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VCE Mathematical Methods ¾
AOS 3 Revision [3.0]
SAC 2 Solutions

54 Marks. 15 Minutes Reading. 75 Minutes Writing.

Section A: SAC Questions (Tech Active) (54 Marks)

- A long time ago in a galaxy far, far away...
- Luke Skywalker is on a mission to destroy the Death Star. He is piloting his X-wing starfighter, but runs into some trouble along the way...

Question 1 (11 marks)

Luke initially plans to take off following a flight path $y = f(x)$, where,

$$f: [0, 10] \to R, f(x) = a \log_e((x+1)^2), \qquad a > 0$$

Assume that the x-axis represents the ground. Luke's angle of attack is the angle of his path relative to the ground.

a. The minimum angle of attack Luke needs to successfully lift off (at the origin) is 30 degrees. If Luke barely manages to lift off, then show that $a = \frac{\sqrt{3}}{6}$. (3 marks)

$f'(x) = \frac{2a}{x+1}$ **1M**

$f'(0) = \tan(30°)$ **1M**

$a = \frac{\sqrt{3}}{6}$ **1A**

Use $a = \frac{\sqrt{3}}{6}$ for the following parts of this question.

b. If Luke reaches space at $x = 10$, what is the difference between his angle of attack when he reaches space and when he lifts off? Give your answer in degrees to one decimal place. (2 marks)

$30 - f'(10)$ **1M**

$= 27.0$ degrees **1A**

c. A missile travelling in a straight line is threatening to intercept Luke's flight at $x = 7$. Find the path of the missile in the form $y = mx + c$ where m and c are real numbers, assuming it intercepts Luke's path at a normal. (2 marks)

$$f'(7) = \frac{\sqrt{3}}{24} \ \textbf{1M}$$

Normal at $x = 7$:

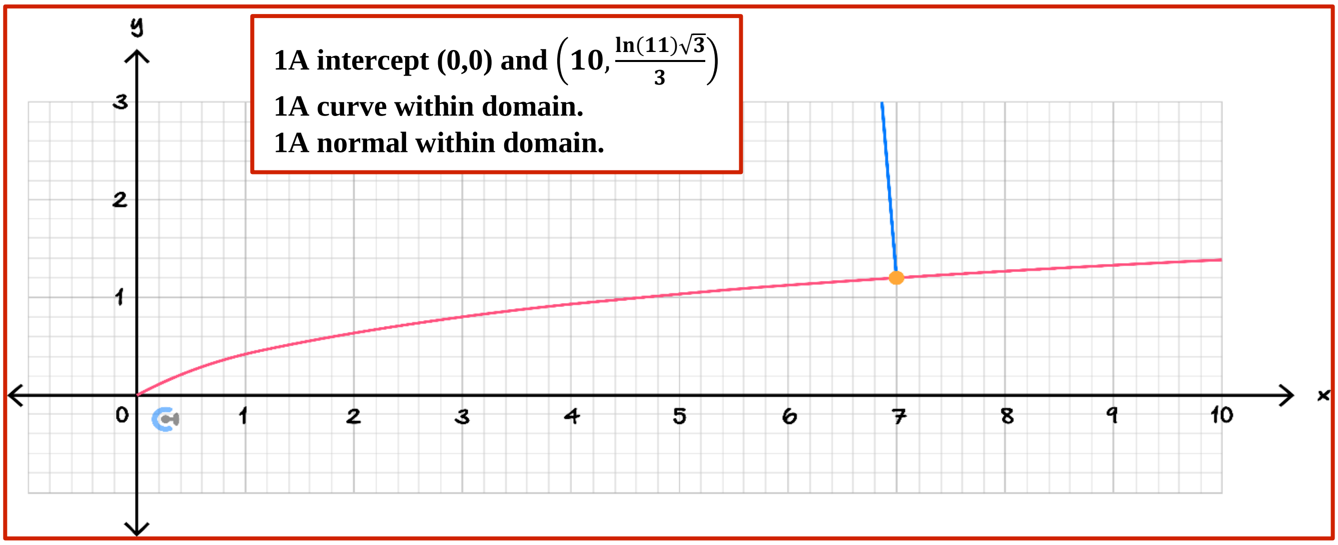
$y = -8\sqrt{3}x + \sqrt{3}(\log_e(2) + 56)$ or with expanded constant. **1A**

d. If the missile comes from an enemy plane located at $y = 3$, and is to stop when it contacts Luke's starfighter, then, find a suitable domain for the missile's path. (1 mark)

$\left[7 - \frac{\sqrt{3}-\log_e(2)}{8}, 7\right]$ **1A**

e. Plot the graph of both the missiles and the flight path $f(x)$ on the axis below. Use the domain from **part d.** for the missile, labelling all intercepts, endpoints and intersections. (3 marks)

1A intercept (0,0) and $\left(10, \frac{\ln(11)\sqrt{3}}{3}\right)$
1A curve within domain.
1A normal within domain.



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Question 2 (18 marks)

Let $f(x) = \frac{\sqrt{3}}{6}\log_e((x+1)^2)$.

Luke runs into some difficulties and must change his take-off plan. Let us model his new path with the function:

$g: [a, 10] \to R, g(x) = \log_e(5(x+1)^2) + c$ where $a > -1$.

a. Describe the series of transformations that transforms the rule of f to that of g. (2 marks)

Dilation of $\frac{6}{\sqrt{3}}$ from the x-axis.
Translation of $\log_e(5) + c$ in the $+$ve y-axis. (1 mark each)

b. If Luke must always be at or above ground level, find the minimum value of a in terms of c. (2 marks)

$g(a) = 0$ **1M**

$a = \frac{e^{\frac{-c}{2}}}{\sqrt{5}} - 1$ **1A**

While taking off, Luke notices that he is low on fuel. A refuelling plane flies up to help him, following the path $g^{-1}(x)$, where g^{-1} is the inverse function of g.

c.

i. Find the rule for g^{-1} in terms of c. (2 marks)

Swap x and y: $g(y) = x$ **1M or equivalent.**

$g^{-1}(x) = \frac{e^{\frac{x-c}{2}}}{\sqrt{5}} - 1$ **1A**

ii. Specify the range and domain of g^{-1}, answering in terms of c where appropriate. (2 marks)

Using range g = domain g^{-1} and vice versa.
Domain g^{-1}: $[0, c + \log(605)]$
Range g^{-1}: $\left[\frac{e^{\frac{-c}{2}}}{\sqrt{5}} - 1, 10\right]$ (1 mark each)

d.

i. State $g'(a)$, the derivative of $g(x)$ at point $(a, g(a))$. (1 mark)

$$g'(a) = \frac{2}{a+1}$$

ii. Hence, find $\frac{d}{dx} g^{-1}(x)$ when $y = a$. (1 mark)

$$\frac{a+1}{2}$$

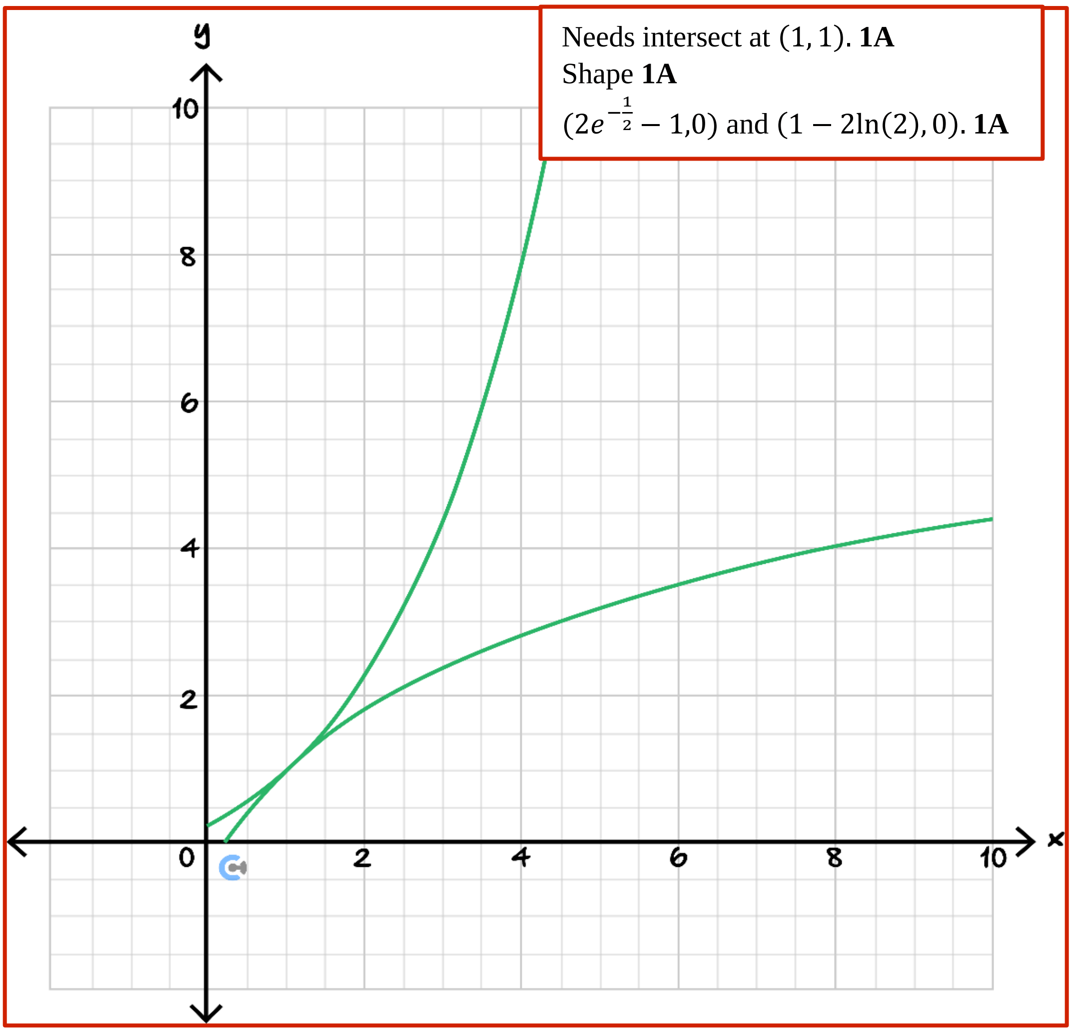
e.

i. For the plane to successfully refuel Luke's starfighter, their paths must be tangential to each other when they meet. Using your result from **part c.** or otherwise, find the value of c which causes the two planes' paths to intersect at a tangent. (3 marks)

$g(x) = x$ **1M**
$g'(x) = 1$ **1M**

$c = 1 - \log_e(20)$ **1A**. (Here, we have to use the fact that if the function and the inverse intersect tangentially, the derivative is 1 at this intersection. Solving simultaneously won't give the exact value.)

ii. Plot the graphs of g^{-1} and g on the axes below for the above value of c, labelling all intercepts, endpoints and intersections. (3 marks)



Needs intersect at $(1, 1)$. **1A**
Shape **1A**
$(2e^{-\frac{1}{2}} - 1, 0)$ and $(1 - 2\ln(2), 0)$. **1A**

iii. Hence, find the values of c for which the two functions have an intersection. Give your answer to 2 decimal places. (2 marks)

$C = -1.99$ is from **part e. i.** where $g(x)$ and its inverse are tangent. 3.59 comes from when the right most endpoints are equal to each other ($c + \log_e(605) = 10$ **1M**)
$[-1.99, 3.59]$ **1A.**

Question 3 (10 marks)

This path seems too impossible for the poor refuelling plane pilot, and so we must improvise once again.

Let us assume Luke takes the flight path $y = h(x)$ as below:

$$h: \left[\frac{\sqrt{5}}{5} - 1, 10\right] \to R, h(x) = \log_e(5(x+1)^2)$$

The refuelling pilot now wants to take a straight-line path that approaches Luke at $x = 4$.

a.

i. Find the tangent line to $h(x)$ at $x = 4$ in the form $y = mx + c$. (2 marks)

$h'(4) = \frac{2}{5}$ **1M**

$y = \frac{2x}{5} + \frac{15\ln(5) - 8}{5}$ **1A**

ii. Use this tangent line to approximate the value of $h(x)$ at $x = 2$. Give your answer to 2 decimal places. (1 mark)

$h(2)$ roughy equals 4.03 by this approximation. **1A**

b. The straight line works quite well, but the pilot is wondering if a quadratic path would be better for the job.

i. State the values of the first and second derivative, $\frac{dh}{dx}$ and $\frac{d^2h}{dx^2}$ at $x = 4$. (2 marks)

$\frac{2}{5}$ **1A** and $-\frac{2}{25}$ **1A** respectively.

ii. Hence, find a quadratic in the form $y = ax^2 + bx + c$, that shares the same value, first derivative and second derivative of $h(x)$ at $x = 4$. (3 marks)

Let $y = P(x)$

$$P(4) = h(4)$$
$$P'(4) = h'(4)$$
$$P''(4) = h''(4)$$

1M equations

Solve a, b, c **1M writing a, b, c values**

$$y = -\frac{1}{25}x^2 + \frac{18}{25}x + \frac{75\ln(5) - 56}{25}$$ **1A**

iii. Use this quadratic to approximate the value of $h(x)$ at $x = 2$. Give your answer to 2 decimal places. (1 mark)

$h(2)$ is roughly 3.87.

iv. Which function better approximates $h(x)$? Hence, which path should the pilot take? (1 mark)

The quadratic path is better as $h(2) = 3.80666 \ldots$ which is closer to quadratic path, and so the pilot should take quadratic path. **1A**

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Question 4 (15 marks)

Finally exiting the atmosphere, Luke Skywalker is now attempting to fire a missile at the Death Star's reactor to destroy the Sith space station.

We can now model Luke's path with $y = L(x)$, where,

$$L: [2, 8] \to R, L(x) = -0.5(x-2)(x-5)(x-8)$$

Assume the Death Star lies on $y = 0$.

a. What is the distance between each point where Luke has the same y-coordinate as the death star? (1 mark)

3 units. **1A**

b. What is Luke's greatest distance from the Death Star in y-direction? Give your answer to 2 decimal places. (2 marks)

$l'(x) = 0$ **1M**

$x = 6.73$

Maximum $= 5.20$ **1A**

The Death Star's reactor core lies at the point $(6, 0)$.

c. What is the nearest distance between Luke and the Death Star's core during his flight? Give your answer to 2 decimal places. (3 marks)

$d(x) = \sqrt{(x-6)^2 + (l(x)-0)^2}$ **1M**

$d'(x) = 0$ **1M**

$x = 5.047$

Minimum $= 0.98$ **1A**

d. What is the angle of elevation of the highest point on Luke's flight path from the point of view of the Death Star's reactor core, taking the positive y-axis to be 0 degrees. Give your answer in degrees to 1 decimal place. (2 marks)

$\tan^{-1}\left(\frac{5.20}{6.73-6}\right)$ **1M**

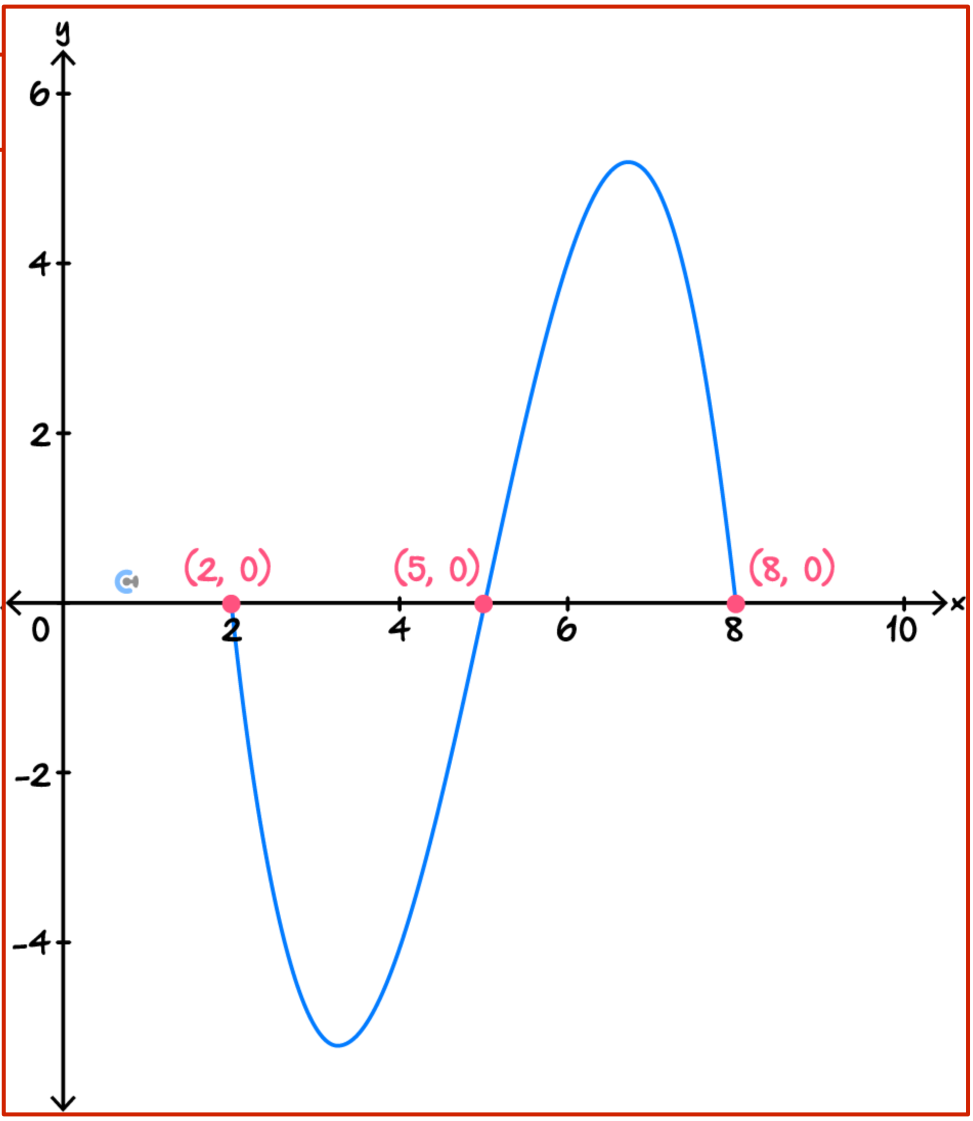
$= 82.0$ degrees **1A**

Luke's missiles can only fire out the front and back of his X-wing in a straight line. This means they travel in a tangent line to his path from when he fires them.

e.

i. Sketch the graph of y against x on the axis below. Only intercepts need to be labelled. (2 marks)

1A Shape
1A Intercepts



ii. Find the coordinates of the point where a missile can be fired and hit the Death Star's reactor by considering when the tangent line to Luke's flight passes through the Death Star. Give your answer to 2 decimal places. (3 marks)

$l'(x) = \frac{l(x)-0}{x-6}$ **1M**

$x = 3.73$ **1M**

$(3.73, -4.70)$ **1A**

iii. What is the shortest distance that a missile can travel and hit the Death Star's reactor? Give your answer correct to one decimal place. (2 marks)

Find the distance between $(6,0)$ and $(3.73, -4.70)$. **1M**
This distance is 5.2 units. **1A**

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