^3$ between $x = 1$ and $x = 3$ using 2 rectangles and:

a. The right-endpoint method.

b. The left-endpoint method.

TIP: Sketch the graph to prevent mistakes.

Space for Personal Notes

Question 2

Approximate the area under $y = x^2 + 4$ between $x = 1$ and $x = 5$ using 4 rectangles and:

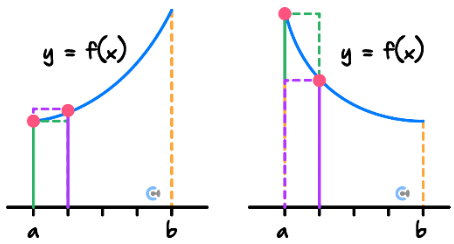
a. The right-endpoint method.

b. The left-endpoint method.

Discussion: For which type of functions will left end point approximation be an underestimation?

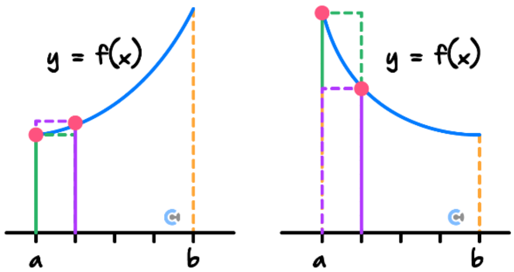
Exploration: Overestimation and Underestimation of Riemann Sum

- Consider an increasing and decreasing function.



- Visualise the left endpoint approximation.
  - Increasing function: ____
  - Decreasing function: ____
- Visualise the right endpoint approximation.
  - Increasing function: ____
  - Decreasing function: ____

Overestimation and Underestimation of Riemann Sum



- **Left endpoint approximation:**
  - Increasing function: Underestimation
  - Decreasing function: Overestimation
- **Right endpoint approximation:**
  - Increasing function: Overestimation
  - Decreasing function: Underestimation

It seems like always one approximation is too big and another approximation is too small!

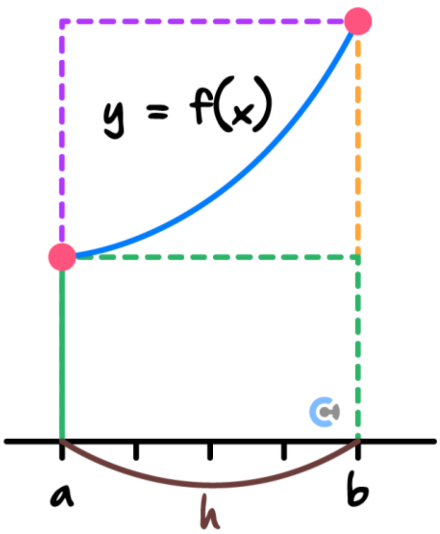
Discussion: How can we find a better estimation using both left and right endpoint approximation?

Space for Personal Notes

Sub-Section: Trapezoidal Approximation

Exploration: Averaging Left and Right End Point Approximation

- Consider overestimating the right endpoint and underestimating the left endpoint approximation.



$$Left = f(a)\,h$$

$$Right = f(b)\,h$$

- If we take the average of the two areas, what shape would the resulting area represent?
- Sketch the shape above!
- Let's calculate the average area!

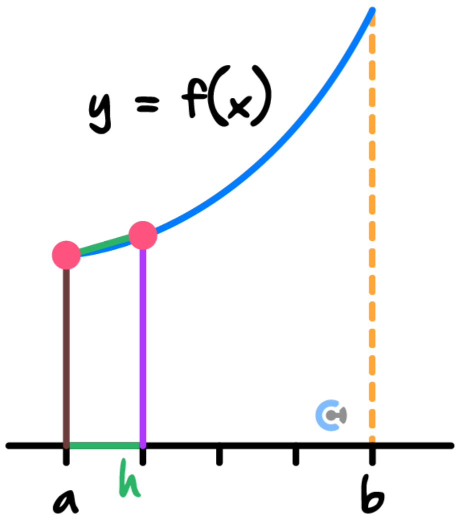
$$Average = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- What does this remind you of? Area of the [trapezium, rectangle, triangle]

Space for Personal Notes

This is called trapezoidal approximation.

Trapezoidal Approximation



- **Trapezoidal Approximations:** Approximates the area under a curve using trapeziums.
- **Formula for each trapezium:** Approximates the area under a curve using trapeziums.

$$Area = \frac{h}{2}(f(a) + f(a+h))$$

- **Step size (h):** Thickness of the trapezium.

$$h = \frac{b-a}{n} \text{ where } n = number\ of\ rectangles$$

- **Formula:** Sum of all the trapeziums

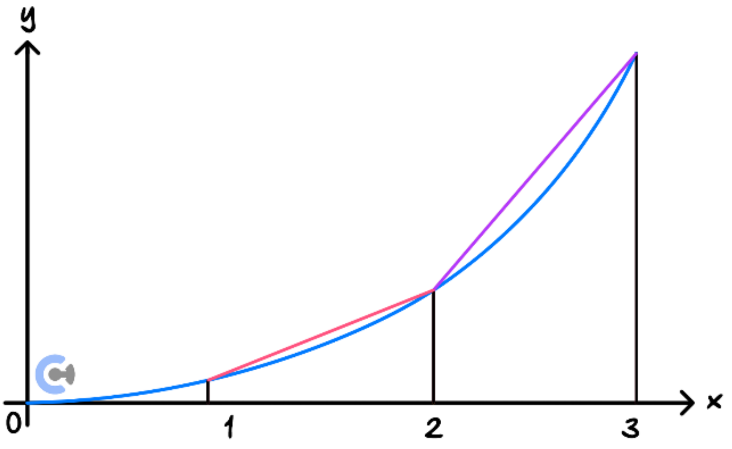
$$Approximation = \frac{b-a}{2n}(f(a) + 2f(a+h) + 2f(a+2h) \cdots + f(b))$$

$$VCAA\ Version = \frac{x_n - x_0}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) \cdots + f(x_n))$$

NOTE: Formula is on the VCAA formula sheet.

Question 3 Walkthrough.

Approximate the area under $y = x^2$ between $x = 1$ to $x = 3$ using the trapezoidal method and the step size of 1.



Your turn!

Question 4

Consider the area under $y = x^3$ between $x = 0$ to $x = 3$.

a. Approximate the area using the trapezoidal method and the step size of 1.

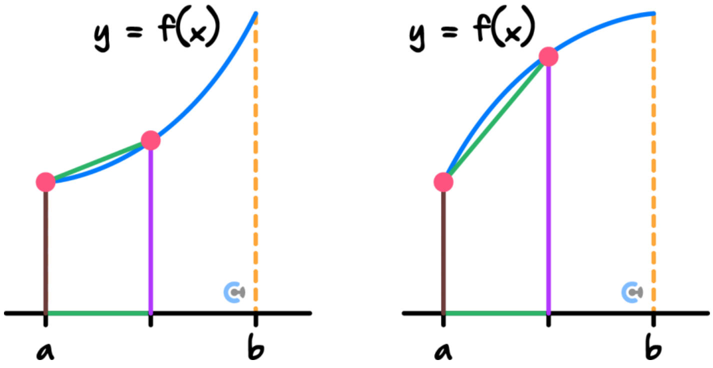
It is known that the left end point approximation with the step size of 1 for the same region is given by 9.

b. Without calculating, state the right endpoint approximation with the step size of 1 for the same region as part.

Discussion: For which type of functions will the trapezoidal approximation be an Underestimation?

Exploration: Overestimation and Underestimation of the Trapezoidal Approximation

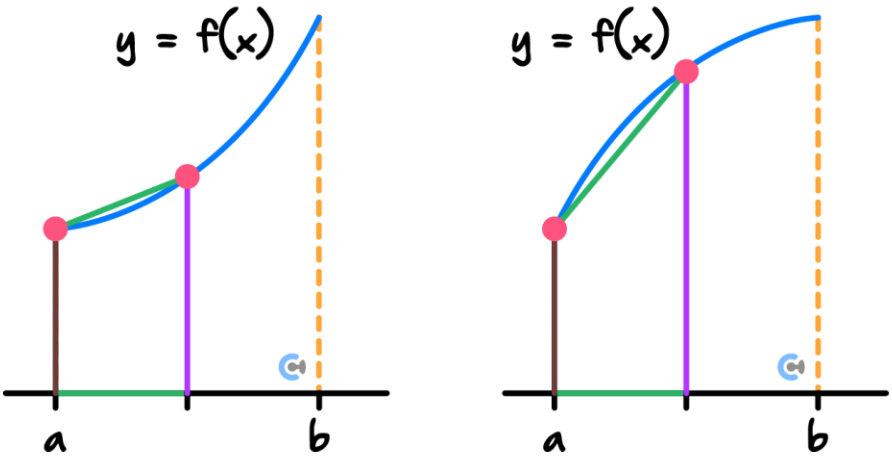
- Consider concave up and down functions.



- Visualise the trapezoidal approximation on the graphs above.
  - Concave Up: ________________________________
  - Concave Down: ______________________________

Overestimation and Underestimation of Trapezoidal Approximation

Definition



- **Trapezoidal approximation:**
  - Concave Up: Overestimation
  - Concave Down: Underestimation

Space for Personal Notes

Section C: Finding the Exact Area

Sub-Section: Step Size and Accuracy

Discussion: How can we increase the accuracy of estimation by changing the step size?

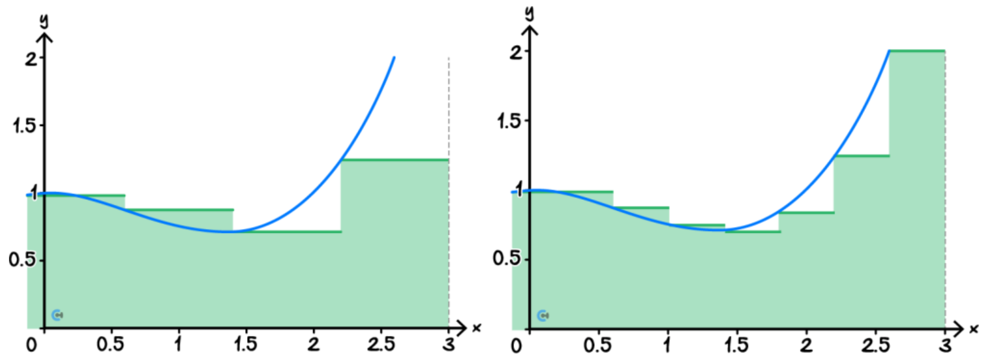
Exploration: Step Size and Accuracy Visualisation

- Scan the QR code below!

- To achieve higher accuracy (less error), do we want more rectangles or less?

- Therefore, to fit more rectangles, should our step size (thickness) be smaller or bigger?

Step Size and Accuracy



- **Step size:** Dictates how accurate the estimation is.

- Smaller step size results in higher accuracy.

Let's extend this idea to find an area with 100% accuracy.

Discussion: How small should the step size be to achieve 100% accuracy?

How can we cut the area into rectangles with a thickness of 0? Rectangles that are infinitely small?

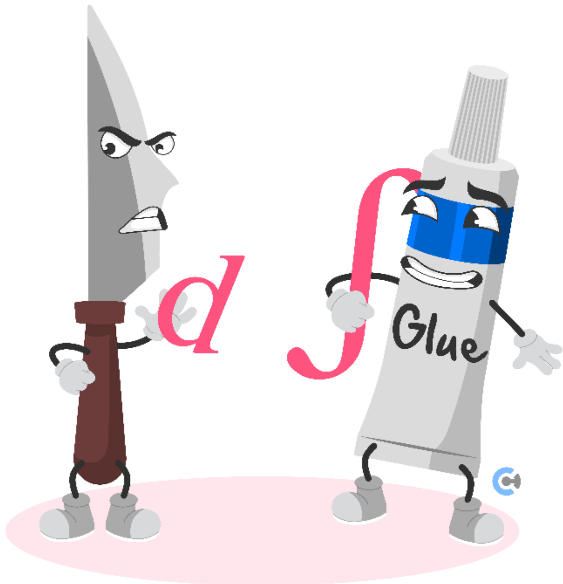
Space for Personal Notes

Sub-Section: Integration for Area

We can use calculus!

Context: Calculus

- Calculus is a study of infinitely small things, originally called “the calculus of infinitesimals”.



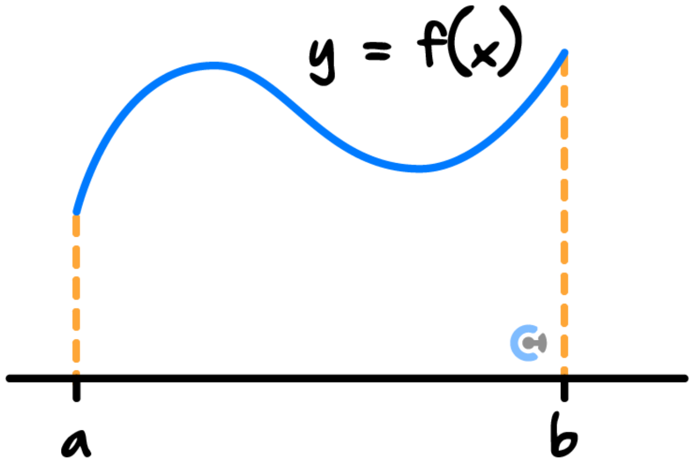
- Differential calculus allows us to_____________________ something into infinitely small thing.
- Integral calculus allows us to __________________________ infinitely small things to something whole.

Space for Personal Notes

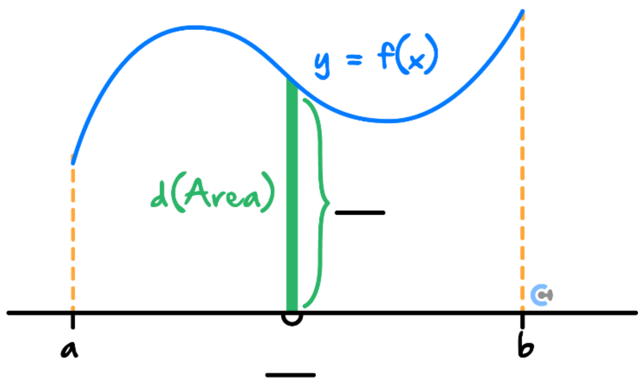
Let's use differentiation and integration for finding the exact area!

Exploration: Calculus for Finding the Exact Area

- Consider the area below the function $f(x)$.



- Let's cut the area into rectangles, but this time, rectangles will be ________________.
- Draw the infinitely thin rectangles above!
- We are differentiating the area!
- Let's define the small cut of the area: $d(Area)$!



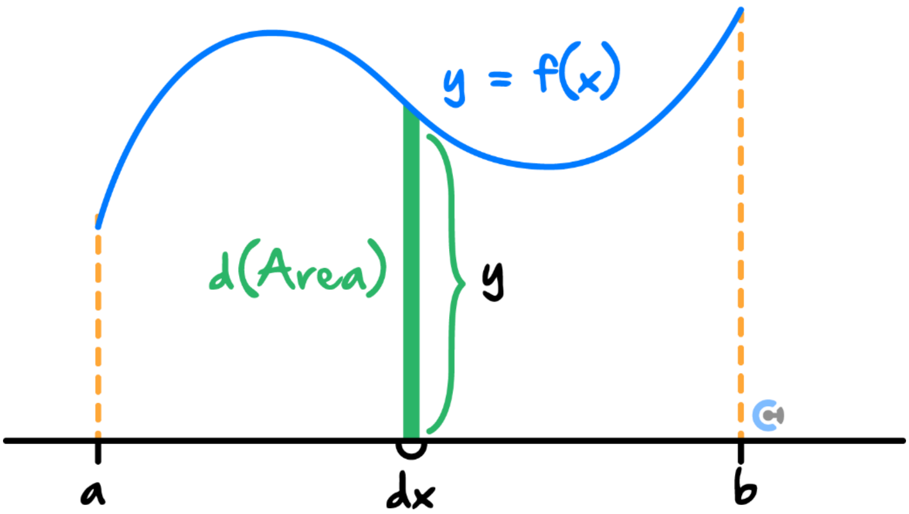
$$d(Area) = \_\_\_\_\_\_\_\_\_\_$$

- How do we find the sum of all $d(Area)s$ from $x = a$ to $x = b$?

$$Area = \_\_\_\_\_\_\_\_\_\_\_\_$$

This is how we find the area under a function!

Calculus for Finding Exact Area



- **Differentiation:** Defines a small cut of the total area.

$$d(Area) = y\,dx = f(x)\,dx$$

- **Integration:** Adds all the small cuts of areas into the entire area.

$$Area = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

NOTE: We use a definite integral to define where to start and stop the sum (area).

Space for Personal Notes

Question 5 Walkthrough.

Find the area bounded under $y = x^2 + 5$ from $x = 1$ to $x = 2$.

TIP: Always visualise the area first!

Question 6

Find the area bounded under $y = e^x + 1$ from $x = 1$ to $x = 3$.

Discussion: Try explaining why the integral of the circumference equals the area to the next person!

(Extension) How about volume and surface area of a sphere?

Space for Personal Notes

Section D: Signed vs Total Areas

Sub-Section: Signed Area

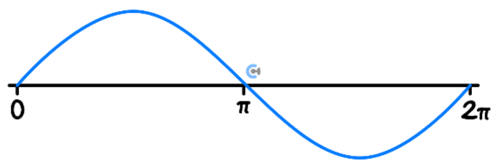
We are adding y dx to find the exact area.

Discussion: What would happen if the y-value is negative?

We call $\int_a^b y\,dx$ signed area!

Exploration: Signed Area

- Consider the function $y = \sin(x)$.



- y-values are positive and negative for $x \in [0, 2\pi]$.
- For which x-values will our signed area be positive?
- For which x-values will our signed area be negative?
- Evaluate the following integral:

$$\int_0^{2\pi} \sin(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Why is the overall signed area 0?

It is because the ________________ and _________________ signed area _________________.

NOTE: Signed areas cancel out when we use one integral!

Definition

<u>Signed Area</u>

$$\textit{Signed Area} = \int_a^b y\,dx$$

- **Signed Area**: Areas with signs due to the sign of the y-value of the function.
- Positive Signed Area: $y > 0$
- Negative Signed Area: $y < 0$
- Positive and negative areas will cancel out.

NOTE: Area below the x-axis $=$ Negative is a dangerous way of thinking! More on this later.

Space for Personal Notes

If y affects the sign, what about dx? (Extension only)

Extension: Sign of dx

- dx stands for instantaneous change in x.

$$\int_{2}^{5} y \, dx$$

- For the above integral, dx would be positive as the x-value is increasing.
- How about this one?

$$\int_{5}^{2} y \, dx$$

- dx would be negative as the x-value is decreasing.
- This is why we always define the area from left to right.

  So, __________________ is always ________________.

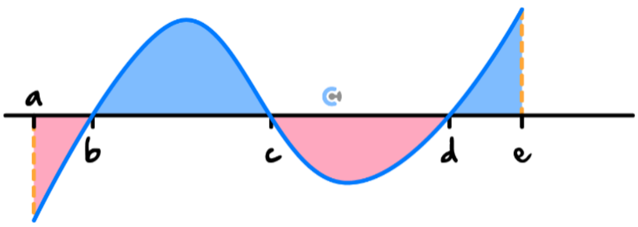
Space for Personal Notes

Sub-Section: Total Area

Do we want a negative area though? [Yes/No]

Exploration: Finding Total Area

- Consider the function below.



We want to find the total positive area: a total area for short!

- What happens when we use one integral? Discuss!
- The positive and negative areas will ______________________!

$$\int_a^e y\,dx$$

- What about this? Discuss!

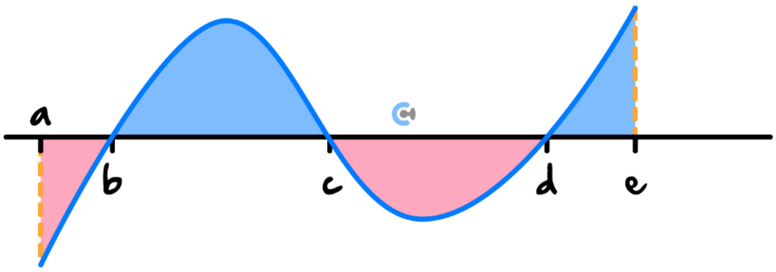
$$\int_a^b y\,dx + \int_b^c y\,dx + \int_c^d y\,dx + \int_d^e y\,dx$$

- We are adding _________________ and ___________________ values.
- It is actually the same as _________________!
- How can we fix the issues we face?
- Write down the integral(s) that finds the total area below!

Total Area = ____

Definition

Total Area



- **Total Area:** Total ____________________ area.
- Require __________________ integrals for areas with ____________________________ y-value.

$$Total\ Area = -\int_a^b y\,dx + \int_b^c y\,dx - \int_c^d y\,dx + \int_d^e y\,dx$$

- **Steps:**
  1. ________________ the integrals for ____________________ and __________________ y-values.
  2. Turn the negative signed areas to positive by putting a _______________________ in front!

Discussion: IMPORTANT! If a question asks for area, how do we know when to find signed vs total area?

Discussion: There is one way in specialist maths to find the total area using one integral. How? Hint: Think about why we need to split the integrals in the first place!

Question 7 Walkthrough.

Find the total area bounded by $y = 12 - 3x^2$ from $x = 1$ to $x = 3$.

TIP: Always sketch the graph of the function.

Question 8

Let $f(x) = x^2 - 4$. Find the area bounded by the graph of $y = f(x)$, the x-axis, the lines $x = 0$ and $x = 4$.

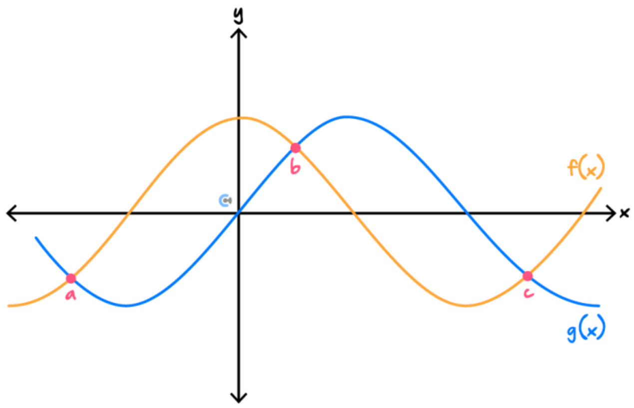
Discussion: Explain to each other the difference between signed and total area!

Space for Personal Notes

Section E: Area Between Two Functions

Sub-Section: Area Between Two Functions

Discussion: How many integrals are required to find the area bounded by $f(x)$ and $g(x)$ from $x = a$ to c?

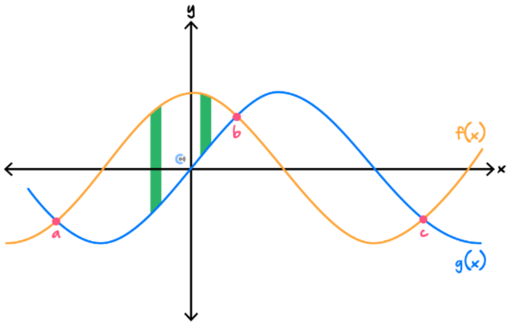


Space for Personal Notes

Is it 2 or 4 integrals? Let's find out!

Exploration: Area between two functions.

- Consider the total area bounded between $f(x)$ and $g(x)$ from $x = a$ to c.

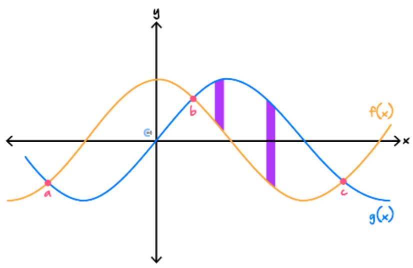


- Using the idea of calculus, we can cut the area into infinitely thin rectangles!
- How can we define the areas of infinitely thin rectangles from $x = a$ to $x = b$? (Positively)

$$\text{Area of the thin rectangle} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Do we need multiple different ways to defining the area from $x = a$ to $x = b$? [Yes / No]

  Hence, how many integrals do we need from $x = a$ to $x = b$? $[0, 1, 2]$



- How can we define the areas of infinitely thin rectangles from $x = b$ to $x = c$? (Positively)

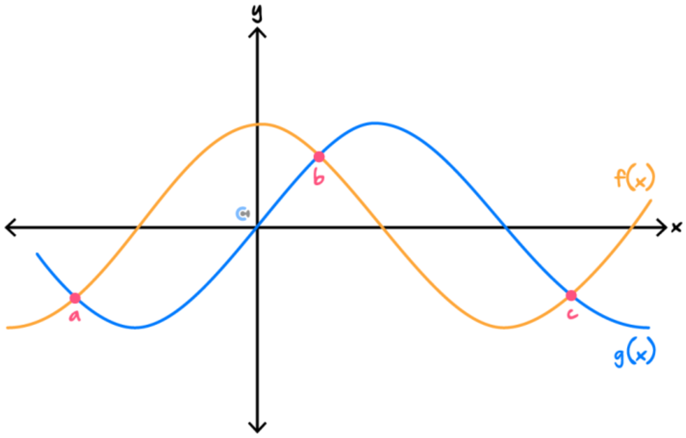
$$\text{Area of the thin rectangle} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Hence, how many integrals do we need from $x = b$ to $x = c$? $[0, 1, 2]$
- Therefore, how can we find the total area bounded between $f(x)$ and $g(x)$ from $x = a$ to c.

$$\text{Total Area} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

We only need two integrals for the above discussion!

<u>Area Between Two Functions</u>



- **Area between two functions**:

  Integrand (inside the integral) is the top function minus the bottom function.

$$Area = \int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx$$

Question 9 Walkthrough.

Find the area enclosed by the curves x^2 and $3x$.

TIP: Always sketch the function.

Question 10

Find the area enclosed by the curves $x^2 - 4x + 3$ and $y = x - 1$.

Sub-Section: Tackling the Misconceptions

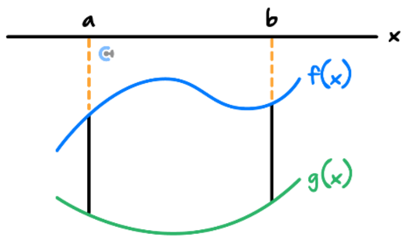
REMINDER: Integral Calculus

- **Integral calculus** allows us to ______________________ infinitely small things to something whole.

Discussion: What makes an integral (overall sum) negative then? What do you have to sum up?

Okay then.. have a go at the following discussion!

Discussion: How do we define the integral for the area between $f(x)$ and $g(x)$ from $x = a$ to b below?
NOTE: We want positive area.



NOTE: The sign of what you are integrating defines whether the integral is positive or negative.

Hence, "Integrals for areas below the x-axis are always negative." True or false?

Misconception

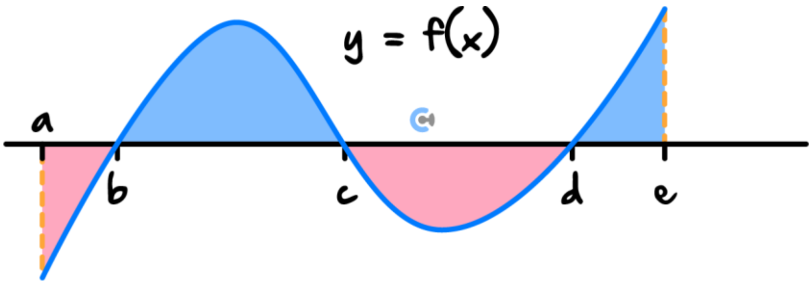
Integrals for areas below the x-axis are always negative.

TRUTH: Integral is negative when we sum up negative values.

Top minus bottom is all we have to do, for every total area!

Exploration: **Top minus bottom for total area**

- Consider the function below.



We can find the total area using the top minus bottom function approach here!

- State the function that models the x-axis.

$$x\text{-}axis: y = \_\_\_\_\_\_\_\_$$

- Use the top minus bottom approach to define the area!

Total Area = ______________________________

- What does it simplify to?

Total Area = ______________________________

- NOTE: We were technically doing the top minus bottom function before for the total area!

Question 11

Find the area enclosed by the curves $-x^2$ and $-4x$.

NOTE: Always only focus on the top minus bottom function.

Space for Personal Notes

Section F: Review

What did we learn today?

Summarise and teach everything we have learnt to your friend next to you!

Discussion: What did we learn in Section B: Approximating the area?

Discussion: What did we learn in Section C: Finding the exact area?

Discussion: What did we learn in Section D: Signed vs Total Area?

Discussion: What did we learn in Section E: Finding the area between two functions?

NOTE: Homework has interesting problem-solving questions! So, make sure to try them!

CONTOUR EDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make the Most of These (Free) Services!

- Experienced Contour tutors ($45+$ raw scores, $99+$ ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
• Book via bit.ly/contour-methods-consult-2025 (or QR code below). • One active booking at a time (must attend before booking the next).	• Message +61 440 138 726 with questions. • Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726