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VCE Mathematical Methods ¾

Circular Functions II [3.3]

Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
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Section A: Compulsory Questions

Sub-Section [3.3.1]: Solve Advanced Trigonometric Equations

Question 1

Find the general solution to the following trigonometric equations over the specified domain.

a. $2\sin(x) - 1 = 0$, for $x \geq 0$.

$\sin(x) = \frac{1}{2}$.

$x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, where $n \in \mathbb{Z}_{\geq 0}$

b. $\tan(2x) = \sqrt{3}$, for $x < 0$.

$2x = \frac{\pi}{3} + n\pi$.

$x = \frac{\pi}{6} - \frac{n\pi}{2}$, where $n \in \mathbb{Z}_{\geq 1}$

c. $\sqrt{2}\cos\left(2\pi + \frac{\pi}{4}\right) = 1$, for $x > 0$.

$\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$2x + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$2x = 2n\pi$ or $2x = 2n\pi - \frac{\pi}{2}$.

$x = n\pi$, where $x \in \mathbb{Z}_{\geq 1}$ or $x = n\pi - \frac{\pi}{4}$, where $x \in \mathbb{Z}_{\geq 1}$.

Question 2

Solve the following trigonometric equations over the specified domain.

a. $2\cos^2(x) - 3\cos(x) + 1 = 0$, for $0 \leq x \leq 2\pi$.

Let $\cos(x) = a$. We solve

$$2a^2 - 3a + 1 = 0$$
$$(2a - 1)(a - 1) = 0$$

$\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$ and $x \in [0, 2\pi]$. Thus

$$x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$$

b. $\sin^2(2x) + \sin(2x) - 2 = 0$, for $x \in \mathbb{R}$.

Let $\sin(2x) = a$. Then we have

$$a^2 + a - 2 = 0$$
$$(a+2)(a-1) = 0$$
$$a = -2, a = 1$$

So due to domain and range we only solve $\sin(2x) = 1$. Thus

$2x = \frac{\pi}{2} + 2n\pi$

$x = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$.

c. $3\sin^2(x) - 6\sin(x) - \cos^2(x) + 3 = 0$, for $0 \le x \le 2\pi$.

Use $\cos^2(x) = 1 - \sin^2(x)$ to rewrite the equation as

$$4\sin^2(x) - 6\sin(x) + 2 = 0$$

Let $a = \sin(x)$ then

$$4a^2 - 6a + 2 = 0$$
$$2a^2 - 3a + 1 = 0$$
$$(2a-1)(a-1) = 0$$

So $\sin(x) = \frac{1}{2}$ or $\sin(x) = 1$. Thus

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}.$$

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Question 3

Find the value(s) of k such that $4\sin^2(x) + k\cos(x) - 2 = 0$ has 2 solutions in the interval $[0, \pi]$.

Use Pythagorean identity to write $4 - 4\cos^2(x) + k\cos(x) - 2 = 0$. Now let $a = \cos(x)$ we have that

$$-4a^2 + ka + 2 = 0$$
$$4a^2 - ka - 2 = 0$$
$$a = \frac{k \pm \sqrt{k^2 + 32}}{8}$$

The period of our function is 2π. The range of $\cos(x)$ is $[-1, 1]$, and $\cos(x) = b$ for $-1 \leq b \leq 1$ will only have one solution in the interval $[0, \pi]$. Thus we will have two solutions in $[0, \pi]$ if both of the following are satisfied:

$$-1 \leq \frac{k + \sqrt{k^2 + 32}}{8} \leq 1 \qquad (1)$$
$$-1 \leq \frac{k - \sqrt{k^2 + 32}}{8} \leq 1 \qquad (2)$$

Inequality (1):

$$-8 \leq k + \sqrt{k^2 + 32} \leq 8$$
$$\implies \sqrt{k^2 + 32} \leq 8 - k$$
$$k^2 + 32 \leq k^2 - 16k + 64$$
$$16k \leq 32$$
$$k \leq 2$$

Inequality (2):

$$-8 \leq k - \sqrt{k^2 + 32} \leq 8$$
$$\implies -\sqrt{k^2 + 32} \geq -8 - k$$
$$k^2 + 32 \leq k^2 + 16k + 64$$
$$0 \leq 16k + 32$$
$$k \geq -2$$

Combining we have that $-2 \leq k \leq 2$. With technology access we can use sliders to check this!

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Question 4 Tech-Active.

a. Find the general solution to the equation $2\cos^2(x) + 3\cos(x) - 2 = 0$.

$$x = 2n\pi \pm \frac{\pi}{3} +, n \in \mathbb{Z}$$

TI:	**Mathematica:**	**Casio:**

solve$(2 \cdot (\cos(x))^2 + 3 \cdot \cos(x) - 2 = 0, x)$
 $x = \frac{(6 \cdot n10 - 1) \cdot \pi}{3}$ or $x = \frac{(6 \cdot n10 + 1) \cdot \pi}{3}$
	In[1]:= Solve[2 (Cos[x])^2 + 3 Cos[x] - 2 == 0, x, Reals] // Expand
	 Out[1]= $\{\{x \to -\frac{\pi}{3} + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\}, \{x \to \frac{\pi}{3} + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}\}\}$
		solve(2(cos(x))^2+3*cos(x)-2=0, x)
		 $\{x = 2 \cdot \pi \cdot \text{constn}(1) - \frac{\pi}{3}, x = 2 \cdot \pi \cdot \text{constn}(2) + \frac{\pi}{3}\}$

b. Hence, find the solutions to $2\cos^2(x) + 3\cos(x) - 2 = 0$ for $x \in [0, 2\pi]$.

$$x = \frac{\pi}{3}, \frac{5\pi}{3}.$$

TI:	**Mathematica:**	**Casio:**
solve$(2 \cdot (\cos(x))^2 + 3 \cdot \cos(x) - 2 = 0, x) \mid 0 \le x \le 2 \cdot \pi$ $x = \frac{\pi}{3}$ or $x = \frac{5 \cdot \pi}{3}$	Solve[2 (Cos[x])^2 + 3 Cos[x] - 2 == 0 && 0 ≤ x ≤ 2 Pi, Reals] $\{\{x \to \frac{\pi}{3}\}, \{x \to \frac{5\pi}{3}\}\}$	solve(2(cos(x))^2+3*cos(x)-2=0, x) | 0≤x≤2*π $\{x = \frac{\pi}{3}, x = \frac{5\pi}{3}\}$

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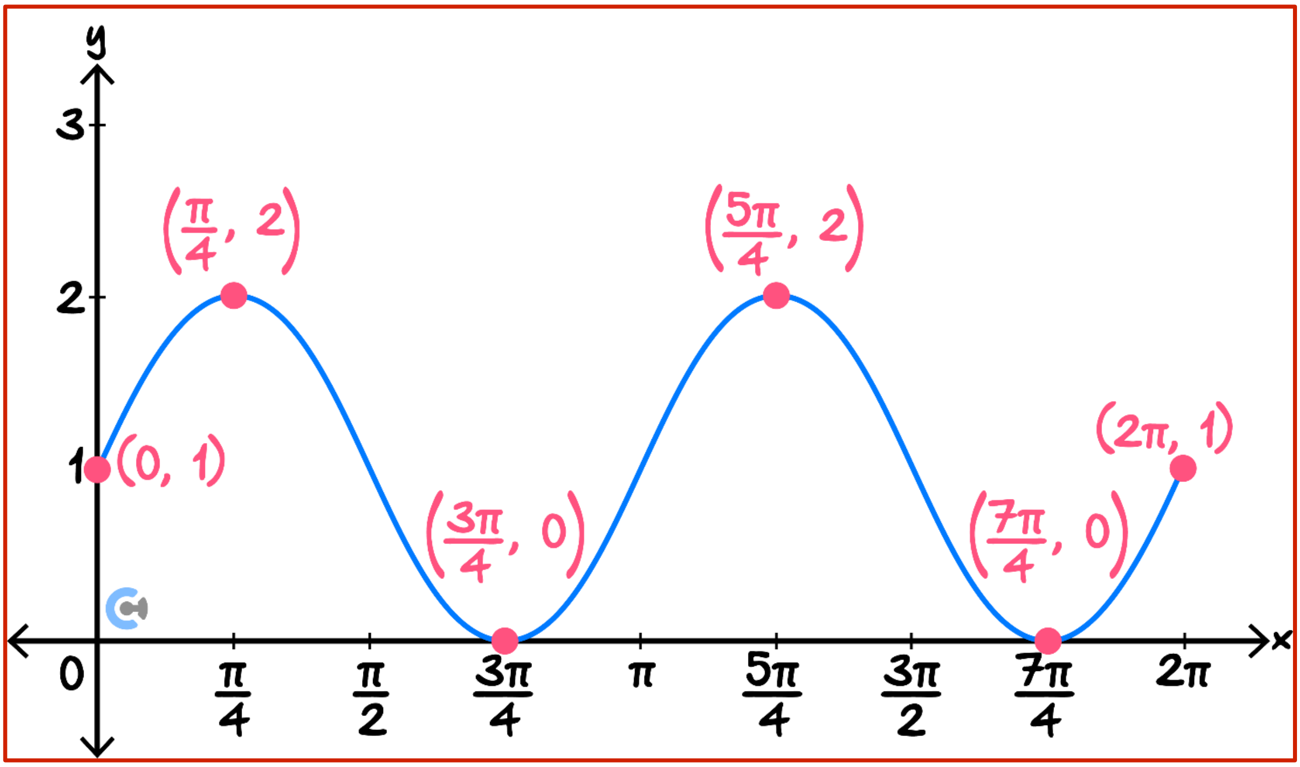
Sub-Section [3.3.2]: Graph Sine, Cosine, and Tangent Functions

Question 5

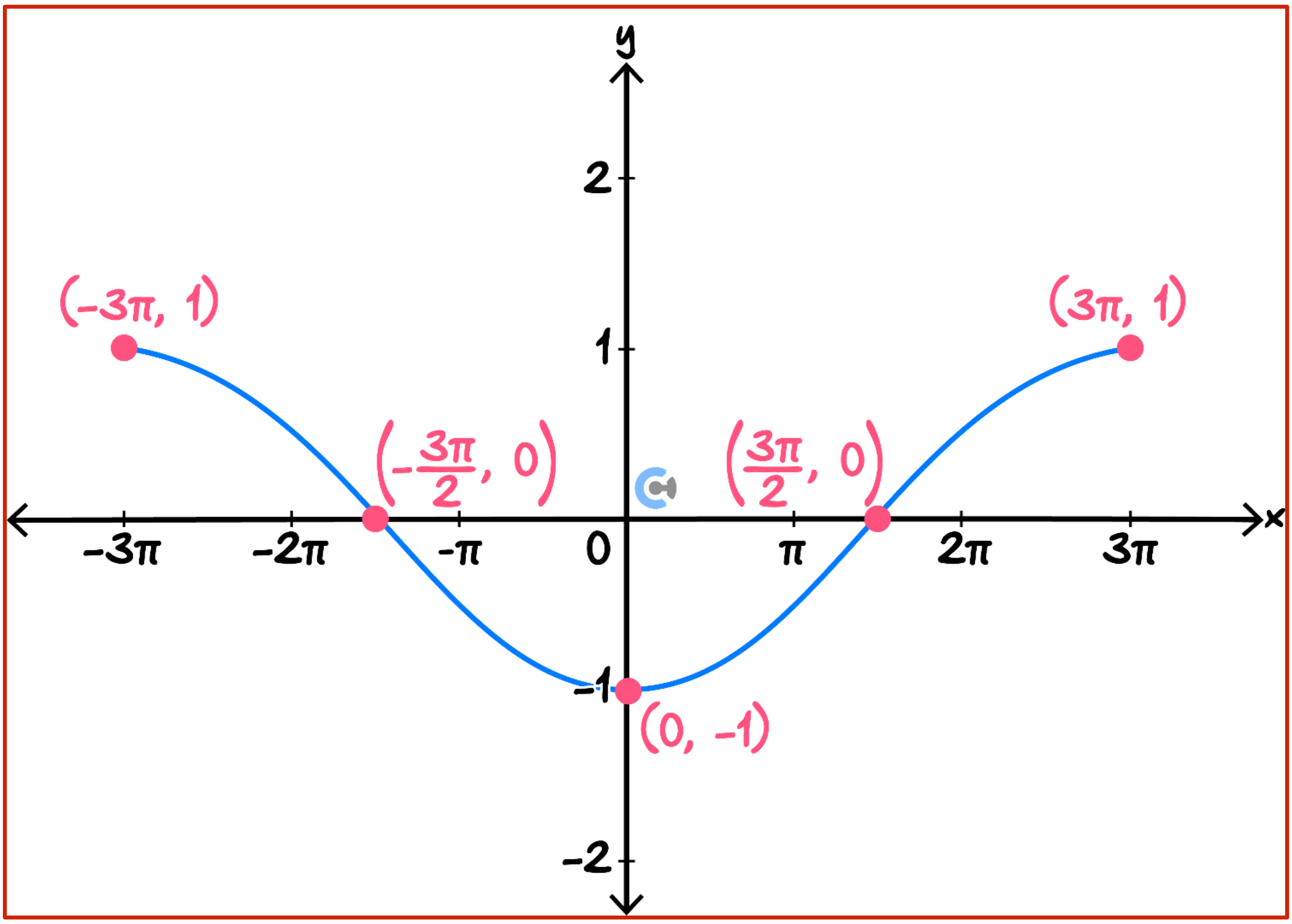
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts, turning points and endpoints with their coordinates, and asymptotes with their equations.

a. $y = \sin(2x) + 1$ for $0 \le x \le 2\pi$.



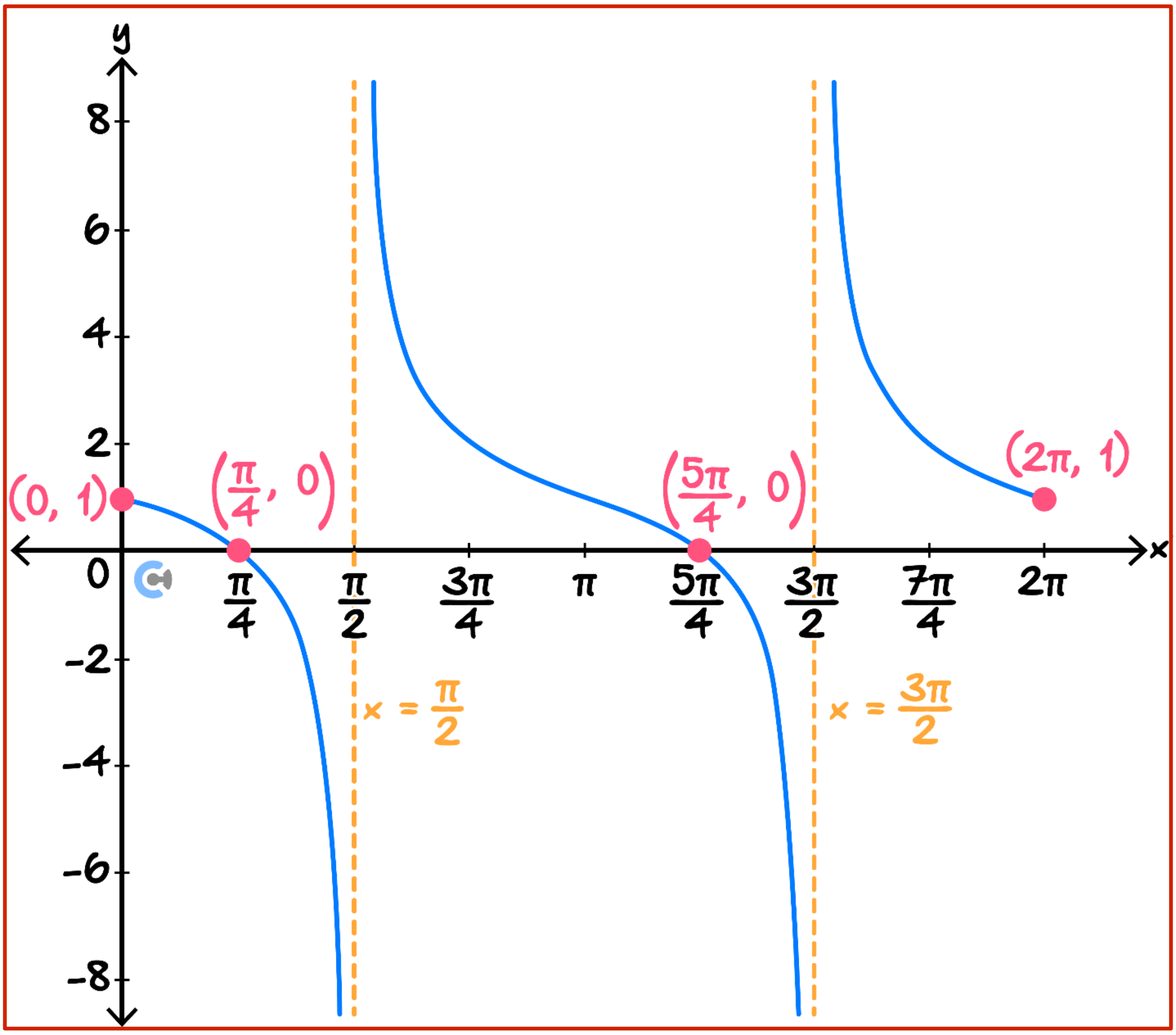
$\sin(2x) = -1 \implies 2x = \frac{3\pi}{2} \implies x = \frac{3\pi}{4}$. Period is π.

b. $y = \cos\left(\frac{x}{3}\right)$ for $-3\pi \leq x \leq 3\pi$.



$\frac{x}{3} = \frac{\pi}{2} \implies x = \frac{3\pi}{2}$ is one intercept. Period is 6π

c. $y = -\tan(x) + 1$ for $0 \le x \le 2\pi$.



Asymptote base angle $\frac{\pi}{2}$, period $= \pi$, x-intercept base angle $\frac{\pi}{4}$.

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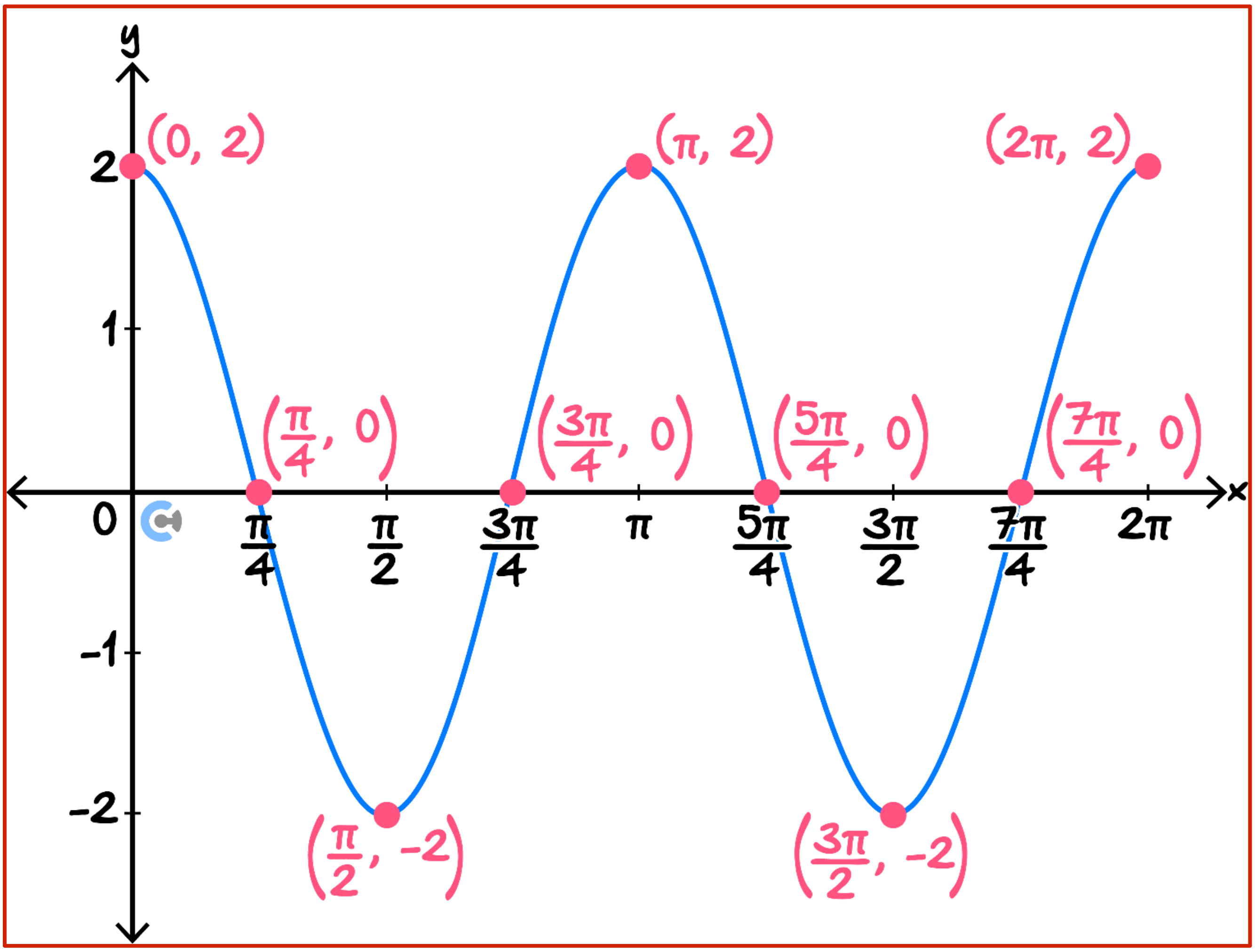
Question 6

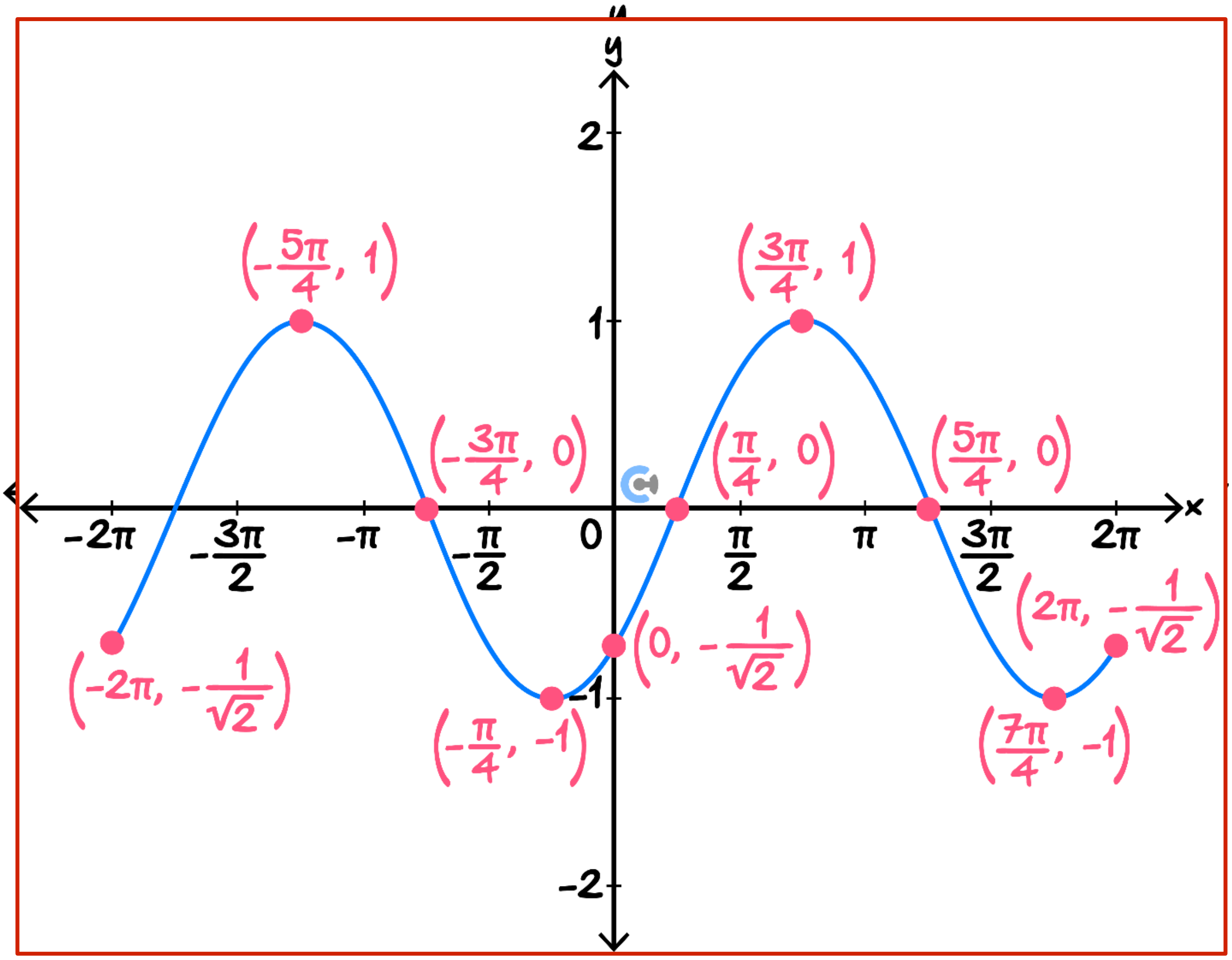
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts, turning points and endpoints with their coordinates and asymptotes with their equations.

a. $y = 2\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ for $0 \leq x \leq 2\pi$.



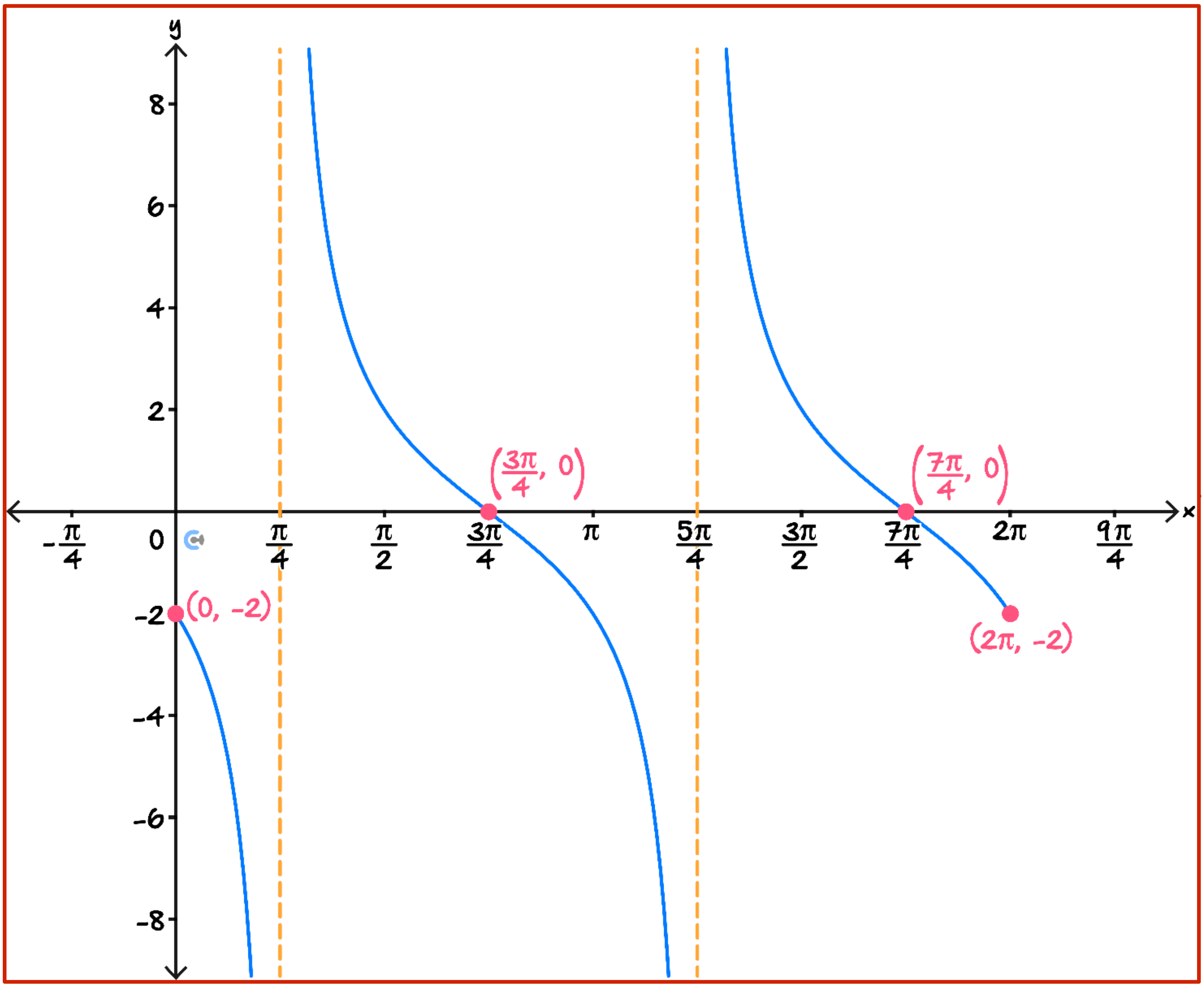
$x = \frac{\pi}{4}, \frac{3\pi}{4}$ base angles for x- intercepts. Period $= \pi$.

b. $y = -\cos\left(x + \frac{\pi}{4}\right)$ for $-2\pi \leq x \leq 2\pi$.



$x + \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2} \implies x = \frac{\pi}{4}, \frac{5\pi}{4}$. Period $= 2\pi$.

c. $y = -2\tan\left(x + \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$.



Period $= \pi$. Asymptote base angle $= \frac{\pi}{4}$. x-intercept base angle $= \frac{3\pi}{4}$.

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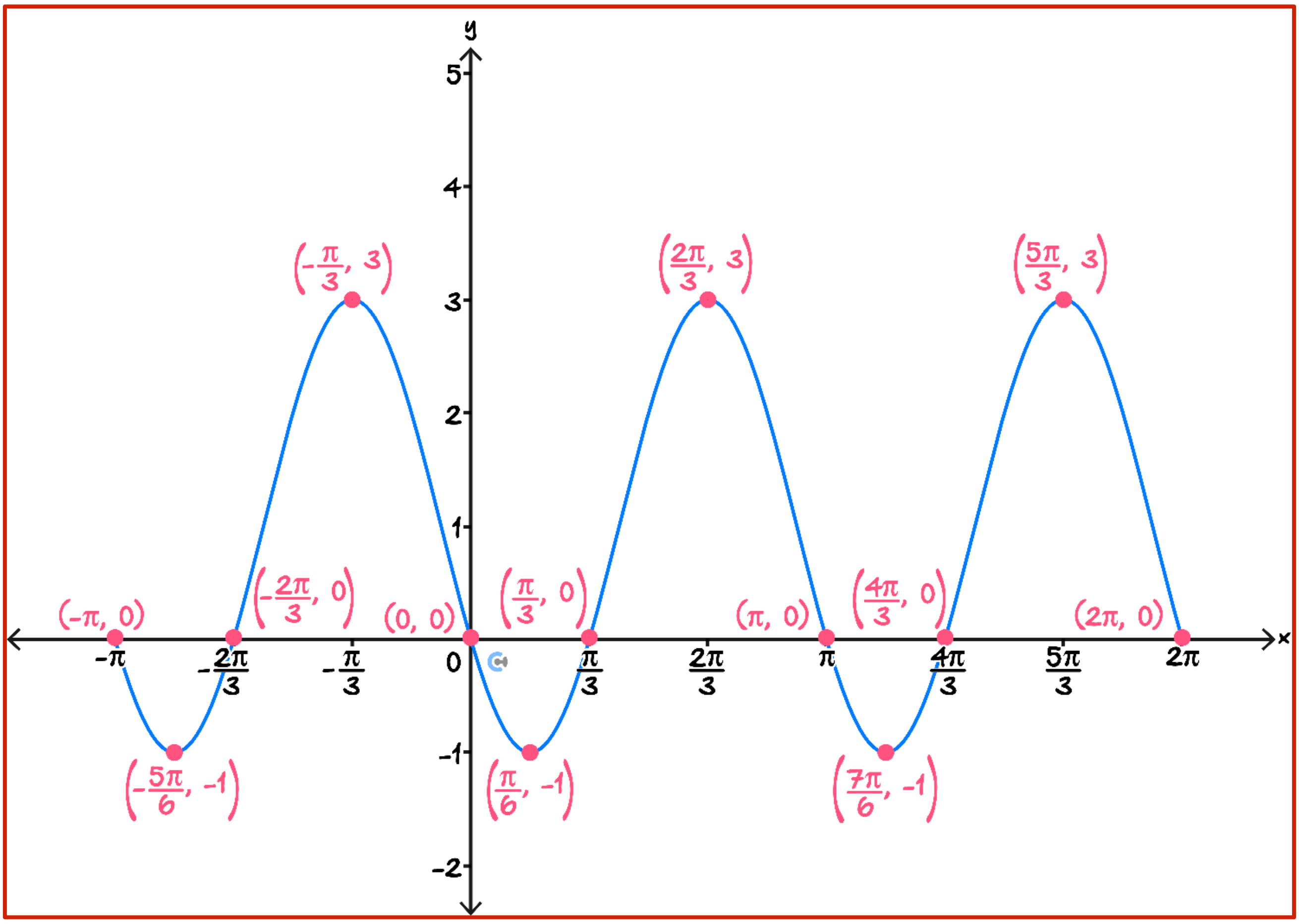
Question 7

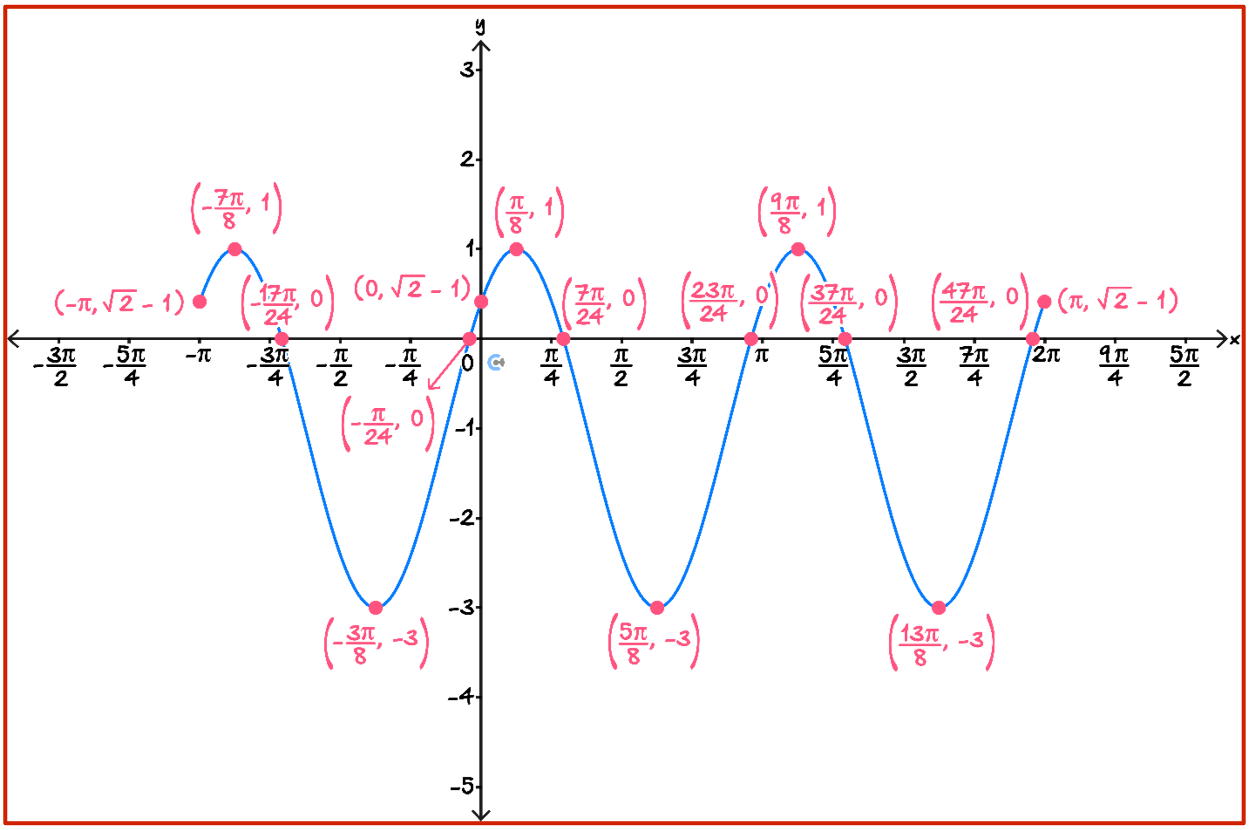
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts and endpoints with their coordinates and asymptotes with their equation.

a. $y = -2\sin\left(2x + \frac{\pi}{6}\right) + 1$ for $-\pi \le x \le 2\pi$.



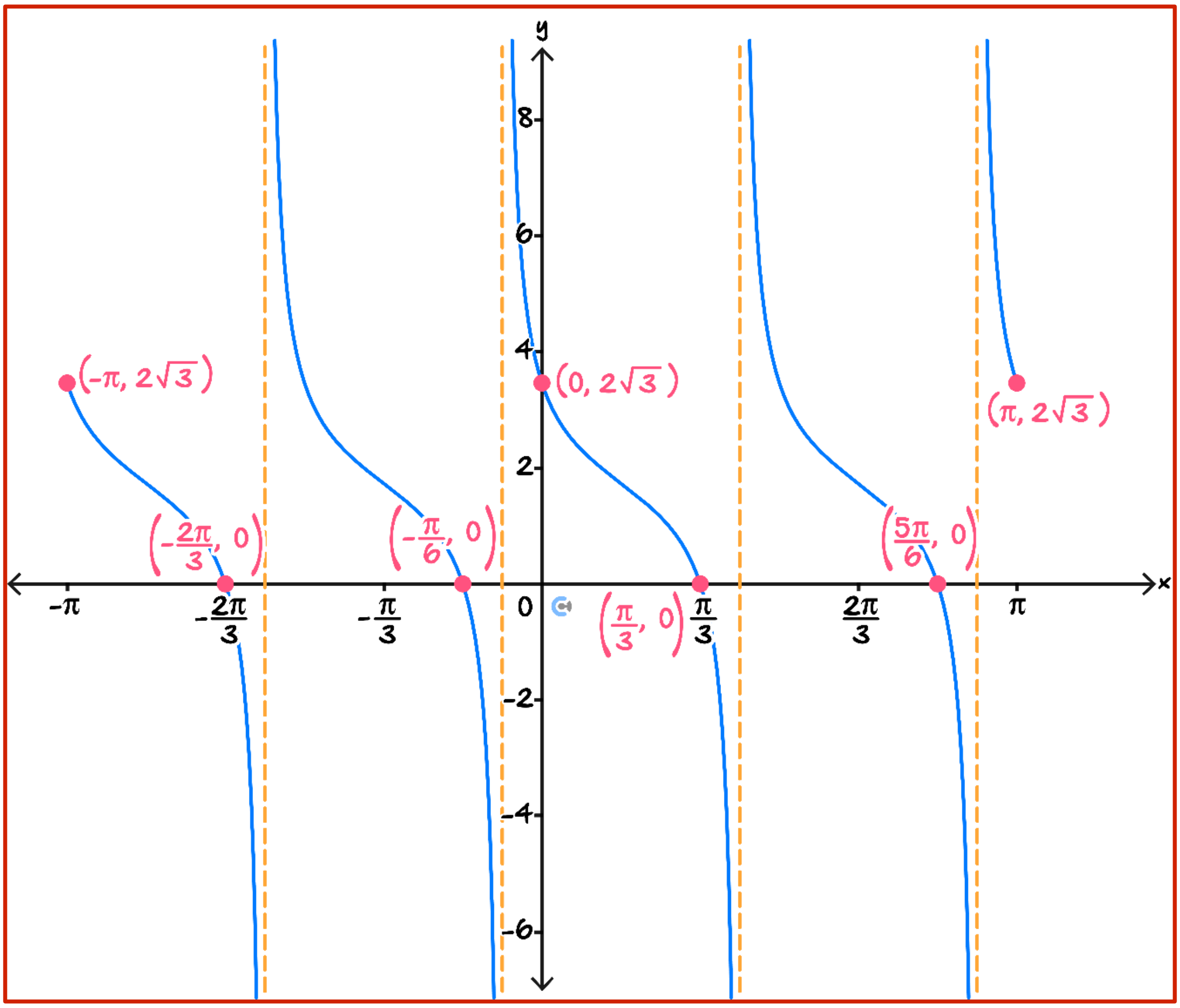
$x = 0, x = \frac{\pi}{3}$ base angles for x-intercepts. Period $= \pi$.

b. $y = 2\cos\left(2x - \frac{\pi}{4}\right) - 1, x \in [-\pi, 2\pi]$.



$$2x - \frac{\pi}{4} = -\frac{\pi}{3}, \frac{\pi}{3} \implies x = -\frac{\pi}{24}, \frac{7\pi}{24}. \text{ Period } = \pi.$$

c. $y = -\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3}$ for $-\pi \leq x \leq \pi$.



Period $= \frac{\pi}{2}$. Asymptotes $x = -\frac{\pi}{12} + \frac{n\pi}{2}$. x-intercepts $\frac{\pi}{3} + \frac{n\pi}{2}$

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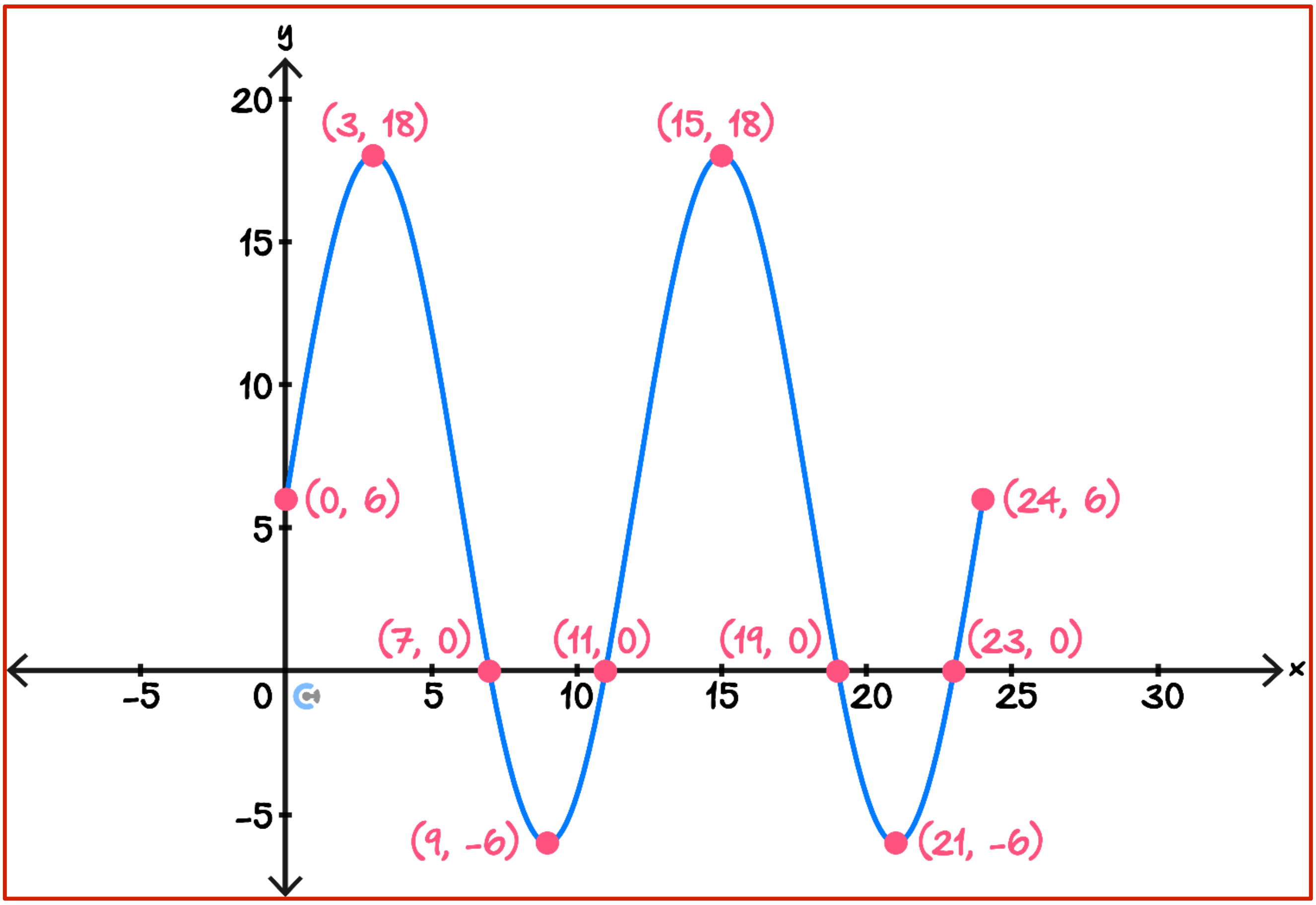
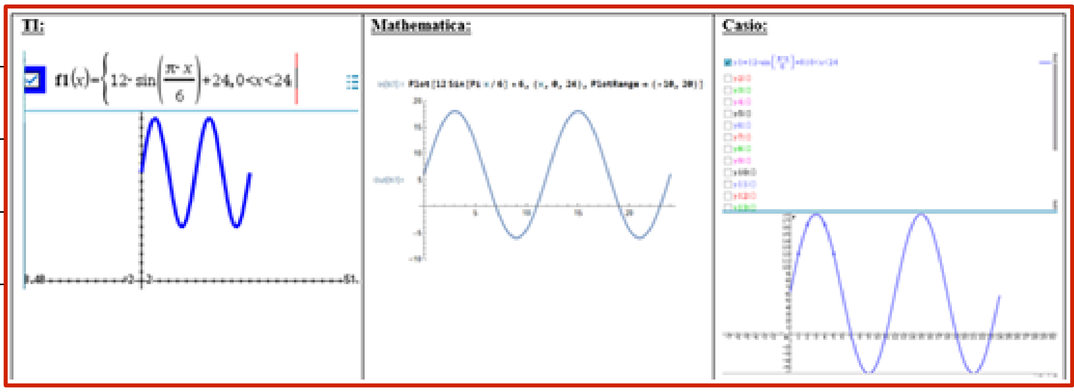
Question 8 Tech-Active.

Sketch the graph of $y = 12\sin\left(\frac{\pi x}{6}\right) + 6$ for $x \in [0, 24]$ on the axes below. Label all axial intercepts and turning points with coordinates.





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Sub-Section [3.3.3]: Fraction of Periods

Question 9

The temperature of a lake throughout the year is modelled by $T(t)$, where:

$$T(t) = 18 + 4\cos\left(\frac{\pi}{6}t\right)$$

Where $T(t)$ represents the temperature (in degrees Celsius) of the lake at t months since January.

Find the fraction of the year during which the temperature is above 20°C.

We need to solve:

$$18 + 4\cos\left(\frac{\pi}{6}t\right) > 20$$
$$4\cos\left(\frac{\pi}{6}t\right) > 2$$
$$\cos\left(\frac{\pi}{6}t\right) > \frac{1}{2}.$$

Thus, solving for t if we have equality:

$$\frac{\pi}{6}t = \pm\frac{\pi}{3} + 2k\pi$$
$$t = \pm 2 + 12k, \quad k \in \mathbb{Z}.$$

Since cosine is positive in quadrants I and IV, the valid range satisfying $\cos\left(\frac{\pi}{6}t\right) > \frac{1}{2}$ within one period is:

$$-2 < t < 2.$$

The periodicity of the function is:

$$T = \frac{2\pi}{\frac{\pi}{6}} = 12.$$

Within one year $(0 \le t \le 12)$, the valid range repeats at $t = 12k$, so the solution intervals are:

$$0 < t < 2 \quad \text{and} \quad 10 < t < 12.$$

The total duration where $T(t) > 20$ is:

$$(2 - 0) + (12 - 10) = 4 \text{ months}.$$

The fraction of the year is:

$$\frac{4}{12} = \frac{1}{3}.$$

Thus, the lake's temperature is above 20°C for $\frac{1}{3}$ of the year.

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Question 10 Tech-Active.

A research team is monitoring the depth of water in a tidal bay. The depth of the water, in metres, is modelled by the function:

$$D(t) = 8 + 3\cos\left(\frac{\pi}{6}t\right)$$

Where $D(t)$ represents the depth of the water t hours after midnight.

a. State the maximum and minimum depth of the water.

The maximum occurs when $\cos x = 1$ and the minimum when $\cos x = -1$:

$$D_{\max} = 8 + 3(1) = 11, \quad D_{\min} = 8 + 3(-1) = 5.$$

The depth varies between 5m (low tide) and 11m (high tide).

b. Determine the first two times after midnight when the water reaches a depth of 10 metres. Give your answers in hours after midnight, correct to two decimal places.

Solve $D(t) = 10 \implies t = 1.61, 10.39$

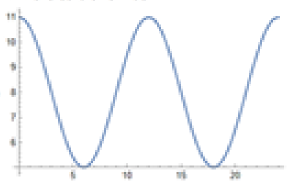
TI:	**Mathematica:**	**Casio:**

Define $d(t)=8+3\cdot\cos\left(\frac{\pi}{6}\cdot t\right)$ Done
solve($d(t)=10,t$)|$0<t<12$ $t=1.60632$ or $t=10.3937$
	In[74]:= d[t_] := 8 + 3 Cos[Pi / 6 t]
	In[75]:= Solve[d[t] == 10 && 0 ≤ t ≤ 12, t] // N
	Out[75]= {{t → 10.3937}, {t → 1.60632}}
		solve(d(t)=10,t)|0<t<12
		{t=10.39367716, t=1.606322837}

c. Find the percentage of a full tidal cycle during which the water depth is above 9 metres. Give your answer correct to two decimal places.

Solve $D(t) = 9 \implies t = 2.35096, 9.64904.$
Then look at the shape of the graph and note that the period is 12.
Thus percentage above 9 metres is $\dfrac{2.35096 + (12 - 9.64904)}{12} \times 100 = 39.18\%$

TI:	**Mathematica:**	**Casio:**

solve($d(t)=9,t$)|$0<t<12$ $t=2.35096$ or $t=9.64904$
$\frac{2.35096+12-9.64904}{12}\cdot 100$ 39.1827
	In[76]:= Plot[d[t], {t, 0, 24}]
	Out[76]= 
	In[77]:= Solve[d[t] == 9 && 0 ≤ t ≤ 12] // N
	Out[77]= {{t → 9.64904}, {t → 2.35096}}
	In[78]:= $\frac{1}{12}$ (2.350959312183644` + (12 - 9.649040687816356`)) * 100
	Out[78]= 39.1827
		solve(d(t)=9,t)|0<t<12
		{t=9.649040688, t=2.350959312}
		$\frac{2.350959312+(12-9.649040688)}{12}$*100
		39.1826552

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Question 11 Tech-Active.

The temperature in a greenhouse fluctuates throughout the day and is modelled by the function:

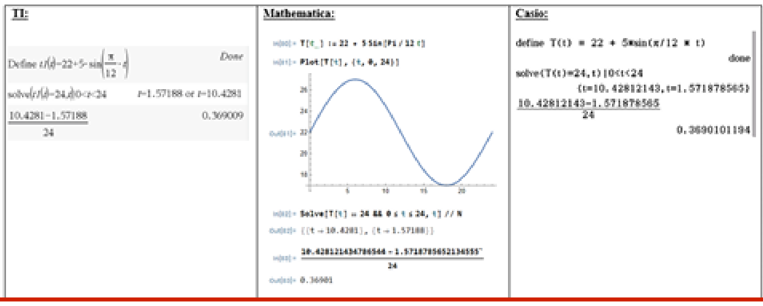
$$T(t) = 22 + 5\sin\left(\frac{\pi}{12}t\right)$$

where $T(t)$ represents the temperature in degrees Celsius, and t is the time in hours after midnight.

a. Find the fraction of a full day during which the temperature exceeds 24°C. Give your answer as a decimal correct to three decimal places.

Solve $T(t) = 24 \implies t = 1.57, 10.42$.

Period is 24 and looking at a graph we see the fraction is $\frac{10.42 - 2.57}{24} = 0.369$.

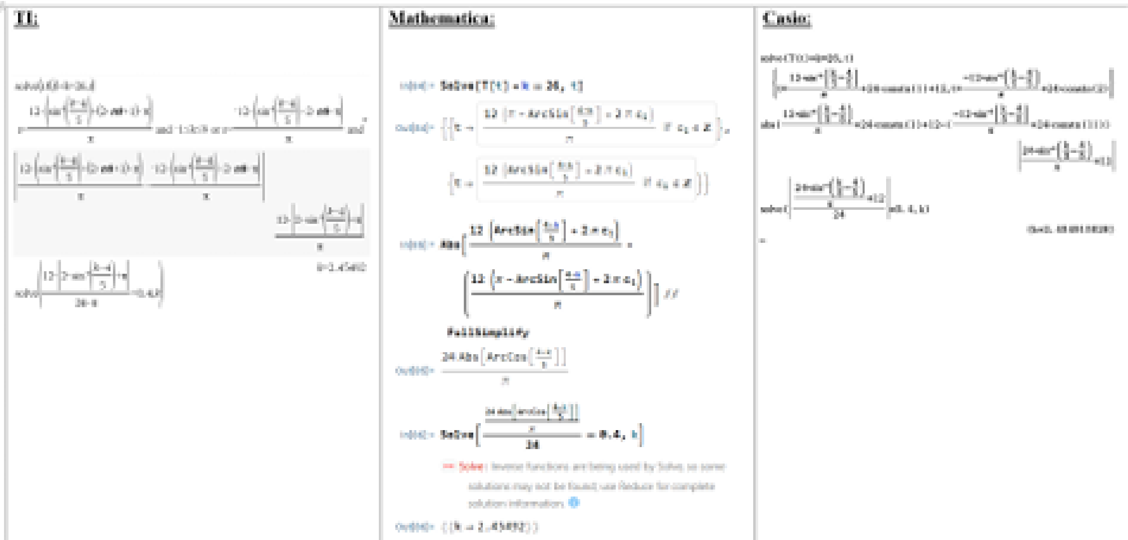


b. Find the value of k such that $T(t) + k$ exceeds 26°C for exactly 40% of the time. Give your answer correct to two decimal places.

We find the distance between two solutions to $T(t) + k = 26$ in terms of k.

Call this distance d, we then solve $\frac{d}{24} = 0.4$ for k.

$k = 2.45$



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Sub-Section: Final Boss

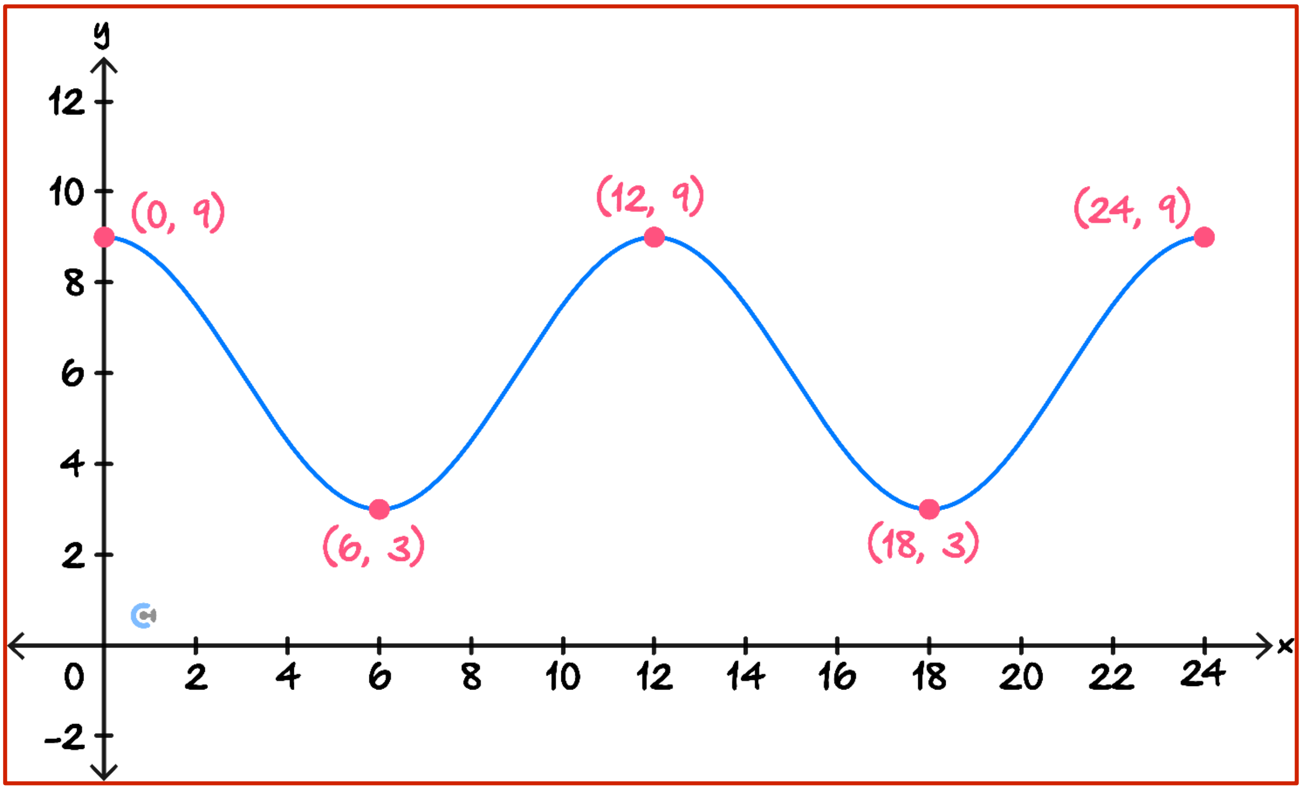
Question 12

The depth of water in a coastal bay varies throughout the day due to tidal motion. The depth of water, in metres, is modelled by the function:

$$D(t) = 6 + 3\cos\left(\frac{\pi}{6}t\right)$$

where t is the time in hours after midnight.

a. Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.



Period $= 12$, max$= 9$, min$= 3$. The graph oscillates between 3 and 9 meters, with key points at $t = 0, 3, 6, 9, 12$.

b. Find the first two times after midnight when the water depth is exactly 7.5 metres.

Solve $6 + 3\cos\left(\frac{\pi}{6}t\right) = 7.5.$

$$\cos\left(\frac{\pi}{6}t\right) = \frac{1}{2}$$
$$\frac{\pi}{6}t = \frac{\pi}{3}, \frac{5\pi}{3}$$
$$t = 2, 10$$

Thus, the first two times are 2am and 10am.

c. Find the fraction of a full tidal cycle when the water depth is below 4.5 metres.

Solve $6 + 3\cos\left(\frac{\pi}{6}t\right) = 5.5.$

$$\cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$$
$$\frac{\pi}{6}t = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$t = 4, 8.$$

By shape of graph below for $\frac{8-4}{12} = \frac{1}{3}$ of the time.

d. The harbour can only accommodate boats when the water depth is at least 5.8 metres. Find the smallest vertical translation k such that $D(t) + k$ ensures this condition is met for at least 75% of the tidal cycle.

Give an exact answer in the form $\frac{a\sqrt{b}-c}{d}$, for positive integers a, b, c, and d.

By looking at the shape of the graph and its symmetry we can deduce that this property will be satisfied if $t = 4.5$ is a solution to the equation, because $\frac{2 \times 4.5}{12} = 75\%$.

$$D(t) + k = 5.8$$

Now $D(4.5) + k = 6 + 3\cos\left(\frac{3\pi}{4}\right) + k = 5.8$. Thus

$$-3 \times \frac{1}{\sqrt{2}} + k = -\frac{1}{5}$$

$$k = \frac{3\sqrt{2}}{2} - \frac{2}{10}$$

$$k = \frac{15\sqrt{2} - 2}{10}$$

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Section B: Supplementary Questions

Sub-Section [3.3.1]: Solve Advanced Trigonometric Equations

Question 13

Find the general solution to the following trigonometric equations over the specified domain.

a. $\sin(2x) = \frac{\sqrt{3}}{2}$, for $x \geq 0$.

$$2x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad 2x = \frac{2\pi}{3} + 2n\pi$$

$$x = \frac{\pi}{6} + n\pi \quad \text{or} \quad x = \frac{\pi}{3} + n\pi, \quad n \in \mathbb{Z}_{\geq 0}$$

b. $\cos\left(x - \frac{\pi}{3}\right) = -\frac{1}{2}$, for $x < 0$.

$$x - \frac{\pi}{3} = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad x - \frac{\pi}{3} = \frac{4\pi}{3} + 2n\pi$$

$$x = \pi - 2n\pi, n \in \mathbb{Z}_{\geq 1} \quad \text{or} \quad x = \frac{5\pi}{3} - 2n\pi, \quad n \in \mathbb{Z}_{\geq}$$

c. $\tan\left(3x + \frac{\pi}{6}\right) = 1$, for $x > 0$.

$$3x + \frac{\pi}{6} = \frac{\pi}{4} + n\pi$$
$$3x = \frac{\pi}{4} - \frac{\pi}{6} + n\pi = \frac{\pi}{12} + n\pi$$
$$x = \frac{\pi}{36} + \frac{n\pi}{3}, \quad n \in \mathbb{Z}_{\geq 0}$$

Question 14

Solve the following equations for $x \in \mathbb{R}$. Note that some solutions will need to be expressed in terms of inverse trigonometric functions.

a. $4\sin^2(x) - 4\sin(x) + 1 = 0$, for $0 \leq x \leq 2\pi$.

Let $\sin(x) = a$, then:

$$4a^2 - 4a + 1 = 0$$
$$(2a-1)^2 = 0$$
$$a = \frac{1}{2}$$

So $\sin(x) = \frac{1}{2}$, hence $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

b. $3\cos^2(2x) + 2\cos(2x) - 1 = 0$, for $x \in \mathbb{R}$.

Let $a = \cos(2x)$, then:

$$3a^2 + 2a - 1 = 0$$
$$(3a-1)(a+1) = 0$$
$$a = \frac{1}{3},\ -1$$

So $\cos(2x) = \frac{1}{3}, -1$.

$$2x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$$
$$2x = 2n\pi \pm \pi$$

Thus all solutions are

$$x = n\pi \pm \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right), n \in \mathbb{Z}$$
$$x = n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$$

c. $\tan^2(x) - \tan(x) - 2 = 0$, for $0 \le x < 2\pi$.

Let $a = \tan(x)$, then:

$$a^2 - a - 2 = 0$$
$$(a-2)(a+1) = 0$$
$$a = 2,\ -1$$

So $\tan(x) = 2$ or $\tan(x) = -1$. Thus

$$x = \tan^{-1}(2), \quad x = \pi + \tan^{-1}(2) \quad (\text{for } \tan(x) = 2)$$
$$x = \frac{3\pi}{4}, \quad x = \frac{7\pi}{4} \quad (\text{for } \tan(x) = -1)$$

Space for Personal Notes

Question 15

Find the value(s) of k such that the equation:

$$3\cos^2(x) + k\sin(x) - 1 = 0$$

has exactly two solutions in the interval $[0, \pi]$.

Use the identity $\cos^2(x) = 1 - \sin^2(x)$:

$$3(1 - \sin^2(x)) + k\sin(x) - 1 = 0$$
$$-3\sin^2(x) + k\sin(x) + 2 = 0.$$

Let $a = \sin(x)$, then:

$$3a^2 - ka - 2 = 0.$$

Solutions for a are:

$$a = \frac{k \pm \sqrt{k^2 + 24}}{6}.$$

Now note that $\sin(x) = b$ will have two solutions in $[0, \pi]$ only if $0 < b < 1$.

The root $a = \dfrac{k - \sqrt{k^2 + 24}}{6} < 0$ so will not contribute any solutions in the interval $[0, \pi]$.

Thus we just require that

$$0 < \frac{k + \sqrt{k^2 + 24}}{6} < 1.$$

Solving:

$$k + \sqrt{k^2 + 24} < 6$$
$$\sqrt{k^2 + 24} < 6 - k$$
$$k^2 + 24 < (6 - k)^2 = 36 - 12k + k^2$$
$$24 < 36 - 12k$$
$$k < 1$$

Thus our answer is $k < 1$. With technology access we can use sliders to check this!

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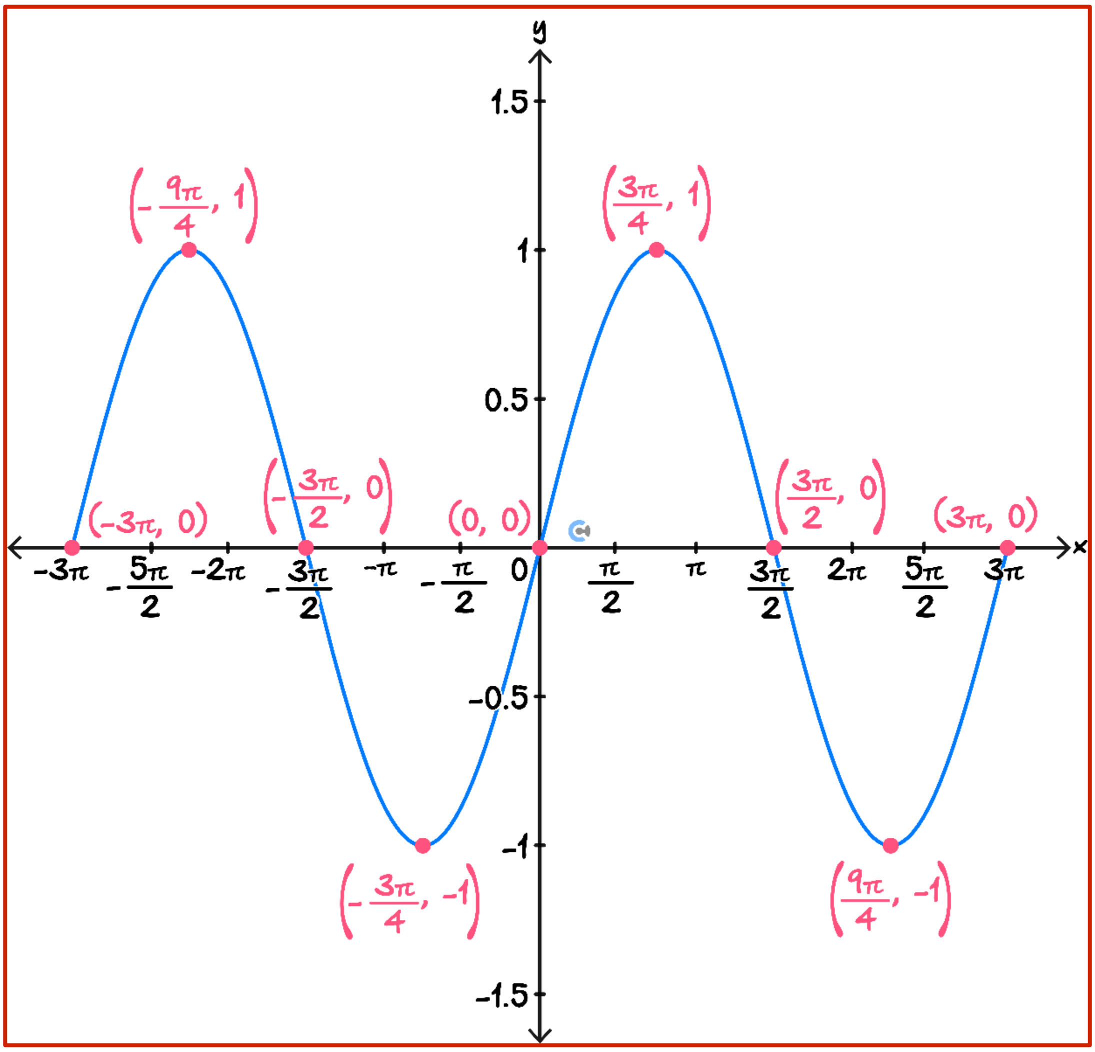
Sub-Section [3.3.2]: Graph Sine, Cosine, and Tangent Functions

Question 16

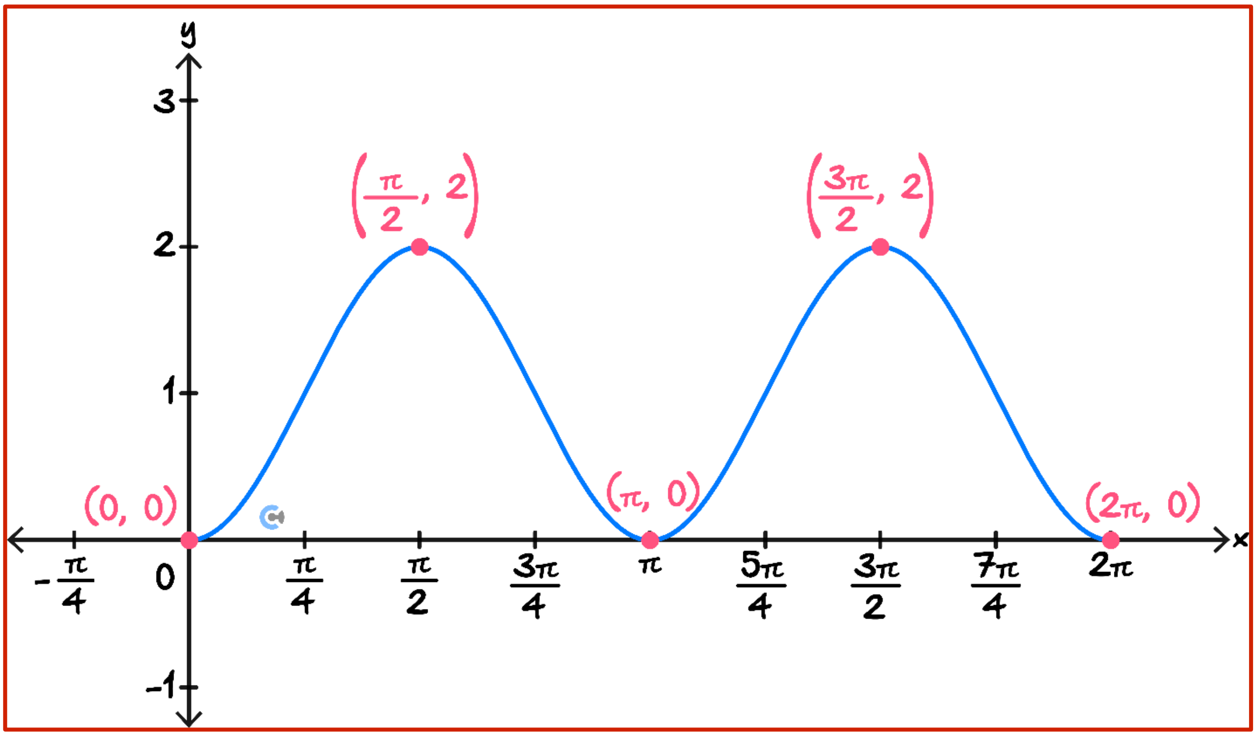
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts, turning points and endpoints with their coordinates, and asymptotes with their equations.

a. $y = \sin\left(\frac{2x}{3}\right)$ for $-3\pi \leq x \leq 3\pi$.



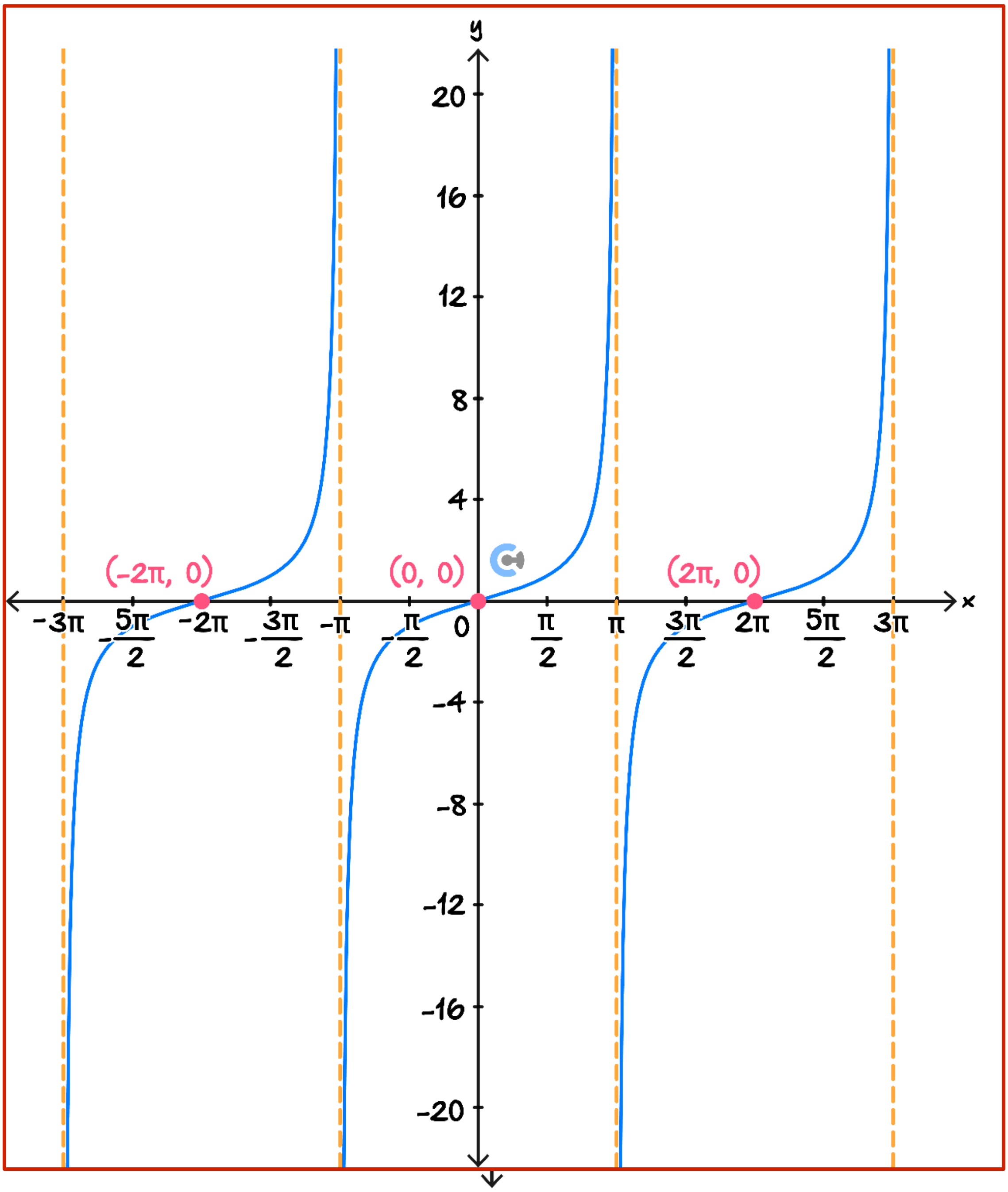
$\frac{2x}{3} = 0 \implies x = 0, \frac{3\pi}{2}$. Period $= 3\pi$.

b. $y = -\cos(2x) + 1$ for $0 \leq x \leq 2\pi$.



$2x = 0 \implies x = 0$ is base angle. Period $= \pi$.

c. $y = \tan\left(\frac{x}{2}\right)$ for $-3\pi \leq x \leq 3\pi$.

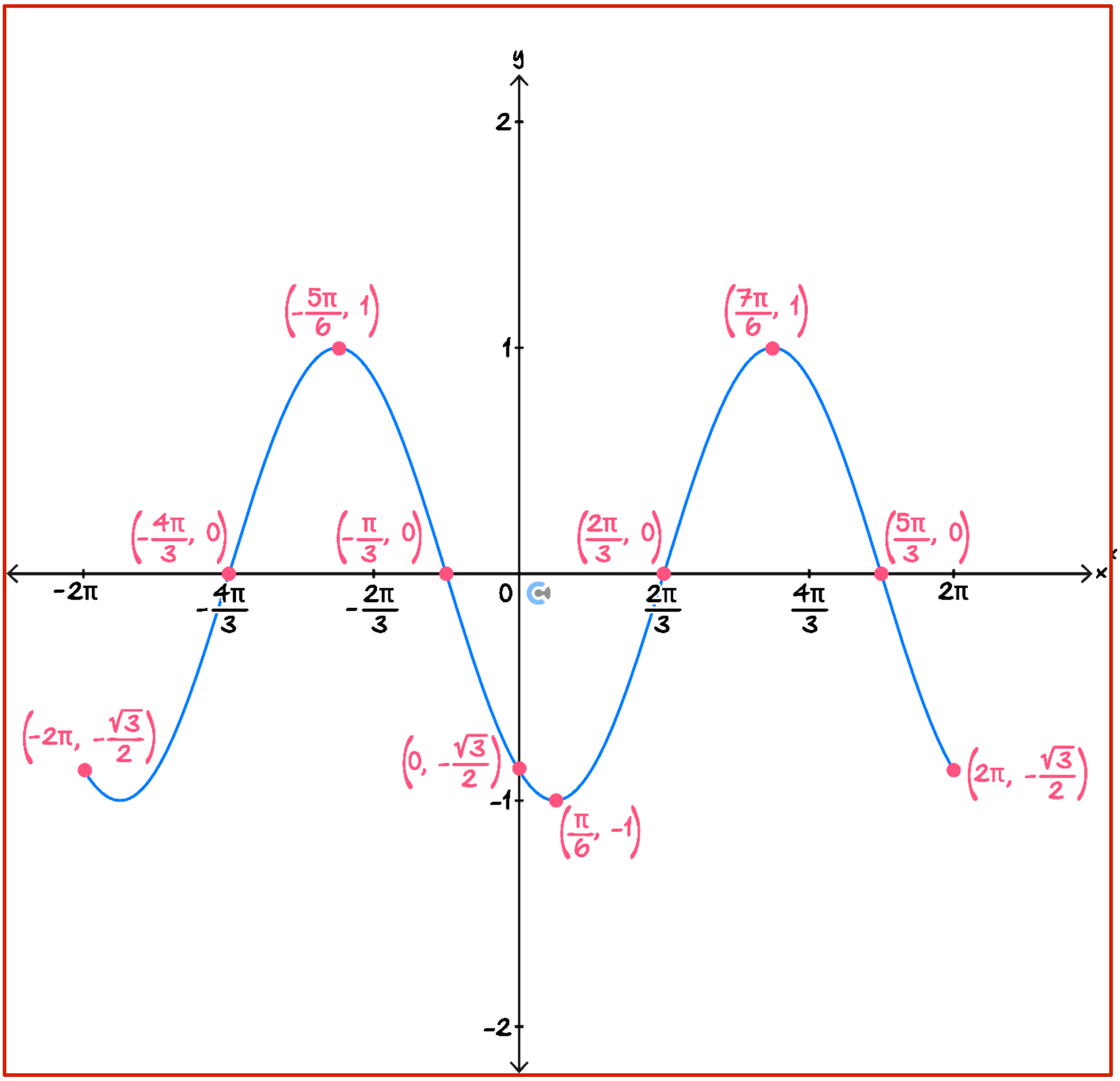


Period $= 2\pi$. Asymptote base angle π, x-intercept base angle 0.

Question 17

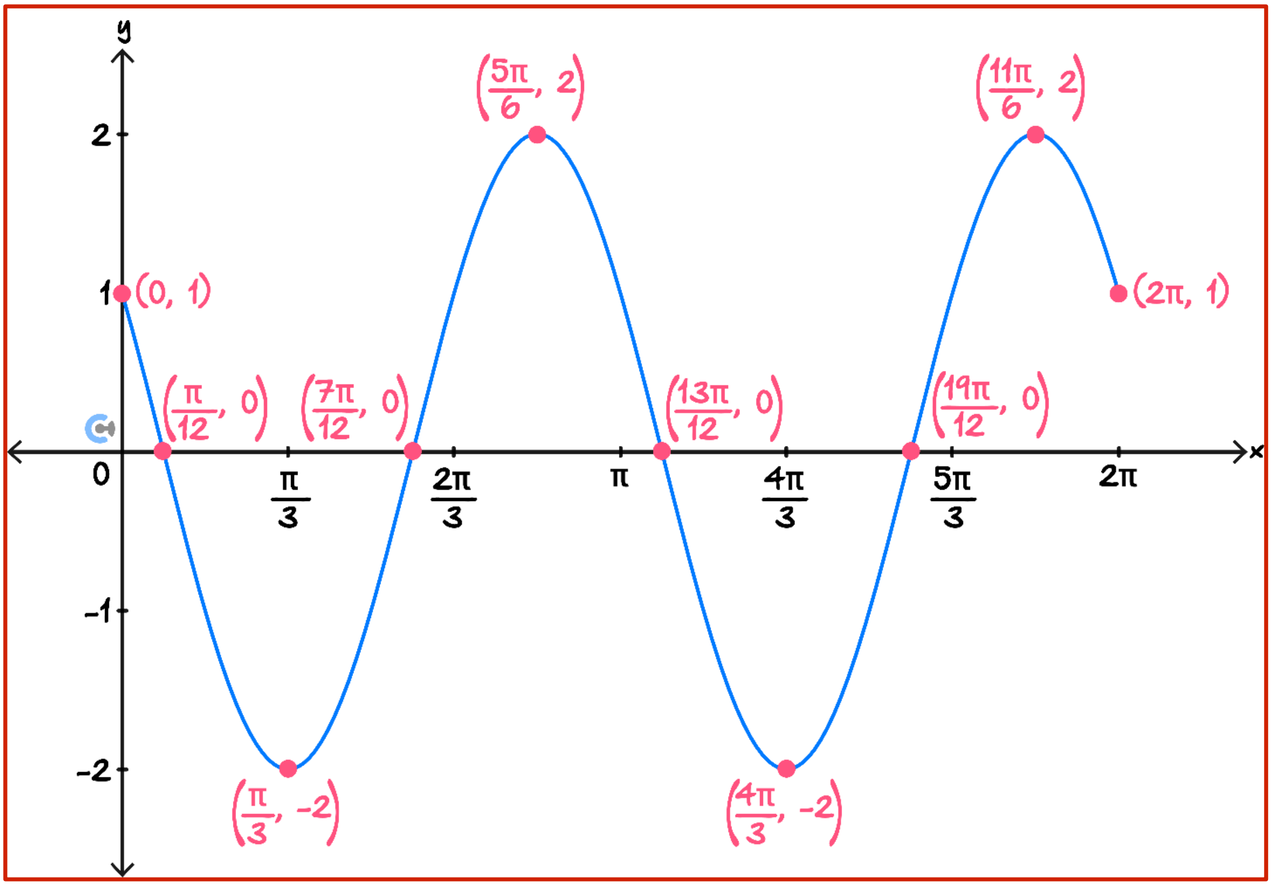
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts, turning points and endpoints with their coordinates and asymptotes with their equations.

a. $y = -\sin\left(x + \frac{\pi}{3}\right)$ for $-2\pi \le x \le 2\pi$.



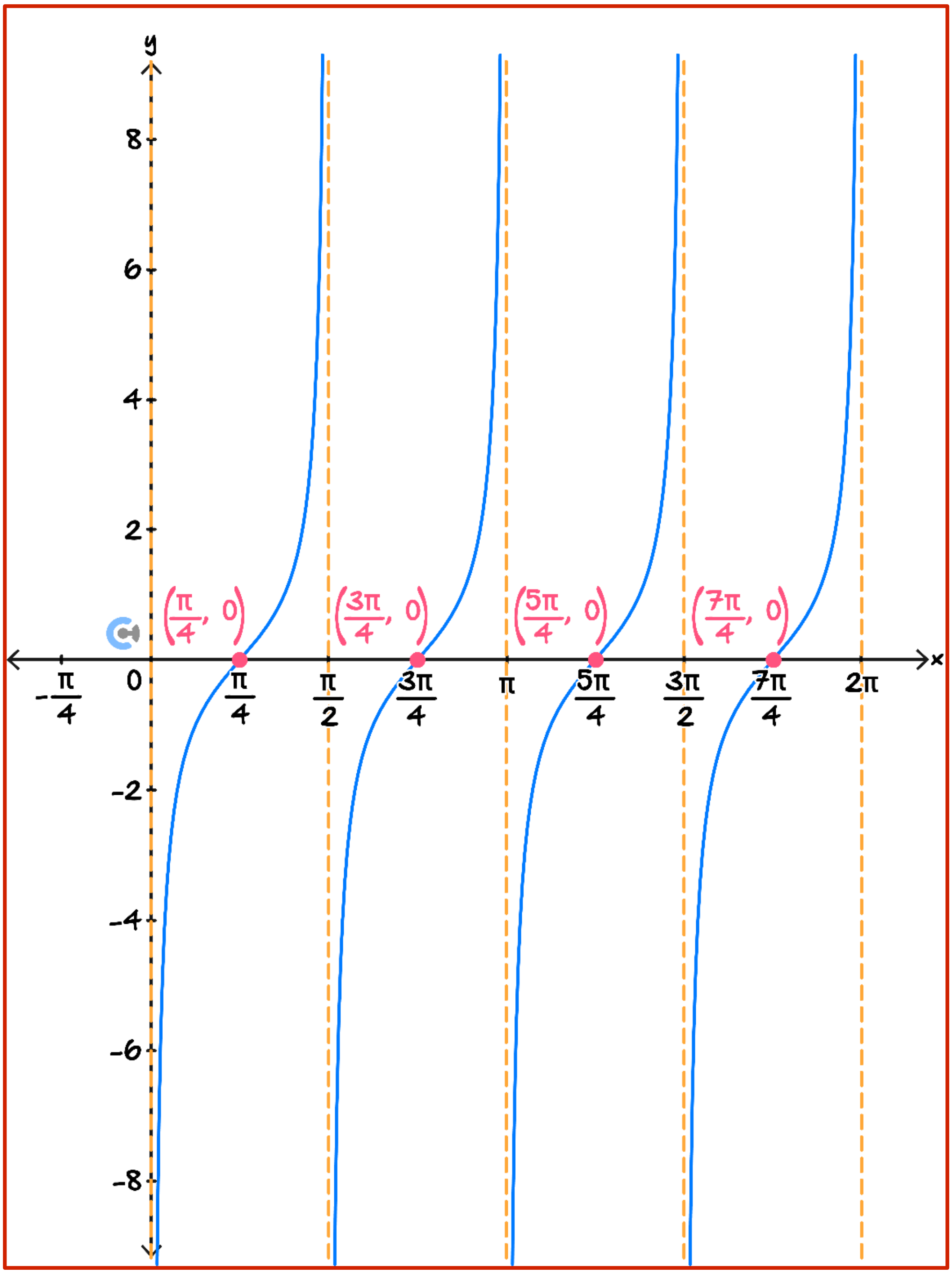
$x = -\frac{\pi}{3}, \frac{2\pi}{3}$ base angles for x-intercepts. Period $= 2\pi$.

b. $y = 2\cos\left(2\left(x + \frac{\pi}{6}\right)\right)$ for $0 \leq x \leq 2\pi$.



$2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2} \implies x = \frac{\pi}{12}, \frac{7\pi}{12}$. Period $= \pi$.

c. $y = \tan\left(2x - \frac{\pi}{2}\right)$ for $0 \leq x \leq 2\pi$.

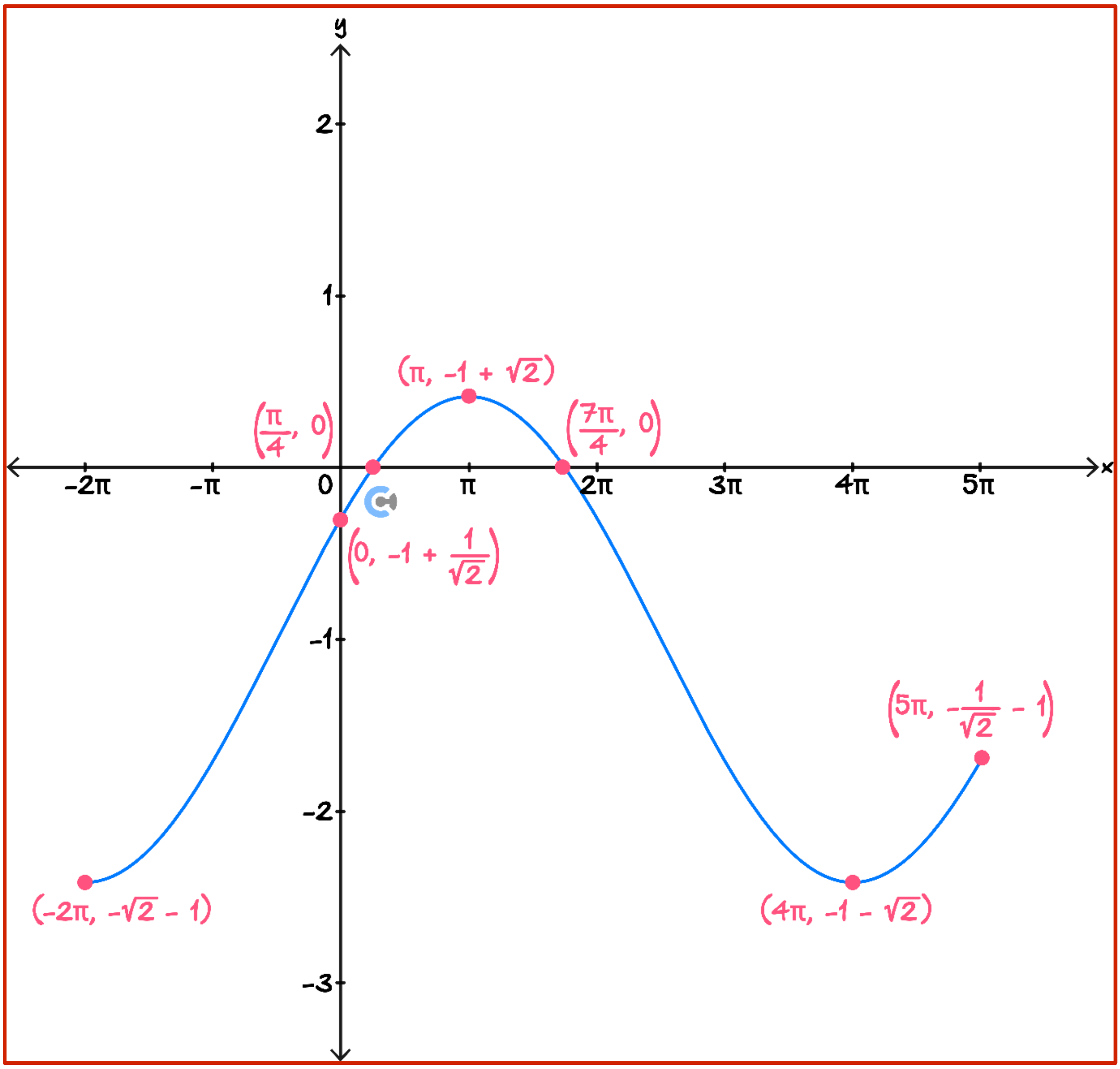


Period $= \frac{\pi}{2}$. Base angle asymptote $= 0$. Base angle x-intercept $= \frac{\pi}{4}$.

Question 18

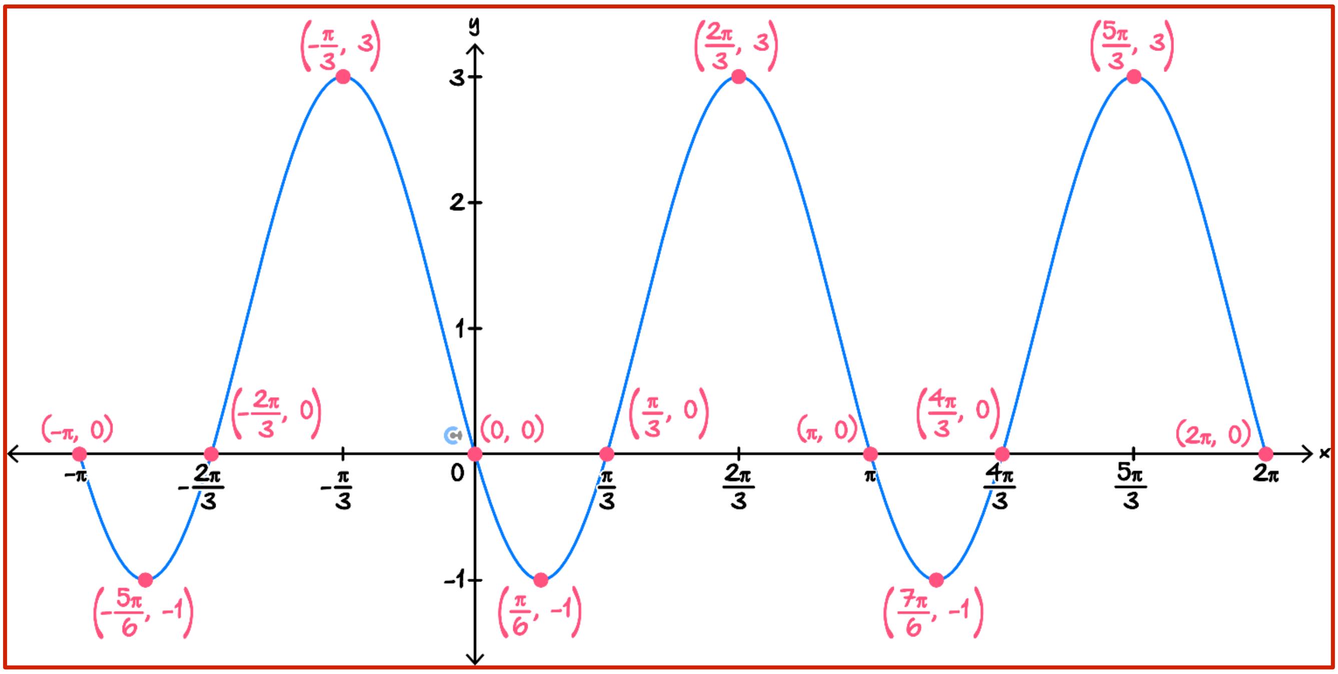
Sketch the graphs of the functions over the specified domain on the given axes. Label all axes intercepts and endpoints with their coordinates and asymptotes with their equation.

a. $y = \sqrt{2}\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) - 1$ for $-2\pi \le x \le 5\pi$.



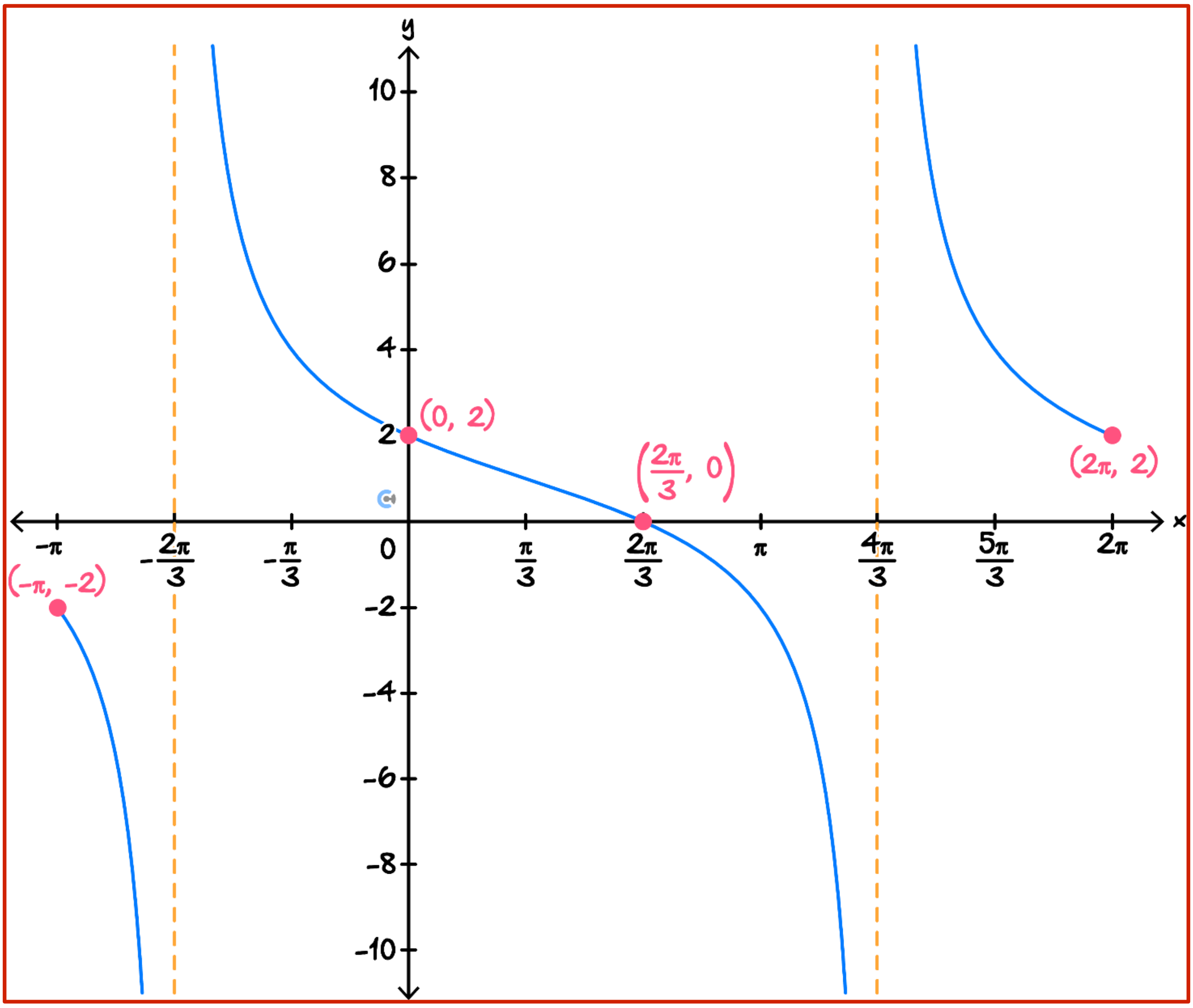
$$\frac{x}{3} + \frac{\pi}{6} = \frac{\pi}{4}, \frac{3\pi}{4} \implies x = \frac{\pi}{4}, \frac{7\pi}{4}$$

b. $y = -2\cos\left(-2x + \frac{\pi}{3}\right) + 1$ for $-\pi \le x \le 2\pi$.



$$-2x + \frac{\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3} \implies x = -\frac{\pi}{6}, -\frac{\pi}{2} \text{ base angles. Period} = \pi.$$

c. $y = -\sqrt{3}\tan\left(\frac{x}{2} - \frac{\pi}{6}\right)$ for $-\pi \leq x \leq 2\pi$.



Period $= 2\pi$. Asymptotes $x = -\frac{2\pi}{3} + 2n\pi$. x-intercepts $\frac{2\pi}{3} + 2n\pi$.

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Sub-Section [3.3.3]: Fraction of Periods

Question 19

The temperature inside a storage container over a 24-hour period is modelled by

$$T(t) = 15 + 6\cos\left(\frac{\pi}{12}t\right),$$

Where $T(t)$ is in degrees Celsius and t is the number of hours since midnight.

Find the fraction of the day during which the temperature exceeds 18°C.

We solve:

$$15 + 6\cos\left(\frac{\pi}{12}t\right) > 18$$
$$\cos\left(\frac{\pi}{12}t\right) > \frac{1}{2}$$

Solve the equation:

$$\frac{\pi}{12}t = \pm\frac{\pi}{3} + 2n\pi \Rightarrow t = \pm 4 + 24n$$

The period is:

$$T = \frac{2\pi}{\frac{\pi}{12}} = 24$$

So within one day, we take:

$$0 < t < 4 \quad \text{and} \quad 20 < t < 24$$

Total time above 18°C is $4 + 4 = 8$ hours. Thus, the fraction is:

$$\frac{8}{24} = \frac{1}{3}$$

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Question 20 Tech-Active.

The depth of a river fluctuates due to tides and is modelled by:

$$D(t) = 10 + 2.5\sin\left(\frac{\pi}{6}t\right),$$

where $D(t)$ is the depth in metres and t is the time in hours since midnight.

a. State the maximum and minimum river depth.

$D_{\text{max}} = 10 + 2.5(1) = 12.5, \quad D_{\text{min}} = 10 + 2.5(-1) = 7.5$

b. Find the first two times after midnight the depth reaches exactly 11 metres. Give answers as hours after midnight correct to two decimal places.

Solve $D(t) = 11$ using CAS.
$t \approx 0.79, 5.21$

c. Determine the percentage of a full tidal cycle during which the depth is greater than 12 metres.

Solve $D(t) = 12$ and find time above this value using a graph.
Let duration above 12 be d, period is 12.
Solve $D(t) = 12 \implies t = 1.77, 4.22 \implies d \approx 4.22 - 1.77 = 2.458$

$$\text{Percentage} = \frac{d}{12} \times 100 \approx 20.5\%$$

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Question 21 Tech-Active.

The brightness inside a room with automated skylights varies during the day and is modelled by

$$B(t) = 300 + 100\sin\left(\frac{\pi}{12}t\right),$$

where $B(t)$ is the brightness in lumens and t is the number of hours since midnight.

a. Find the fraction of a full day during which the brightness exceeds 350 lumens.

Solve $B(t) = 350 \Rightarrow t = 2, 10$
Fraction of day:

$$\frac{10-2}{24} = \frac{1}{3}$$

b. Find the value of k such that $B(t) + k$ exceeds 400 lumens for exactly 30% of the time. Give your answer to two decimal places.

Solve $B(t) + k = 400$, find duration d where we are above 400 by subtracting two adjacent solutions and taking the magnitude.
Set $\frac{d}{24} = 0.3$, solve for k.
$k \approx 41.22$. It is quite simple to then verify your solution by using this value of k and finding the fraction above 400.

```
In[47]:= b[t_] := 300 + 100 Sin[Pi * t / 12]

In[52]:= Solve[b[t] + k == 400, t]

Out[52]= {{t -> 12 (π - ArcSin[(100-k)/100] + 2 π c1)/π  if c1 ∈ Z},
          {t -> 12 (ArcSin[(100-k)/100] + 2 π c1)/π  if c1 ∈ Z}}

In[53]:= Abs[12 (π - ArcSin[(100-k)/100] + 2 π c1)/π - 12 (ArcSin[(100-k)/100] + 2 π c1)/π] //
          Simplify

Out[53]= Abs[12 - 24 ArcSin[1 - k/100]/π]

In[54]:= Solve[Abs[12 - 24 ArcSin[1 - k/100]/π]/24 == 0.3, k]

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use
Reduce for complete solution information.

Out[54]= {{k -> 41.2215}, {k -> 41.2215}}
```

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