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VCE Mathematical Methods ¾

Circular Functions Exam Skills [3.4]

Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 3-Pg 12
Supplementary Questions	Pg 13-Pg 22

Section A: Recap

Contour Check

- Learning Objective: [3.4.1] – Period for sum of trigonometric functions and equivalent general solutions

Key Takeaways

- Multiple Forms of a General Solution

$$a + Period \cdot n = b + Period \cdot n$$

If the difference of a and b is a multiple of period.

- Period For Sum/Difference of Circular Functions
  - When we add two circular functions,

$$Period\ of\ the\ sum = LCM\ of\ two\ periods$$

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Section B: Compulsory Questions

Sub-Section: Basics (Tech Free)

Question 1 [3.4.1]

Find the period of $\sin(2x) - \cos(3x)$.

$\sin(2x)$ has period of π and $\cos(3x)$ has period of $\frac{2\pi}{3}$. [1M]
The LCM is 2π. So the period is 2π

Question 2 [3.3.3]

Consider the function $f(x) = 2\sin(2x) + 2$. Find the fraction of a period that $f(x) > 3$.

We solve $f(x) = 3 \implies \sin(2x) = \frac{1}{2}$ [1M]
$2x = \frac{\pi}{6}, \frac{5\pi}{6}$.
So $x = \frac{\pi}{12}, \frac{5\pi}{12}$. [1M]
Period is π. Above 3 for $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.
So fraction of a period that $f(x) > 3$ is $\frac{1}{3}$. [1A]

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Question 3 [3.2.3]

Solve the equation $\sin\left(\frac{x}{3}\right) = -\frac{1}{2}$ for $x \in [2\pi, 6\pi]$.

$$\frac{x}{3} = \frac{7\pi}{6}, \frac{11\pi}{6} \quad [1M]$$
$$x = \frac{7\pi}{2}, \frac{11\pi}{2}. \quad [1A]$$

Question 4

a. Find a general solution to the equation $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{3}$. **[3.2.3]**

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$
$$x - \frac{\pi}{3} = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi. \quad [1M]$$
$$x = \frac{2\pi}{3} + 2n\pi, \pi + 2n\pi, \text{ where } n \in \mathbb{Z}. \quad [1A]$$

b. Subu is marking a student's answer to **part a.,** the student gave the answer:

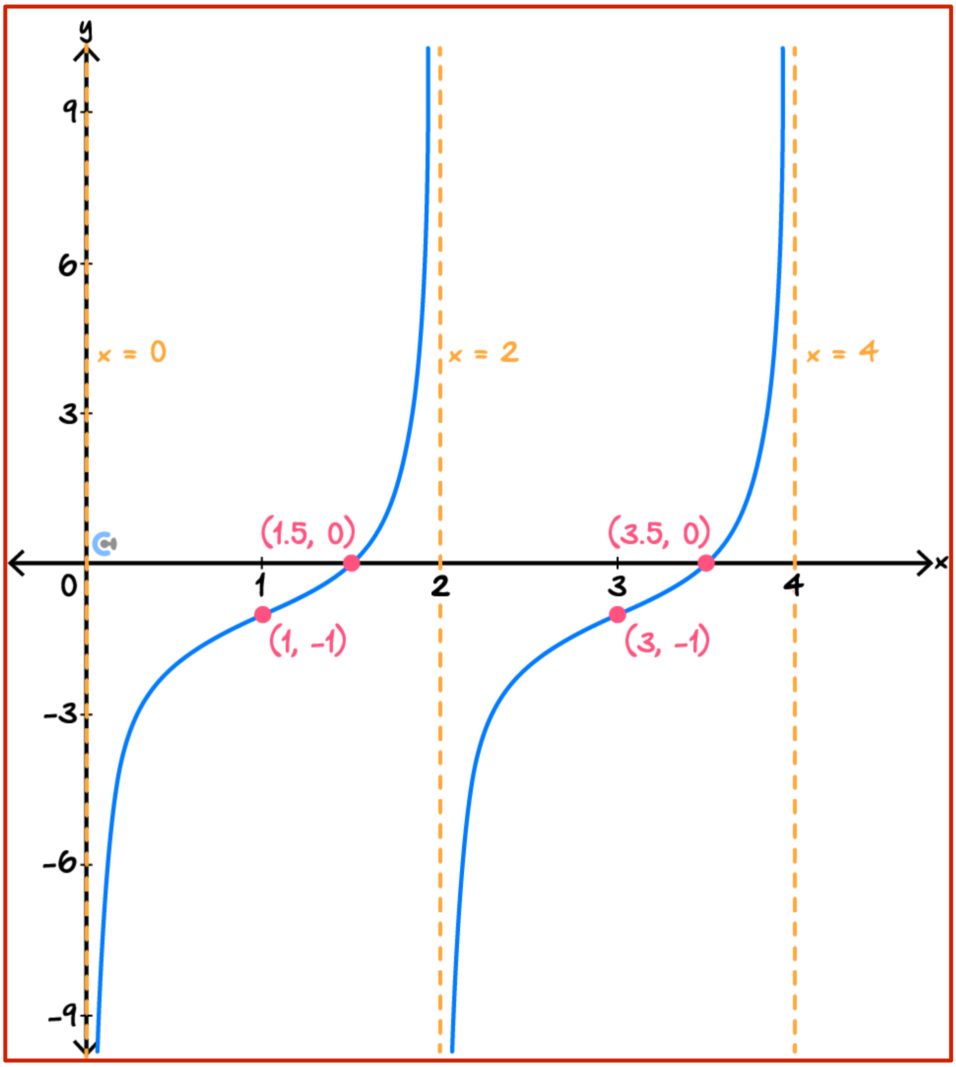
$$x = 2n\pi - \frac{4\pi}{3}, 2n\pi - \pi, \text{ where } n \in \mathbb{Z}.$$

Is the student's answer correct? Explain. **[3.4.1]**

The answer is correct, because each of the answers differ by a multiple of 2π from our answer in **part a.**, so they are equivalent general solutions.

Question 5 [3.3.2]

Sketch the graph of $y = \tan\left(\frac{\pi}{2}(x-1)\right) - 1$ for $x \in (0,4)$. Label all axes intercepts with coordinates and asymptotes with their equations.



Sub-Section: Problem Solving (Tech-Active)

Question 6

The temperature T (in degrees Celsius) inside a greenhouse at t hours after midnight on a typical November day is modelled by the formula:

$$T = 25 - 4\cos\left(\frac{\pi(t-3)}{12}\right), \text{ for } 0 \leq t \leq 24.$$

Use this model to answer the following:

a. State the maximum and minimum temperatures reached inside the greenhouse during the day. **[3.3.2]**

Max is 29 and Min is 21 [1A each]

b. At what time is the maximum temperature reached? **[3.2.3]**

$t = 15$, so 3pm. [1A]

c. Determine the time(s) when the temperature is exactly 23°C. **[3.2.3]**

Solve $T(t) = 23$ for $0 \leq t \leq 24$ [1M]
$t = 7, 23$. So at 7am and 11pm. [1A]

d. Consider the hottest continuous 4-hour period in the greenhouse. What is the minimum temperature reached during this period? Give your answer to two decimal places. **[3.3.2]**

By symmetry hottest 4 hours period will be 2 hours either side of the max which occurs when $t = 15$. [1M]
So coolest temperature is $T(13) = T(17) \approx 28.46^\circ$ [1A]

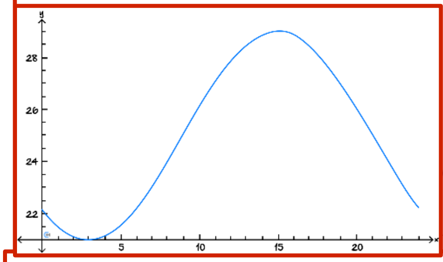
e. In the greenhouse, there is an automatic watering system that activates when the rate of change of temperature with respect to time is at least $+0.2$°C per hour. It switches off once the rate drops below this threshold.

i. Use calculus to find an expression for the rate of change of temperature with respect to time.

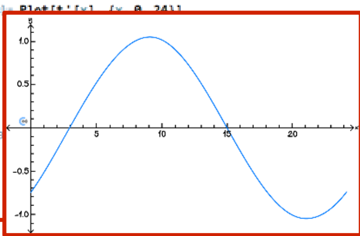
$$T'(t) = \frac{1}{3}\pi \sin\left(\frac{\pi}{12}(t-3)\right) \quad [1A]$$

We require that $T'(t) \geq 0.2$ [1M]
Solving $T'(t) = 0.2$ for $0 \leq t \leq 24$ yields $t \approx 3.73, 14.27$. [1M]
Then looking at the shape of the derivative graph we conclude that $t \in [3.73, 14.27]$.
[1A]

```
In[63]:= t[x_] := 25 - 4 Cos[Pi (x - 3)/12]
In[64]:= t[x]
Out[64]= 25 - 4 Cos[1/12 π (-3 + x)]
In[65]:= Solve[t[x] == 29 && 0 ≤ x ≤ 24]
Out[65]= {{x → 15}, {x → 15}}
In[66]:= Solve[t[x] == 23 && 0 ≤ x ≤ 24]
Out[66]= {{x → 7}, {x → 23}}
In[67]:= Plot[t[x], {x, 0, 24}]
```



```
In[68]:= t[13]
Out[68]= 25 + 2 √3
In[69]:= t[17]
Out[69]= 25 + 2 √3
In[70]:= t[13] // N
Out[70]= 28.4641
In[71]:= t'[t]
Out[71]= 1/3 π Sin[1/12 π (-3 + t)]
In[72]:= Reduce[t'[x] ≥ 0.2 && 0 ≤ x ≤ 24, x]
Reduce: Reduce was unable to solve the system with inexact coefficients. The a
Out[72]= 3.73402 ≤ x ≤ 14.266
In[73]:= Plot[t'[x], {x, 0, 24}]
```

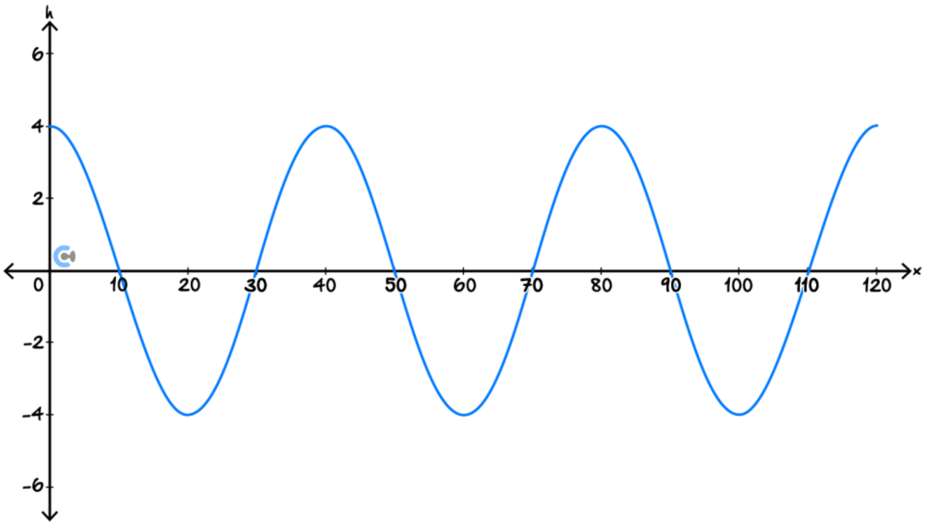


```
define t(x) = 25 − 4*cos(π/12 * (x−3))                 done
solve(t(x)=29,x)|0<x<24                                {x=15}
solve(t(x)=23,x)|0<x<24                                {x=7,x=23}
t(13)                                                  28.46410162
d/dx(t(x))                                             (sin((x−3)·π/12)·π)/3
solve((sin((x−3)·π/12)·π)/3=0.2,x)|0<x<24              {x=3.734021842,x=14.26597816}
```

```
Define t(x)=25−4·cos(π/12·(x−3))                        Done
solve(t(x)=29,x)|0<x<24                                 x=15.
solve(t(x)=23,x)|0<x<24                                 x=7. or x=23.
t(13)                                                   28.4641
d/dx(t(x))                                              (−π·cos(π·x/12+π/4))/3
solve((−π·cos(π·x/12+π/4))/3=0.2,x)|0<x<24              x=3.73402 or x=14.266
```

Question 7

An environmental scientist is monitoring the shape of sand dunes along a stretch of beach in Western Australia. She observes that the profile of the dunes can be modelled by a sine curve. Let h cm represent the height of the sand above sea level and x m represent the horizontal distance along the beach.



a. What is the period of this curve? **[3.3.2]**

40 **[1A]**

b. What is the amplitude of the curve? **[3.3.2]**

4 **[1A]**

c. The height of the dunes can be modelled by a function of the form $h = a\sin(n(x - b))$. Write down an equation for the height of the dunes. **[3.3.2]**

$h = 4\sin\left(\frac{\pi}{20}(x - 30)\right)$ **[1A]**

b term could be any $30 + 40n, n \in \mathbb{Z}$

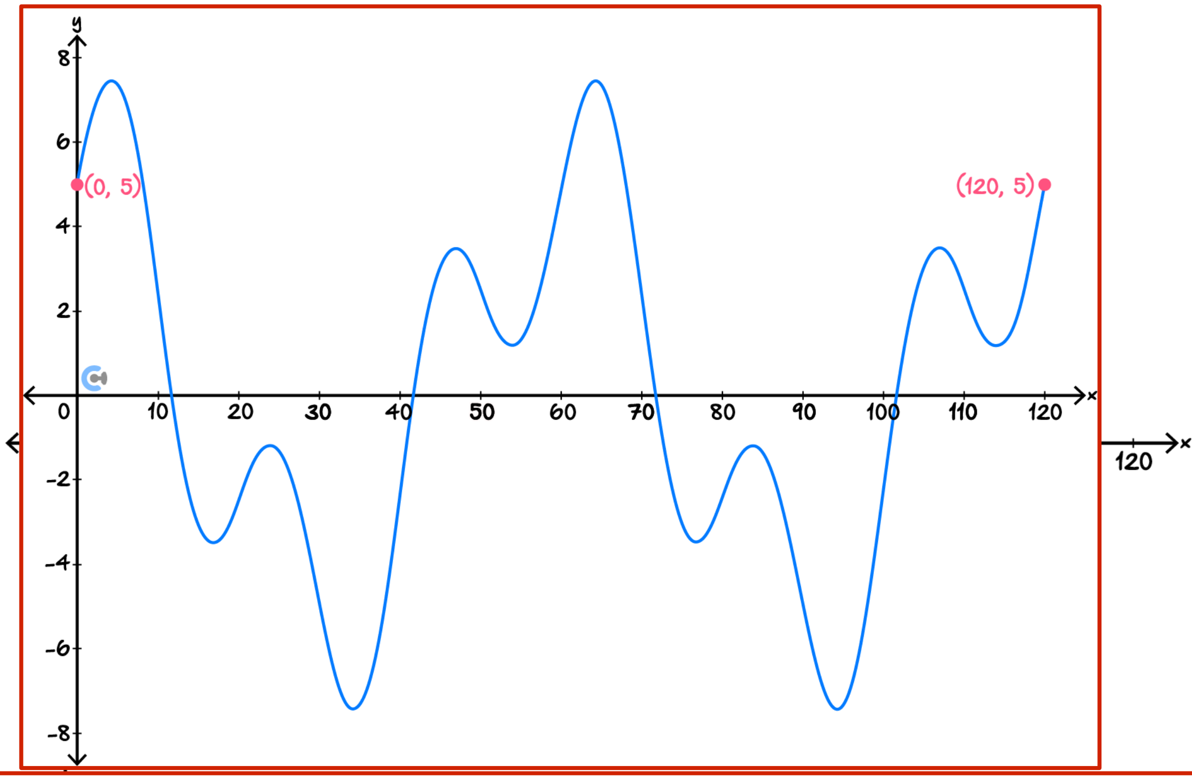
d. If this pattern continues for 2 km, how many dune troughs would a person walk through? **[3.3.2]**

$\frac{2000}{40} = 50$ dunes. **[1A]**

Further along the beach, the wind pattern changes, causing the shape of the dunes to be modelled by a more complex function:

$$h = 3\sin\left(\frac{\pi}{10}x\right) + 5\cos\left(\frac{\pi}{30}x\right)$$

e. Sketch this function over the domain $0 \le x \le 120$. You only need to label the endpoints with coordinates. **[3.3.2]**



[1M shape, 1M endpoints labelled, 1M approximately correct local maxes and mins]

f. What is the period of this new function? **[3.4.1]**

LCM of the two individual periods. LCM of 20 and 60. Period is 60. [1A]

g. What are the absolute maximum and minimum values of this function? Give your answers correct to two decimal places. **[3.3.2]**

We maximise and minimize the function over 1 period. The max is 7.43 and the min is -7.43. [1A]

h. Give the coordinates, correct to two decimal places, of all local maximum points during the first cycle. **[3.2.3]**

We solve $h'(x) = 0$ and look at the graph to identify which are maxima.
$h'(x) = 0 \implies x = 4.23, 16.84, 23.93, 34.23, 46.84, 53.93.$ [1M]
Local maxima occur at $(4.23, 7.43)$, $(23.93, -1.19)$ and $(46.84, 3.47)$. [1A]

Question 8

The temperature, A°C, inside a cabin at t hours after 3 PM is given by the rule:

$$A = 20 - 4\cos\left(\frac{\pi}{12}t\right), \text{ for } 0 \le t \le 24.$$

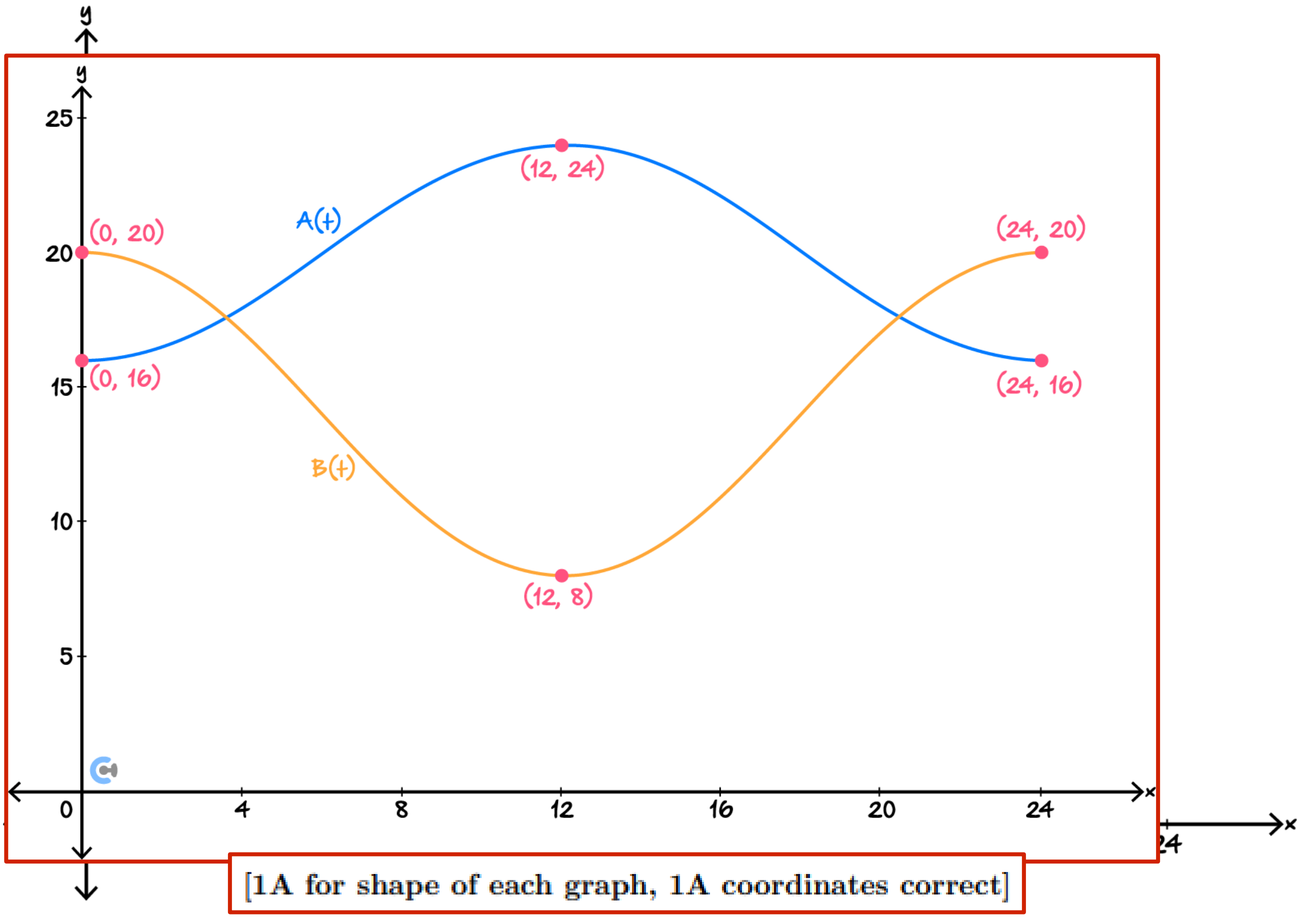
The temperature, B°C, outside the cabin at the same time is given by:

$$B = 14 + 6\cos\left(\frac{\pi}{12}t\right), \text{ for } 0 \le t \le 24.$$

a. Find the temperature inside the cabin at 9 AM. **[3.2.1]**

$t = 12 + 6 = 18.$ $A(18) = 20°.$ [1A]

b. Sketch the graphs of $y = A(t)$ and $y = B(t)$ on the axes below. Label all endpoints and stationary points with coordinates. **[3.3.2]**



[1A for shape of each graph, 1A coordinates correct]

c. Hence, state the values of t for when the temperature inside the cabin is greater than the temperature outside. **[3.2.3]**

We solve $A(t) = B(t)$ to get $t = 3.54, 20.46$. [1M]
Then by looking at the graph we have $t \in (3.54, 20.46)$ [1A]
(note round brackets because we said *greater*)

d. Let $D(t)$ be the function that represents the difference between the temperature inside the cabin and outside the cabin.

i. State the rule for $D(t)$.

$$D(t) = 6 - 10\cos\left(\frac{\pi}{12}t\right)$$

ii. A sudden heat wave occurs and now the temperature outside is given by $K + 6\cos\left(\frac{\pi}{12}t\right)$. The temperature inside the cabin remains unchanged.

Find the value of K if the temperature outside is warmer than inside the cabin for exactly 16 hours. **[3.3.3]**

$D(t) = 20 - K - 10\cos\left(\frac{\pi}{12}t\right)$.

We want $D(t) > 0$ for exactly 8 hours over one period. [1M]

Solve $D(t) = 0 \implies t_1 = \frac{12}{\pi}\arccos\left(\frac{20-K}{10}\right), t_2 = 24 - \frac{12}{\pi}\arccos\left(\frac{20-K}{10}\right)$ [1M]

Then $t_2 - t_1 = 24 - \frac{24}{\pi}\arccos\left(\frac{20-K}{10}\right) = 8 \implies K = 25$ [1A] .

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Section C: Supplementary Questions

Sub-Section: Exam 1 Questions

Question 9

a. State the range and period of the function.

$$h : \mathbb{R} \to \mathbb{R}, h(x) = 5 + 6\cos\left(\frac{\pi x}{2}\right)$$

The period is $\frac{2\pi}{\pi/2} = 4$ [1A]

The range is $[-1, 11]$. [1A]

b. Solve the equation:

$$\sin\left(3x - \frac{\pi}{6}\right) = \frac{1}{2}, \text{ for } x \in [0, \pi].$$

We have $3x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$ [1M]

$3x = \frac{\pi}{3}, \pi$

$x = \frac{\pi}{9}, \frac{\pi}{3}$

The period is $\frac{2\pi}{3} = \frac{6\pi}{9}$ so all solutions in $x \in [0, \pi]$ are:

$$x = \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}, \pi. \quad [1A]$$

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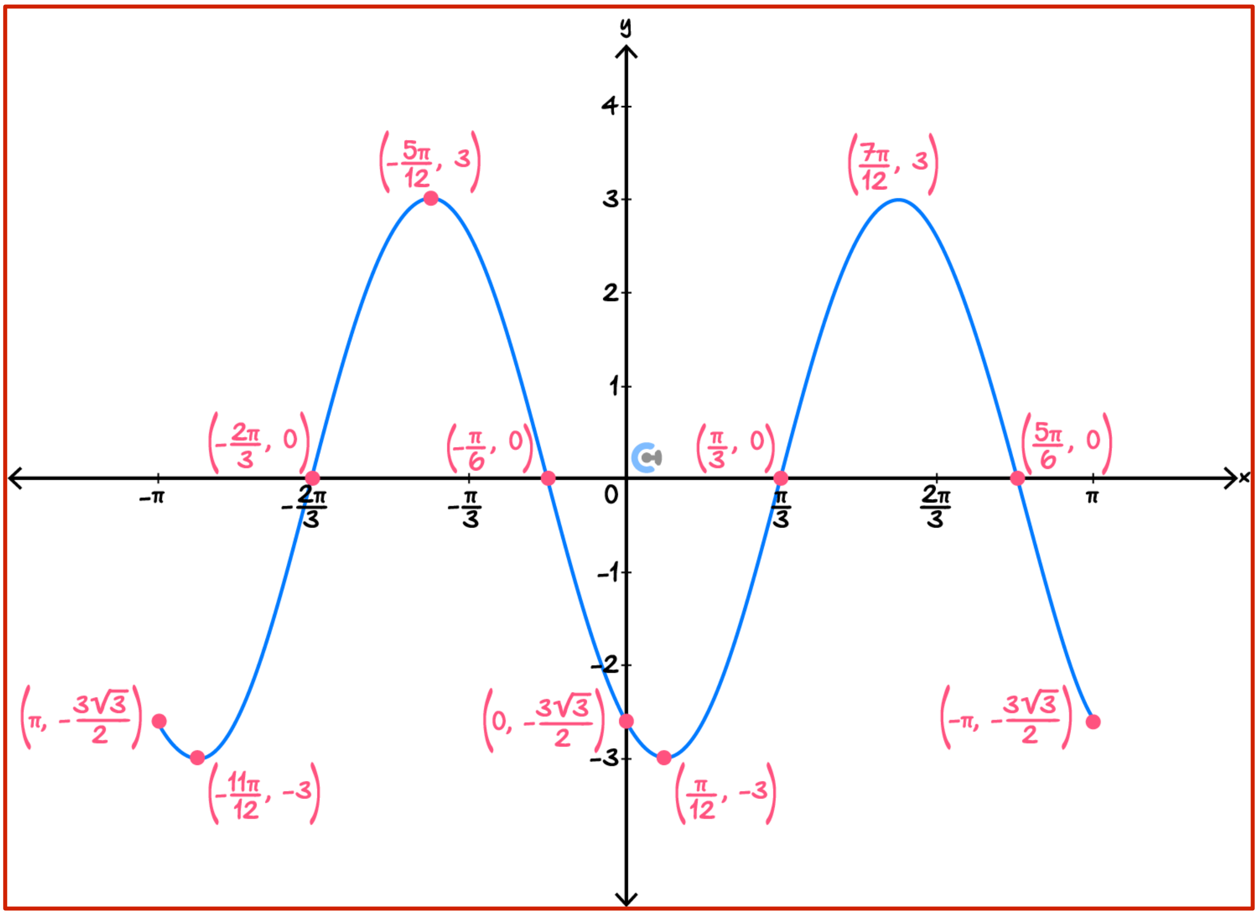
Question 10

For the function $f : [-\pi, \pi] \to \mathbb{R}, f(x) = -3\sin\left(2x + \frac{\pi}{3}\right)$.

a. Write down the amplitude and period of the function.

Period is π and amplitude is 3.

b. Sketch the graph of the function f on the set of axes below. Label axes intercepts with their coordinates. Label endpoints of the graph with their coordinates.



[1M shape, 1M endpoints, 1M intercepts]

Question 11

a. Show that $(2x - 1)(4x^2 + 2x - 1) = 8x^3 - 4x + 1$.

$$(2x-1)(4x^2+2x-1) = 8x^3 + 4x^2 - 2x + (-4x^2 - 2x + 1)$$
$$= 8x^3 - 4x + 1 \quad [1M]$$

b. Show that $\frac{\tan^2(x)+1}{\tan^2(x)-1} = \frac{1}{1-2\cos^2(x)}$.

We proceed as follows:

$$\begin{aligned}\frac{\tan^2(x)+1}{\tan^2(x)-1} &= \frac{\tan^2(x)+1}{\tan^2(x)-1} \times \frac{\cos^2(x)}{\cos^2(x)} \quad [1M] \\ &= \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)-\cos^2(x)} \\ &= \frac{1}{\sin^2(x)-\cos^2(x)} \quad [1M] \\ &= \frac{1}{(1-\cos^2(x))-\cos^2(x)} \\ &= \frac{1}{1-2\cos^2(x)} \quad [1A]\end{aligned}$$

c. Hence, using the previous two results, solve the equation:

$$\frac{\tan^2(x)+1}{\tan^2(x)-1} = 4\cos(x) \text{ for } 0 \leq x \leq \frac{\pi}{2}.$$

You may use the fact that $\cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right) = \frac{2\pi}{5}$.

Let $a = \cos(x)$, then we have that

$$\frac{1}{1-2a^2} = 4a$$
$$4a - 8a^3 = 1$$
$$8a^3 - 4a + 1 = 0 \quad [1M]$$

Then from part a. we have $8a^3 - 4a + 1 = 0 \implies (2a-1)(4a^2+2a-1) = 0$.
Solve

$$4a^2 + 2a - 1 = 0$$
$$a = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4} \quad [1M]$$

Note that both of these are in the range $[-1, 1]$.
Therefore we solve $\cos(x) = \frac{1}{2}$, $\cos(x) = \frac{-1-\sqrt{5}}{4}$, $\cos(x) = \frac{-1+\sqrt{5}}{4}$ for $0 \leq x \leq \frac{\pi}{2}$.
[1M]
$x = \frac{\pi}{6}, \frac{2\pi}{5}$ [1A]

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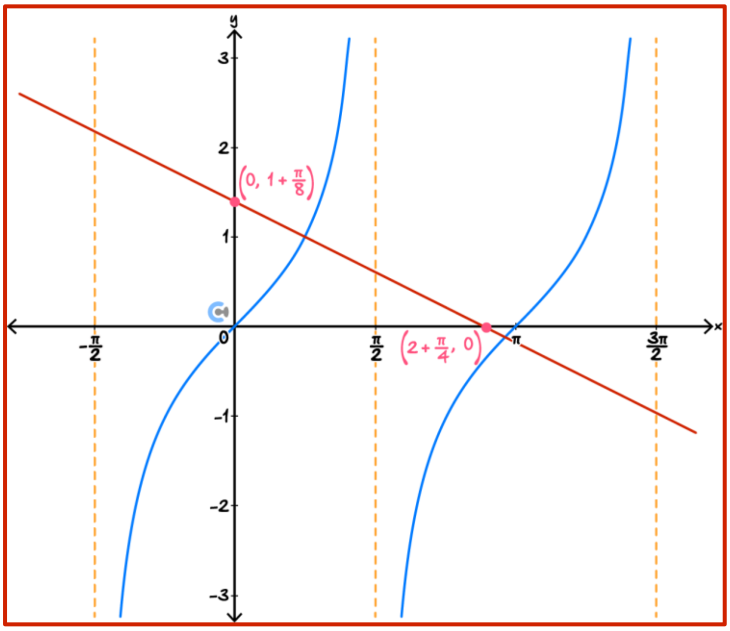
Sub-Section: Exam 2 Questions

Question 12

The graph of $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}$, where $f(x) = \tan(x)$ is shown below.



a.

i. Find $f'\left(\frac{\pi}{4}\right)$.

2. [1A]

ii. Find the equation of the **normal** to the graph of $y = f(x)$ at the point where $x = \frac{\pi}{4}$.

Line with gradient of $-\frac{1}{2}$ that passes through $\left(\frac{\pi}{4}, 1\right)$. [1M]

$y = \frac{8+\pi}{8} - \frac{x}{2}$ [1A]

[1A for each intercept, 1M straight line through $\left(\frac{\pi}{4}, 1\right)$]

iii. Sketch the graph of this normal on the axes above. Give the exact axis intercepts.

b. Find the exact values of $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such that $f'(x) = f'\left(\frac{\pi}{4}\right)$.

We solve $f'(x) = 2$. [1M]
$x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$. [1A]

c. Let $g(x) = f(x - a)$. Find the exact value of $a \in (-1, 1)$ such that $g(1) = 1$.

We solve $\tan(1-a) = 1$ for $a \in (-1, 1)$. [1M]
$a = 1 - \frac{\pi}{4}$. [1A]

d. Let $h: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}, h(x) = \sin(x) + \tan(x) + 2$.

i. Find $h'(x)$.

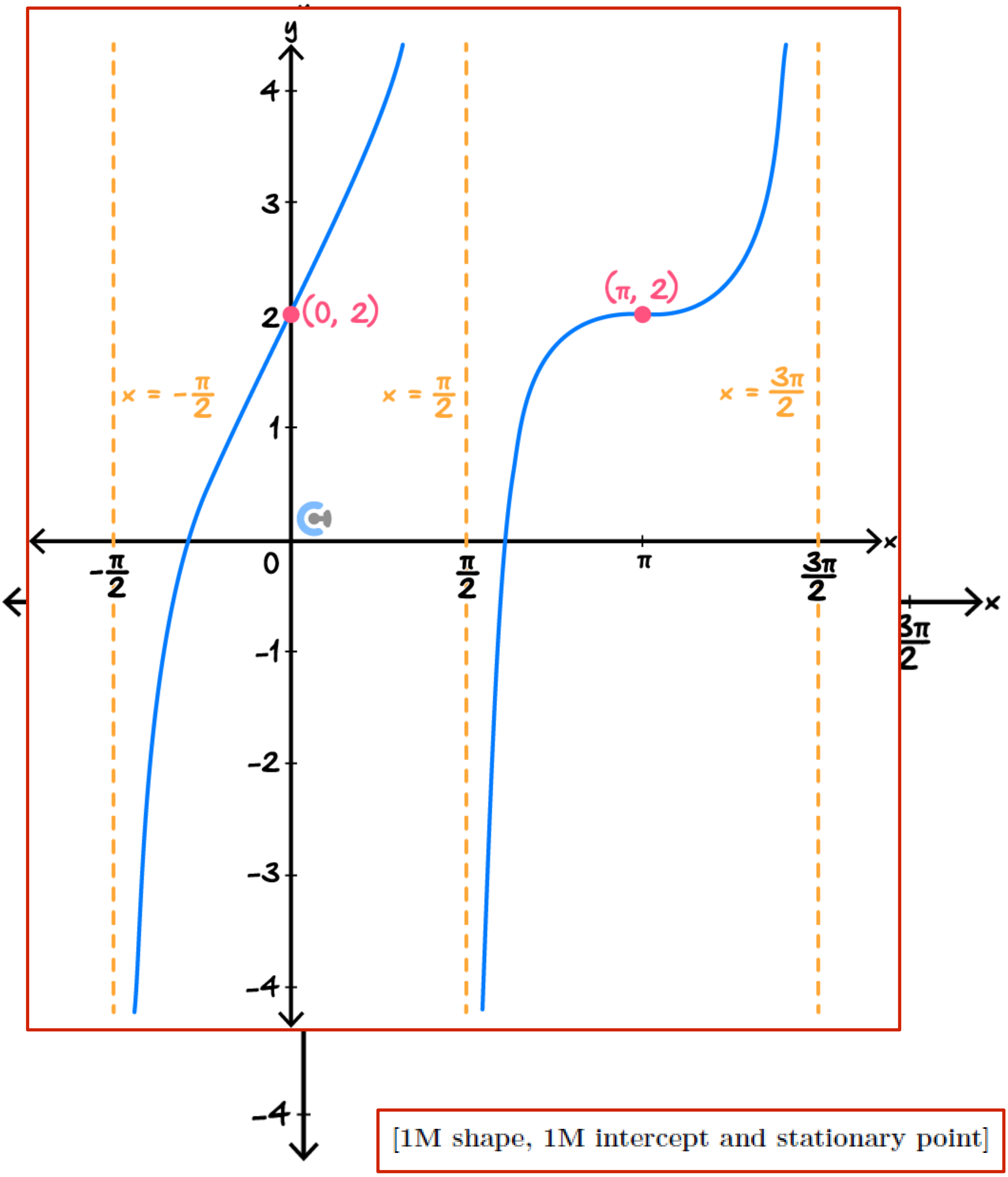
$h'(x) = \cos(x) + \sec^2(x) = \cos(x) + \frac{1}{\cos^2(x)}$ [1A]

ii. Solve the equation $h'(x) = 0$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. (Give exact values.)

We solve $\cos(x) + \sec^2(x) = 0$ over the domain. [1M]
$x = \pi$ only. [1A]

e. Sketch the graph of $y = h(x)$ on the axes below.

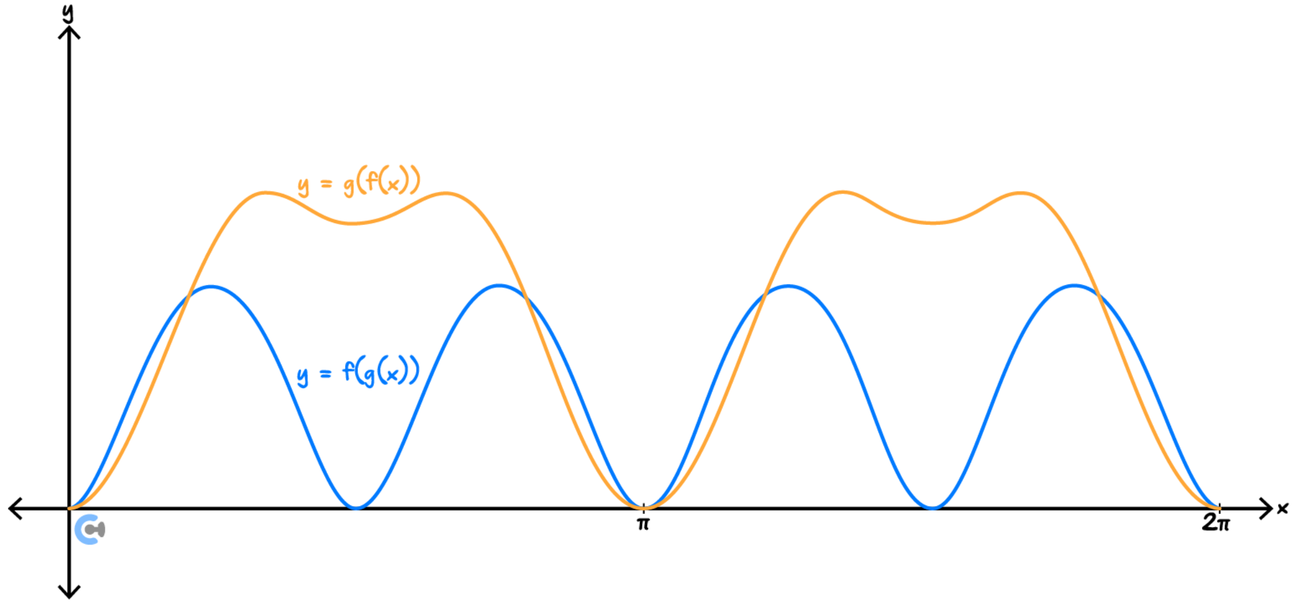
- Give the exact coordinates of any stationary points.
- Label each asymptote with its equation.
- Give the exact value of the y-intercept.



[1M shape, 1M intercept and stationary point]

Question 13

The graph below shows the compositions $g \circ f$ and $f \circ g$, where $f(x) = \sin^2(x)$ and $g(x) = \sin(2x)$.



a.

i. The graph of $y = (g \circ f)(x)$ has a local maximum whose x-value lies in the interval $\left[0, \frac{\pi}{2}\right]$.

Find the coordinates of this local maximum, correct to one decimal place.

$(1.1, 1)$. **[1A]**

ii. State the range of $g \circ f$ for $x \in [1, 2]$. Give your answers correct to one decimal place.

$[0.9, 1]$

b.

i. State the period of $f \circ g$.

$$\text{Period} = \frac{\pi}{2}.$$

ii. Find the derivative of $f \circ g$.

$$(f \circ g)'(x) = 4\sin(\sin(2x))\cos(2x)\cos(\sin(2x))$$

iii. Hence, find two equations that when solved give the $x$coordinates for turning points of $f \circ g$.

$\sin(\sin(2x)) = 0$. **[1A]**
$\cos(2x) = 0$. **[1A]**
Note that the equation $\cos(\sin(2x)) = 0$ has no real solutions.

iv. Hence, find the x-values of the stationary points of $f \circ g$ where $x \in [0, \pi]$.

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$

v. Find the range of $f \circ g$ where $x \in [0, 2\pi]$.

$$[0, \sin^2(1)].$$

c. Let $f_1 : (-\pi, 0) \to \mathbb{R}, f_1(x) = \sin^2(x)$.

Find all values of x in the interval $(0, 2\pi)$ for which the composition $f_1 \circ g$ is defined.

We require that ran $g \subseteq$ dom f_1.
ran $g = [-1, 1]$. We have that $[-1, 0) \subseteq (-\pi, 0)$. **[1M]**
So find when ran $g = [-1, 0)$. Solve $-1 \leq \sin(2x) < 0$ for $x \in (0, 2\pi)$.
$x \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. **[1A]**

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