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VCE Mathematical Methods ¾

AOS 3 Revision [3.0]

SAC 4 Solutions

58 Marks. 5 Minutes Reading. 80 Minutes Writing.

Section A: SAC Questions (58 Marks)

INSTRUCTION:

- For the purpose of this task, assume that ATAR is continuous and does not increase in increments of 0.05. For example, an ATAR of 95.0156 may exist.

Question 1 (17 marks)

Ms Gradey wants to teach her students about the importance of sleep for them to do their academic best. To do this, she models a graph of ATAR vs average nightly sleep. This model is given by the function:

$$a(t) = 30e^{\frac{t-6}{3}} + 40 \quad t \in [0, 9]$$

where t is the time in hours that students slept on average per night, and $a(t)$ is a student's ATAR.

a. State the transformations applied to the graph of e^t to turn it into the graph of $a(t)$. (4 marks)

Dilation factor 30 from t-axis.
Dilation of factor 3 from the y-axis.
Translation of 6 units in positive t direction.
Translation of 40 units in positive y direction.
1A each

b. For how many hours must a student sleep on average per night to get an ATAR of 99.95? (2 marks)

$a(t) = 99.95$ **1M**
$t = 3(\ln(1199) - \ln(600) + 2)$ **1A**

Ms Gradey believes that, as per her model, the hours of sleep at which a student's performance begins to pick up is when the rate of change of ATAR per hour of sleep is equal to 10.

c. Show that $a'(t) = 10e^{\frac{t}{3}-2}$. (2 marks)

$$a'(t) = 30 \cdot \frac{1}{3} e^{\frac{t-6}{3}} \quad \textbf{1M}$$
$$= 10e^{\frac{t-6}{3}} \quad \textbf{1M}$$
$$= 10e^{\frac{t}{3}-2}$$

d. Hence, show that the average hours of sleep at which the gradient of $a(t)$ is equal to 10 is $t = 6$. (2 marks)

$$a'(t) = 10e^{\frac{t-6}{3}}$$
$$10e^{\frac{t-6}{3}} = 10 \quad \textbf{1M}$$
$$e^{\frac{t-6}{3}} = 1$$
$$\frac{t-6}{3} = 0 \quad \textbf{1M}$$
$$t = 6$$

However, despite this being the point where performance is substantially improved, Ms Gradey also wants to highlight to her students the power of her exponential model to show how getting even more sleep after this point gives even higher performance yields. To do this, she decides to implement a tangent.

e. Find the tangent line of $a(t)$ at $t = 6$ to model what would happen if ATAR per hour of sleep would increase at a constant rate of 10 after this point. (1 mark)

$$a'(6) = 10 \quad \textbf{1M}$$
$$y = 10t + 10 \quad \textbf{1A}$$

f. Hence, give a piece-wise function, $a_2(t)$, to show this new model in **part e.** (2 marks)

$$a_2(t) = \begin{cases} 30e^{\frac{t-6}{3}} + 40 & t \in [0, 6] \\ 10t + 10 & t \in (6, 9] \end{cases} \quad \textbf{1A rule 1A domains}$$

g. Find the average hours of sleep needed per night to achieve a 99.95 ATAR based on this model. Round your answer to the nearest integer. (2 marks)

$10t + 10 = 99.95$ **1M**

$t = 9$ **1A**

h. For someone who achieved a 99.95 ATAR, and whose sleep to performance can be modelled by $a(t)$, find the difference in ATAR between this person and someone who slept the same amount but whose performance is modelled by $a_2(t)$, correct to 3 decimal places. (2 marks)

solve$(a(t)=99.95,t)$	$t=8.07694$
Define $a2(t)=\begin{cases} a(t), & 0\le t\le 6 \\ 10\cdot t+10, & 6<t\le 9 \end{cases}$	*Done*
$a2(8.0769404994339)$ **1M**	90.7694
$99.95-90.769404994339$ **1M**	9.180595 **1A** for 9.181

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Question 2 (11 marks)

Despite her spectacular model, Ms Gradey doesn't think it'll be enough to convince her rebellious students to sleep well. She thus decides to take some *creative liberty* with her data to make the returns of sleeping well appear too good for her students to pass up. To do this, she decides to reconstruct her model with the following piecewise function:

$$h(t) = \begin{cases} 3e^{t-6} + 40 & t \in [0,6] \\ 15(t-k)^2 + l & t \in (6,9] \end{cases}$$

where t is the average sleep per night in hours, and $h(t)$ is ATAR.

a. Find the coordinate at which the function changes from an exponential to a polynomial, given that the graph of $h(t)$ is continuous. (1 mark)

$(6, 43)$ **1A**

b. Find the derivative of $h(t)$ in terms of k. (2 marks)

$$h'(t) = \begin{cases} 3e^{t-6} & t \in (0,6) \\ 30t - 30k & t \in (6,9) \end{cases}$$ **1A rule 1A domains**

To trick her students, Ms Gradey wants to ensure that the transition from the first equation of the piecewise function to the second is seamless, so the students don't notice she has changed the results to convince them (i.e., the graph of $h(t)$ is smooth).

c. Find the value of k and l such that the graph of $h(t)$ is smooth everywhere. (4 marks)

Define $h1(t)=3\cdot e^{t-6}+40$ *Done*

Define $h2(t)=15\cdot(t-k)^2+l$ *Done*

Define $dh1(t)=\frac{d}{dt}(h1(t))$ *Done*

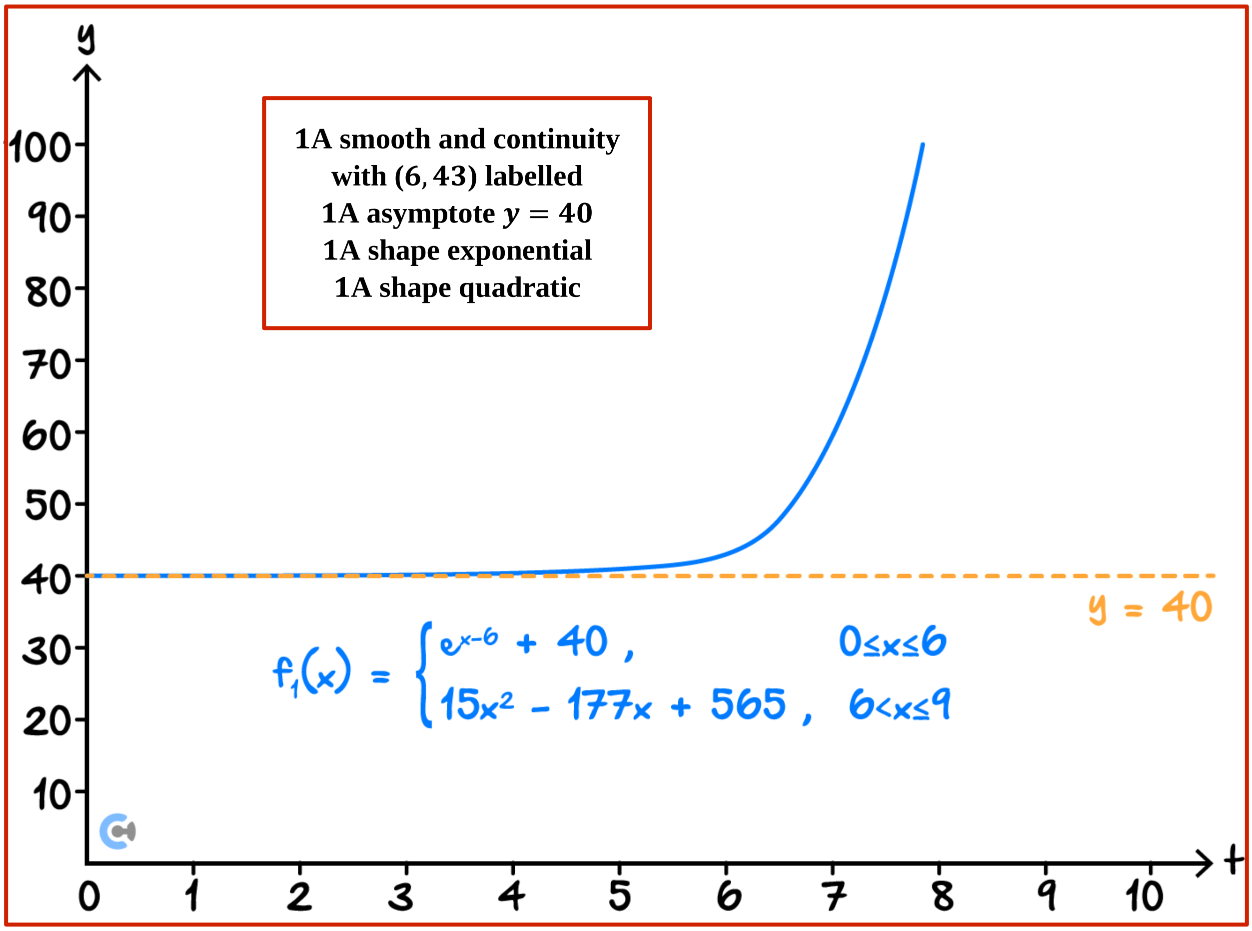
Define $dh2(t)=\frac{d}{dt}(h2(t))$ *Done* **1M**

solve$(h1(6)=h2(6)$ and $dh1(6)=dh2(6),k,l)$

1M $k=\frac{59}{10}$ and $l=\frac{857}{20}$

$k = \frac{59}{10}$ **1A**, $l = \frac{857}{20}$ **1A**

d. Using the values of k and l found in **part c.**, graph the piecewise function $h(t)$ on the axis below, labelling asymptotes and the point where the equation switches. (4 marks)



1A smooth and continuity with $(6, 43)$ labelled
1A asymptote $y = 40$
1A shape exponential
1A shape quadratic

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Question 3 (12 marks)

Rei, one of Ms Gradey's bright but rebellious students, decides to take matters into his own hands and constructs his model for the effects of sleep on a person's ATAR. Rei understands that sleep is necessary, but he believes that there is only so much sleep you can get before the tradeoffs with study time start to occur. As such, he models the effects of time slept vs ATAR by the following equation:

$$r(t) = 99.95 - 30e^{\frac{6-t}{3}}$$

where t is the average time slept in hours, and $r(t)$ is the student ATAR.

a. Based on the equation of $r(t)$, what ATAR does the model approach over time? (1 mark)

The model approaches an ATAR of $\mathbf{99.95}$ over time. **1A**

b. The model used in $r(t)$ can be thought of as a series of transformations applied to Ms Gradey's original model from **Question 1**, $a(t)$. Describe the transformations applied to $a(t)$ such that $r(t)$ is produced. (3 marks)

Reflection in r-axis
Reflection in t-axis
Translation of 59.95 units up.
1A each

To understand the effects of sleep time vs ATAR Ms Gradey decides to look at the rate of change of ATAR with respect to time.

c. Show that $r'(t) = 10e^{-\frac{t-6}{3}}$. (2 marks)

$$r(t) = 99.95 - 30e^{\frac{6-t}{3}} = 99.95 - 30e^{-\frac{t-6}{3}} \quad \mathbf{1M}$$

$$r'(t) = -30 \cdot \left(-\frac{1}{3}\right) e^{-\frac{t-6}{3}} \quad \mathbf{1M}$$

$$= 10e^{-\frac{t-6}{3}}$$

d. State the series of transformations that map the equation of $r(t)$ to the equation of $r'(t)$. (3 marks)

Vertical translation down by 99.95 units.
Reflection in the x-axis.
Dilation by a factor of $\frac{1}{3}$ from the x-axis.
1A each

Rei begins to notice that the longer a student sleeps, the amount that the ATAR increases by diminishing. It is known that if the difference in ATAR between point $(t, r(t))$ and $(t + 1, r(t + 1))$ is less than 0.5, then sleeping longer than t hours are not worth the wasted study time.

e. Find the coordinates of the point where a student should stop sleeping. Give your answer correct to two decimal places. (3 marks)

$\text{solve}(r(t+1)-r(t)<0.5,t)$ **1**	$t>14.50107$ **1**
$r(14.501072872102)$	98.18614 **1A**

Question 4 (10 marks)

In her endeavours to improve her students' sleep, and in turn, their academic performance, Ms Gradey finds out about a drug that supposedly optimises 6 hours of sleep to provide better sleep quality by causing the consumer to have increased levels of deep sleep. The increased levels of deep sleep due to the drug can be modelled by the equation:

$$d(t) = \frac{-t^2}{3240} + \frac{t}{9}$$

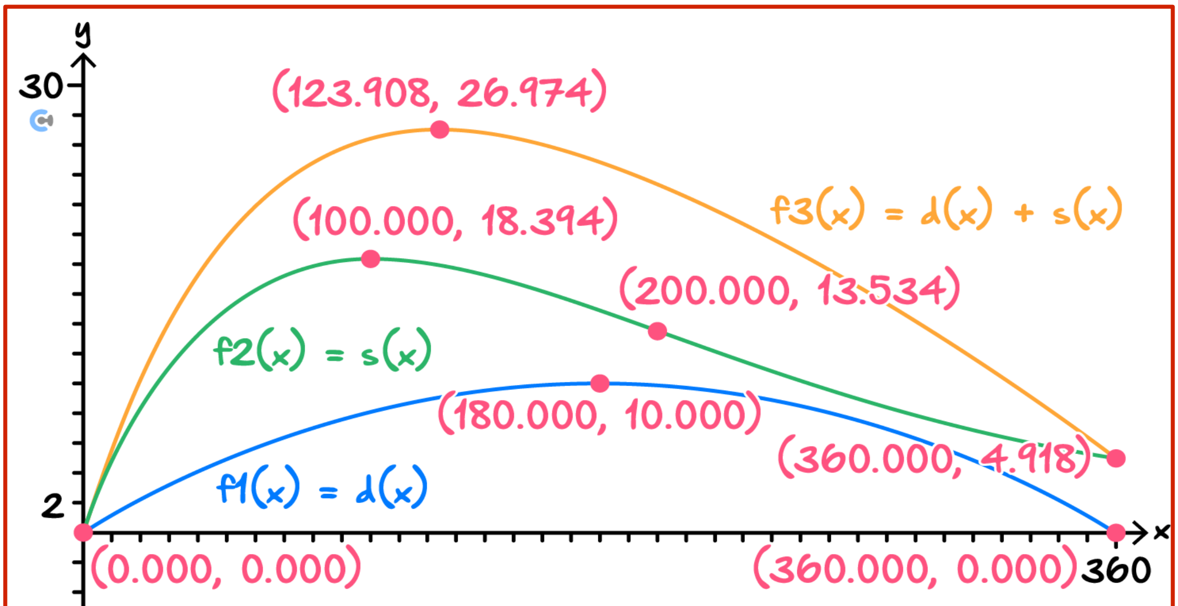
where t is the time asleep in minutes, and $d(t)$ is the boosted levels of deep sleep caused by the drug.

With this, the graph of sleep quality of someone under the effects of the drug can be modelled by:

$$s_D(t) = s(t) + d(t)$$

Where $s_D(t)$ is the sleep quality / 'deepness' of sleep of someone under the effect of the drug and $s(t)$ is the sleep quality equation without the drug given by $s(t) = 0.5t \cdot e^{-0.01t}$.

a. On the axes below, draw the graph of $s(t)$ over $t \in [0, 360]$, clearly labelling it as $s(t)$. Label the axial intercept as well as the stationary point and points of inflection, correct to three decimal places. (4 marks)



Part (a): 1A TP; 1A POI; 1A endpoints; 1A shape
Part (b): 1A TP; 1A intercepts; 1A Shape
Part (c): 1A TP; 1A Endpoints; 1A Shape

b. On the same axis, draw the graph of $d(t)$ over $t \in [0, 360]$, clearly labelling it as $d(t)$. Label the turning point and axial intercepts. (3 marks)

c. Using the addition of ordinates, draw the graph of $s_D(t)$ over $t \in [0, 360]$, clearly labelling it as $s_D(t)$. Label the turning point and axial intercept, correct to three decimal places. (3 marks)

Question 5 (8 marks)

To see how much her class listened to her about how important sleep is, two weeks later, Ms Gradey gave her class a short quiz about sleep and its effects on the human body, with the first question asking for the students' own average sleep per night over the past two weeks. With the data she gathered, Ms Gradey managed to model her students' results by their average sleep using the following logarithmic model:

$$p(t) = \ln(e^{40}t^{20}) \quad t \in (0,10]$$

where t is the average number of hours slept per night for the past 2 weeks and $p(t)$ is the percentage scored by the students.

a. Rewrite $p(t)$ using log laws into the form $A\ln(x) + k$. (1 mark)

$p(t) = 20\ln(t) + 40$

b. What is the highest score achieved by a student according to the model, in percentage, correct to 2 decimal places? Also, state the value of t when this occurs. (2 marks)

$t = 10$ **1A**,
max 86.05 **1A**

Ms Gradey was expecting the results to follow an exponential model instead of a logarithmic one. She thus decides to look into the inverse of $p(t)$.

c. Find p^{-1}. (3 marks)

Swap y and t: $t = 40 + 20\ln(y)$ **1M**

$y = e^{\frac{t-40}{20}}$

$p^{-1}(t) = e^{\frac{t-40}{20}}$ **1A**

Domain of inverse: $(-\infty, 20\ln(10) + 40]$ **1A**

Ms Gradey sees that this equation could definitely not model the scores since the scores are incredibly low under this equation. She thinks a translation upward will make the model better.

d. Find the value of k such that a translation of k in the positive y direction applied to $p^{-1}(t)$ would cause the graph to intersect with $p(t)$ at $t = 10$. (2 marks)

$p(10) = p^{-1}(10) + k$ **1M**

$k = 20\ln(10) - e^{\frac{-3}{2}} + 40$ **1M**

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