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VCE Mathematical Methods ¾
Circular Functions I [3.2]
Workbook

Outline:

Introduction to Circular Functions	Pg 2-15	**Symmetry**	Pg 16-28
➤ Radians and Degrees		➤ Supplementary Relationships	
➤ Unit Circle		➤ Complementary Relationships	
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➤ Pythagorean Identities		➤ Particular Solutions	
➤ Exact Values		➤ General Solutions	

Learning Objectives:

- ☐ **MM34 [3.2.1]** - Find Exact Values of Circular Functions
- ☐ **MM34 [3.2.2]** - Apply Pythagorean Identity and Symmetrical Properties
- ☐ **MM34 [3.2.3]** - Find Particular and General Solutions

Section A: Introduction to Circular Functions

Sub-Section: Radians and Degrees

Radians and Degrees

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$1^o = \left(\frac{\pi}{180}\right)^c$$

$$180^o = \pi^c$$

Question 1

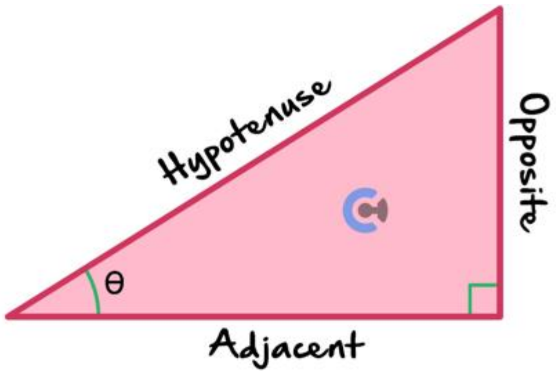
a. Find $\left(\frac{\pi}{4}\right)^c$ in degrees.

b. Find 12^o in radians.

Sub-Section: Unit Circle

REMINDER:

- SOHCAHTOA



$$\sin = opposite/hypotenuse$$

$$\cos = adjacent/hypotenuse$$

$$= opposite/adjacent$$

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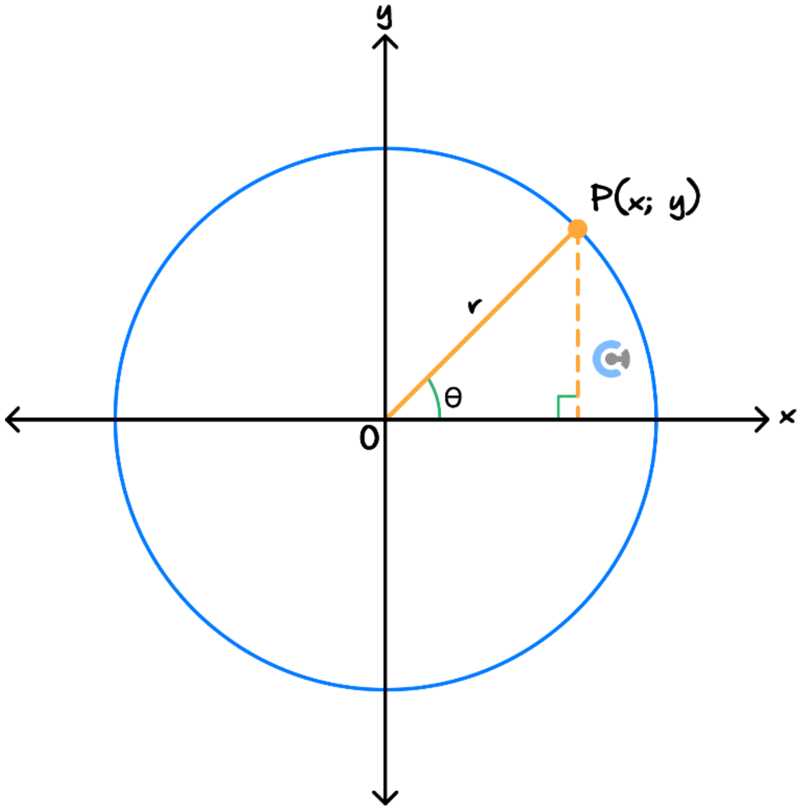
What is a Unit Circle and How Do We Use It?

<u>Exploration</u>: Unit Circle

- The unit circle is simply a circle of radius ____________.
- Angles are measured from the ________________ direction of the x-axis in an _____________________ direction.
- It can be divided into **four quadrants:**



- We can use SOHCAHTOA to define $\sin, \cos$ and $\tan$ on the unit circle.

$$\sin(\theta) = [X\ value, Y\ value\ , Gradient]$$

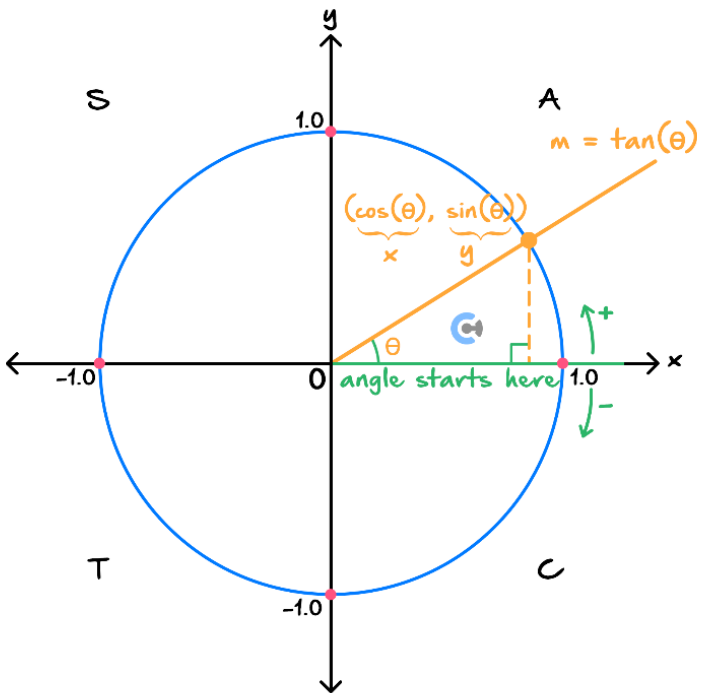
$$\cos(\theta) = [X\ value, Y\ value, Gradient]$$

$$\tan(\theta) = [X\ value, Y\ value, Gradient]$$

Discussion: Using what we have defined above, for which quadrant is $\cos$, $\sin$ and tangent positive?

Unit Circle

- The unit circle is simply a circle of radius 1.



$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = gradient$$

Discussion: How much does the angle need to change for $\cos$ and $\sin$ to repeat its value?

Discussion: How much does the angle need to change for the tangent to repeat its value?

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Sub-Section: Period

Period of a Trigonometric Function

$$\textit{period of } \sin(nx) \textit{ and } \cos(nx) \textit{ functions} = \frac{2\pi}{n}$$

$$\textit{period of } \tan(nx) \textit{ functions} = \frac{\pi}{n}$$

$$\textit{where } n = \textit{coefficient of } x$$

Discussion: Why do we divide by n? Consider $\cos(2x)$. Why is the period only π instead of 2π?

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Question 2

Find the period of each of the following trigonometric functions:

a. $p(x) = \tan\left(\frac{x}{2}\right)$.

b. $q(x) = \cos\left(\frac{3}{2}x + \frac{\pi}{3}\right)$.

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Question 3 Extension.

Find the period of $f(x) = \tan(3x) + \cos^2(2x)$.

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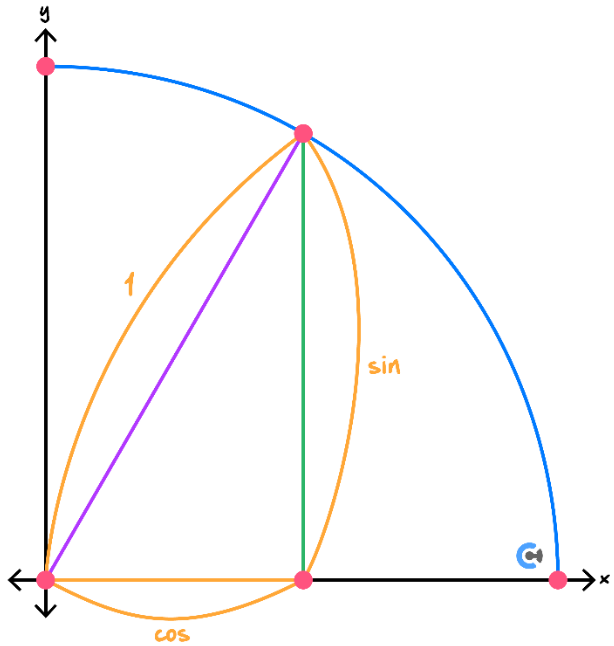
Sub-Section: Pythagorean Identities

Discussion: What does $\sin^2(\theta) + \cos^2(\theta)$ equal to?

Pythagorean Identities



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

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Question 4 Walkthrough.

Find the value of cos (x) given that $\sin(x) = -\frac{2}{3}$ and x is 2^{rd} quadrant.

Question 5

Find the value of $\tan(x)$ given that $\cos(x) = -\frac{1}{3}$ and x is in the 2^{nd} quadrant.

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Question 6 Extension.

Find the value of $\sin(x)\cos(x)$ given that $\tan(x) = \frac{3}{4}$.

NOTE: Consider the quadrant to determine signs.

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Sub-Section: Exact Values

Let's quickly recall the exact values!

The Exact Values Table

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

TIP: Use the fact that $\sin$ is the y-value, $\cos$ is the - value and the tangent is the gradient to remember the values well!

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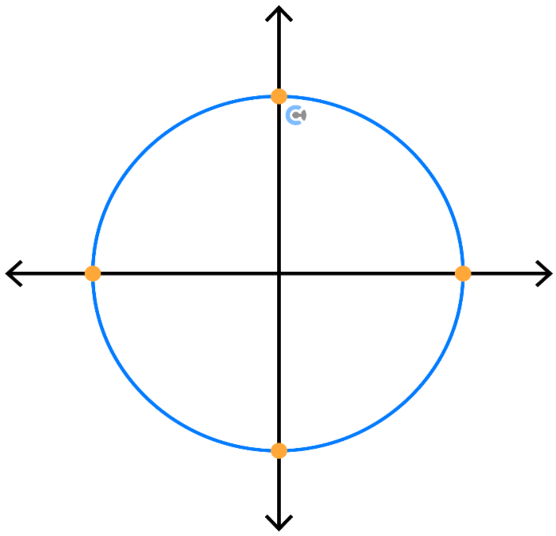
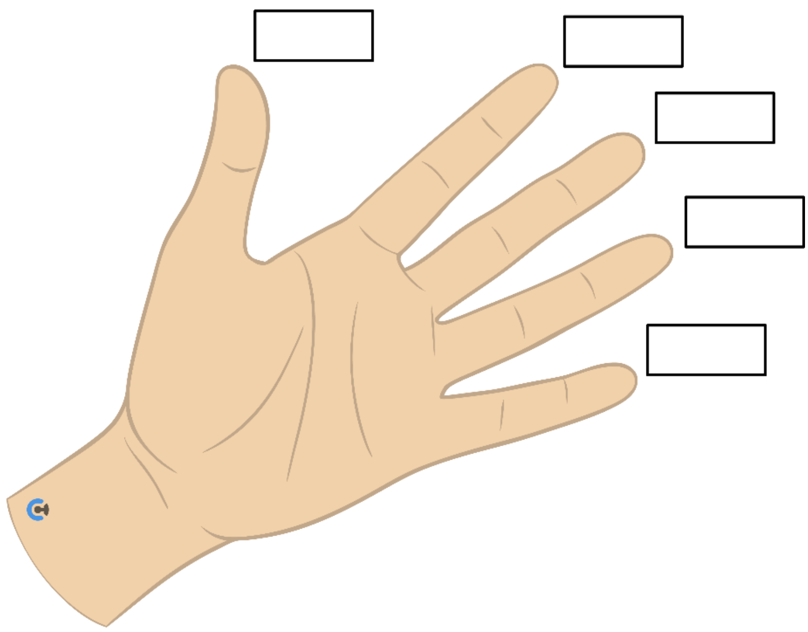
Is there a quick method to remember the exact values?

Exploration: Exact values

- Hold your left palm up facing yourself.
- Imagine the thumb to align with the positive y-axis, so the thumb is ________.
- Imagine the pinky to align with the positive x-axis, so the pinky is _______.
- Can you guess what angles the index finger, middle finger and ring finger represent?
- Label the angles on a unit circle as well as the finger.





$$\sin(\theta) = \frac{\sqrt{(the\ number\ of\ fingers\ below)}}{2}$$

$$\cos(\theta) = \frac{\sqrt{(the\ number\ of\ fingers\ above)}}{2}$$

<u>Active Recall:</u> The Exact Values Table

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$					
$\cos(x)$					
$\tan(x)$					

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Section B: Symmetry

Sub-Section: Supplementary Relationships

Exploration: Supplementary Relationships

- Scan the following QR code on your device and complete the identities below.
  - **Second Quadrant:** _______________.
    - Reflection around _______________-axis.

$$\cos(\pi - \theta) = +/- \cos(\theta)$$

$$\sin(\pi - \theta) = +/- \sin(\theta)$$

$$\tan(\pi - \theta) = +/- \tan(\theta)$$

  - **Third Quadrant:** _______________.
    - Reflection around _______________-axis.

$$\cos(\pi + \theta) = +/- \cos(\theta)$$

$$\sin(\pi + \theta) = +/- \sin(\theta)$$

$$\tan(\pi + \theta) = +/- \tan(\theta)$$

- **Fourth Quadrant:** ______________.
  - Reflection around ______________-axis.

$$\cos(-\theta) = +/- \cos(\theta)$$

$$\sin(-\theta) = +/- \sin(\theta)$$

$$\tan(-\theta) = +/- \tan(\theta)$$

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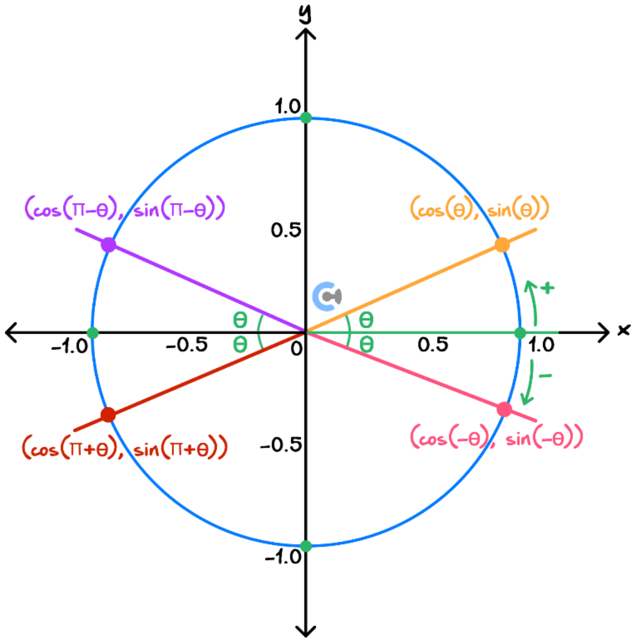
Let's Summarise

Definition

Supplementary Relationships



- Look at the quadrant to find the correct sign.
  - **Second Quadrant** $(\pi - \theta)$:
    - Reflection around y-axis

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\sin(\pi - \theta) = +\sin(\theta)$$

$$\tan(\pi - \theta) = -\tan(\theta)$$

  - **Third Quadrant** $(\pi + \theta)$:
    - Reflection around x- and y-axis

$$\cos(\pi + \theta) = -\cos(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta)$$

$$\tan(\pi + \theta) = +\tan(\theta)$$

- **Fourth Quadrant** $(-\boldsymbol{\theta})$:
  - Reflection around x-axis.

$$\cos(-\boldsymbol{\theta}) = +\cos(\boldsymbol{\theta})$$

$$\sin(-\boldsymbol{\theta}) = -\sin(\boldsymbol{\theta})$$

$$\tan(-\boldsymbol{\theta}) = -\tan(\boldsymbol{\theta})$$

- **Steps:**

1. Equate to $\cos/\sin/\tan(\theta)$.
2. Determine the sign $(\pm)$ by considering the quadrant.

Question 7 Walkthrough.

If $\sin(x) = \frac{5}{13}$, where x is a first quadrant angle, evaluate the following:

a. $\sin(\pi - x)$.

b. $\cos(\pi + x)$.

c. $\tan(\pi + x)$.

Try the Following Question!

Question 8

If $\sin(\theta) = -0.6$ where θ is a third quadrant angle, evaluate the following:

a. $\sin(\pi + \theta)$.

b. $\cos(\pi + \theta)$.

c. $\tan(\pi - \theta)$.

Question 9

If $\cos(x) = \frac{2}{3}$, evaluate $2\cos^2(x) + \tan^2(x) + \sin^2(x)$.

Sub-Section: Complementary Relationships

What happens to the angle if you reflect it around $y = x$?

Exploration: Complementary Relationships

- Complementary relationships involve reflection around ____________.
- Implying that we are finding the ____________.
- Given that $x = \cos$, $y = \sin$, $gradient = \tan$.
- What would happen to $\cos/\sin/\tan$ when we have a complementary relationship?

$$\cos \to \_\_\_\_\_\_\_\_\_\_$$

$$\sin \to \_\_\_\_\_\_\_\_\_\_$$

$$\tan \to \_\_\_\_\_\_\_\_\_\_$$

- Scan the following QR code on your device and complete the identities below.
  - First Quadrant ____________

$$\cos\left(\frac{\pi}{2} - \theta\right) = +/- \sin(\theta)$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=+/-\cos(\theta)$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=+/-\frac{1}{\tan(\theta)}$$

- Second Quadrant ____________

$$\cos\left(\frac{\pi}{2}+\theta\right)=+/-\sin(\theta)$$

$$\sin\left(\frac{\pi}{2}+\theta\right)=+/-\cos(\theta)$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=+/-\frac{1}{tan(\theta)}$$

- Third Quadrant ____________

$$\cos\left(\frac{3\pi}{2}-\theta\right) = +/- \sin(\theta)$$

$$\sin\left(\frac{3\pi}{2}-\theta\right) = +/- \cos(\theta)$$

$$\tan\left(\frac{3\pi}{2}-\theta\right) = +/- \frac{1}{tan(\theta)}$$

- Fourth Quadrant ____________

$$\cos\left(\frac{3\pi}{2}+\theta\right) = +/- \sin(\theta)$$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = +/- \cos(\theta)$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = +/- \frac{1}{tan(\theta)}$$

<u>Discussion:</u> What do all the complementary relationship angles start from?

Let's summarise!

Definition

Complementary Relationships

- First Quadrant $\left(\frac{\pi}{2}-\theta\right)$

$$\cos\left(\frac{\pi}{2}-\theta\right)=+\sin(\theta)$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos(\theta)$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=+\frac{1}{\tan(\theta)}$$

- Second Quadrant $\left(\frac{\pi}{2}+\theta\right)$

$$\sin\left(\frac{\pi}{2}+\theta\right)=+\cos(\theta)$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin(\theta)$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=-\frac{1}{\tan(\theta)}$$

- Third Quadrant $\left(\frac{3\pi}{2}-\theta\right)$

$$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2}-\theta\right)=\frac{1}{\tan(\theta)}$$

- Fourth Quadrant $\left(\frac{3\pi}{2}+\theta\right)$

$$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2}+\theta\right) = +\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2}+\theta\right) = -\frac{1}{\tan(\theta)}$$

- **Steps:**

1. Note the complementary relationship by identifying a vertical angle $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
2. Equate to the opposite trigonometric function $\cos/\sin/\frac{1}{\tan(\theta)}$.
3. Determine the sign $(\pm)$ by considering the quadrant.

NOTE: Steps are identical to the supplementary relationship.

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Question 10 Walkthrough.

If $\sin(\alpha) = \frac{1}{4}$ and $\cos(\beta) = -\frac{2}{3}$ where α, β are second quadrant angles, evaluate the following.

a. $\cos\left(\frac{\pi}{2} - \alpha\right)$

b. $\sin\left(\frac{\pi}{2} + \beta\right)$

c. $\sin\left(\frac{3\pi}{2} - \alpha\right) - \cos\left(\frac{3\pi}{2} + \beta\right)$

Question 11

If $\sin(\alpha) = 0.6$ and $\cos(\beta) = 0.4$ where α, β are first quadrant angles, evaluate the following.

a. $\cos\left(\frac{\pi}{2} + \alpha\right)$

b. $\sin\left(\frac{\pi}{2} - \alpha\right)$

c. $\sin\left(\frac{3\pi}{2} + \alpha\right) - \cos\left(\frac{3\pi}{2} - \beta\right)$

Question 12 Extension.

Consider solving the equation:

$$\sin(x) + \cos(y) = \frac{\tan(x)}{\tan(y)}$$

Where x and y are such that $x + y = \frac{\pi}{2}$.

Find an equation in terms of $\sin(x)$ only, that can be solved.

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Section C: Particular and General Solutions

Sub-Section: Particular Solutions

Active Recall: Period of Trigonometric Function

$$period\ of\ \sin(nx)\ and\ \cos(nx)\ functions = \_\_\_\_\_\_\_\_\_\_$$

$$period\ of\ \tan(nx)\ functions = \_\_\_\_\_\_\_\_\_\_$$

$$where\ n = coefficient\ of\ x$$

Discussion: How often would the solution to $\sin(x) = \frac{1}{2}$ repeat?

Particular Solutions

- Solving trigonometric equations **for finite solutions.**
- **Steps:**
  1. Make the trigonometric function the subject.
  2. Find the necessary angle for one period.
  3. Solve for x by equating the necessary angles to the inside of the trigonometric functions.
  4. Add and subtract the period to find all other solutions in the domain.

Question 13 Walkthrough.

Solve the following equations for x over the domains specified.

$$2\sin(2x) + \sqrt{3} = 0 \text{ for } x \in [0, 2\pi]$$

<u>Active Recall:</u> **Particular Solutions**

- Solving trigonometric equations **for finite solutions.**
- **Steps:**
  1. Make the trigonometric function the ____________.
  2. Find the necessary ____________ for one period.
  3. Solve for x by equating the necessary angles to the ____________ of the trigonometric functions.
  4. Add and subtract the ____________ to find all other solutions in the domain.

Question 14

Solve the following equations for x over the domains specified.

a. $\sin(4x) = -1$ for $x \in [-\pi, \pi]$

b. $2\cos\left(2x - \frac{\pi}{2}\right) + 1 = 0$ for $x \in [0, 2\pi]$

Question 15 Walkthrough.

Solve the following equations for x over the domains specified.

$$2\tan(2x+\pi)+2\sqrt{3}=0 \text{ for } x \in [0, 2\pi]$$

<u>Discussion:</u> **Why do we need to find one angle only for tangents?**

Question 16

Solve the following equations for x over the domains specified.

$$\sqrt{3}\tan\left(x-\frac{\pi}{3}\right)-1=0 \text{ for } x\in(0,3\pi)$$

Question 17 Extension.

Solve the following equation:

$$3\sin^2(2x) + \cos^2(2x) = 3\sin(2x), \text{for } x \in [-\pi, \pi]$$

Sub-Section: General Solutions

Discussion: How many solutions would there be for $x \in R$?

General Solutions

- Finding ______________________ solutions to a trigonometric equation.
- **Steps:**

1. Make the trigonometric function the ____________.
2. Find the necessary ____________ for one period.
3. Solve for x by equating the necessary angles to the ____________ of the trigonometric functions.
4. Add _______________ where $n \in Z$.

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Question 18 Walkthrough.

Find the general solutions to the following equations:

$$2\sin\left(2x+\frac{\pi}{2}\right)-1=0$$

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Active Recall: General Solutions

- **Steps:**

1. Make the trigonometric function the ___________.
2. Find the necessary ___________ for one period.
3. Solve for x by equating the necessary angles to the ___________ of the trigonometric functions.
4. Add ______________ where $n \in Z$.

Question 19

Find the general solutions to the following equations:

$$-2\sin\left(3x + \frac{\pi}{2}\right) = \sqrt{2}$$

Question 20 Extension.

Find the general solution to the equation:

$$4\cos^2(x) - 3\sin(x) = 2$$

Give an exact answer that includes the use of the $\sin^{-1}(\ldots)$ function.

Question 21 Walkthrough.

Find the general solutions to the following equations:

$$\tan\left(\frac{1}{2}x - \pi\right) - \frac{1}{\sqrt{3}} = 0$$

NOTE: We only need to find one angle for tangents!

Question 22

Find the general solutions to the following equations:

$$\sqrt{3} - \tan\left(3\left(x + \frac{\pi}{3}\right)\right) = 0$$

Question 23 Extension.

How many solutions does the equation $\tan(2x) = \sqrt{x}$ have for $x \in [0, 50\pi]$?

Contour Check

Learning Objective: [3.2.1] - Find exact values of circular functions

Key Takeaways

- The Exact Values Table:

x	$0\ (0^o)$	$\frac{\pi}{6}\ (30^o)$	$\frac{\pi}{4}\ (45^o)$	$\frac{\pi}{3}\ (60^o)$	$\frac{\pi}{2}\ (90^o)$
$\sin(x)$					
$\cos(x)$					
$\tan(x)$					

Learning Objective: [3.2.2] - Apply Pythagorean identity and symmetrical properties

Key Takeaways

- **Pythagorean Identity:**

$$\sin^2(\theta) + \cos^2(\theta) = \_\_\_\_\_\_\_\_$$

- **Supplementary Relationships:**
- Look at the quadrant to find the correct sign.
  - **Second Quadrant** $(\pi - \theta)$
    - Reflection around y-axis.

$$\cos(\pi - \theta) = \_\_\_\_\_\_\_\_$$

$$\sin(\pi - \theta) = \_\_\_\_\_\_\_\_$$

$$\tan(\pi - \theta) = \_\_\_\_\_\_\_\_$$

  - **Third Quadrant** $(\pi + \theta)$
    - Reflection around x and y-axis.

$$\cos(\pi + \theta) = \_\_\_\_\_\_\_\_$$

$$\sin(\pi + \theta) = \_\_\_\_\_\_\_\_$$

$$\tan(\pi + \theta) = \_\_\_\_\_\_\_\_$$

- **Fourth Quadrant** $(-\theta)$
  - Reflection around x-axis.

$$\cos(-\theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\sin(-\theta) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan(-\theta) = \_\_\_\_\_\_\_\_\_\_$$

- **Steps**
  1. Equate to $\cos/\sin/\tan(\theta)$.
  2. Determine the sign $(\pm)$ by considering the quadrant.

- **Complementary Relationships:**
  - **First Quadrant** $\left(\frac{\pi}{2} - \theta\right)$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

  - **Second Quadrant** $\left(\frac{\pi}{2} + \theta\right)$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

- Third Quadrant $\left(\frac{3\pi}{2} - \theta\right)$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\cos\left(\frac{3\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

- Fourth Quadrant $\left(\frac{3\pi}{2} + \theta\right)$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\cos\left(\frac{3\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = \_\_\_\_\_\_\_\_\_\_$$

- Steps:

1. Note complementary relationship by identifying a vertical angle $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
2. Equate to the opposite trigonometric function $\cos/\sin/\ \frac{1}{\tan(\theta)}$.
3. Determine the sign $(\pm)$ by considering the quadrant.

☐ Learning Objective: [3.2.3] - Find particular and general solutions

Key Takeaways

☐ **Particular Solutions:**

- Solving trigonometric equations **for finite solutions.**
- **Steps:**
  1. Make the trigonometric function the ____________.
  2. Find the necessary ____________ for one period.
  3. Solve for x by equating the necessary angles to the ____________ of the trigonometric functions.
  4. Add and subtract the ____________ to find all other solutions in the domain.

☐ **General Solutions:**

- Finding ______________________ solutions to a trigonometric equation.
- **Steps:**
  1. Make the trigonometric function the ____________.
  2. Find the necessary ____________ for one period.
  3. Solve for x by equating the necessary angles to the ____________ of the trigonometric functions.
  4. Add _______________ where $n \in Z$.

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