Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾

AOS 3 Revision [3.0]

SAC 1 Solutions

52 Marks. 15 Minutes Reading. 75 Minutes Writing.

Section A: SAC Questions (Tech Active) (52 Marks)

Question 1 (10 marks)

Supersonic cars race against each other in a 100 km straight stretch of land in the desert. The position x km from the starting point after t minutes is given by:

$$x(t) = \frac{3}{4} t^{\frac{4}{3}} - 360t - 3(t - 60)^{\frac{4}{3}} + c$$

where $t \geq 60$.

a. Find the exact value of c, assuming the car starts at $x = 0$ when $t = 60$. (2 marks)

Require $x(60) = 0$ [1M]
$0 = \frac{3}{4}(60)^{4/3} - 360(60) - 3(60 - 60)^{4/3} + c$
$0 = \frac{3}{4}(60)^{4/3} - 21600 - 0 + c$
$c = 21600 - \frac{3}{4}(60)^{4/3}$ [1A]
(or $c = 21600 - 45 \cdot (20)^{1/3}$)

b. Find the domain of $x'(t)$. (1 mark)

Domain $(60, \infty)$ [1A]

c. The velocity is given by $v(t) = x'(t)$. Find $v(t)$. (2 marks)

$v(t) = \frac{d}{dt}\left(\frac{3}{4}t^{4/3} - 360t - 3(t-60)^{4/3} + c\right)$ [1M]
$v(t) = \frac{3}{4} \cdot \frac{4}{3}t^{1/3} - 360 - 3 \cdot \frac{4}{3}(t-60)^{1/3}$
$v(t) = t^{1/3} - 4(t-60)^{1/3} - 360$ [1A]

d. The acceleration is given by $a(t) = v'(t)$. Find $a(t)$. (2 marks)

$a(t) = \frac{d}{dt}\left(t^{1/3} - 4(t-60)^{1/3} - 360\right)$ [1M]
$a(t) = \frac{1}{3}t^{-2/3} - 4 \cdot \frac{1}{3}(t-60)^{-2/3}$
$a(t) = \frac{1}{3t^{2/3}} - \frac{4}{3(t-60)^{2/3}}$ [1A]

e. Hence, investigate the speed of the car for $t > 60$. Is there a maximum speed? Justify your answer.

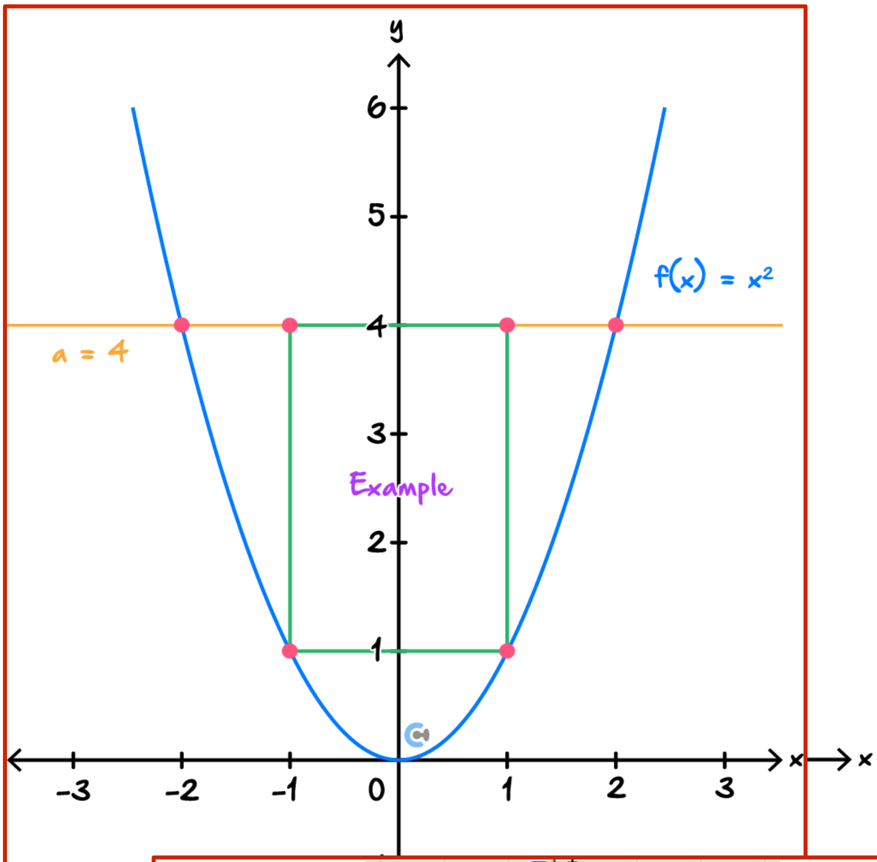
Note: Speed is the magnitude (size) of velocity. That is, speed $= |v(t)|$. (3 marks)

Setting $a(t) = 0$ yields no solution in the domain $t > 60$ **[1M]**
Analysis shows $a(t) = \frac{1}{3}[t^{-2/3} - 4(t-60)^{-2/3}] < 0$ for all $t > 60$.
Since $a(t) = v'(t) < 0$, $v(t)$ is always decreasing and tends to $-\infty$. Speed $= |v(t)|$ increases without bound.**[1M]**
There is no maximum speed **[1A]**

Question 2 (16 marks)

Consider the two functions: $f(x) = x^2$ and $g(x) = a$, where $a \in \mathbb{R}^+$. A rectangle is inscribed between these two functions, such that one edge lies on the line $y = g(x)$ whilst the two opposite vertices lie on the graph of $y = f(x)$.

a. On a set of axes, sketch $f(x), g(x)$ for $a = 4$. An example of the inscribed rectangle is described. Label the point(s) of intersection between $f(x)$ and $g(x)$ in terms of a. (5 marks)



Sketch $f(x)$ **[1A]**. Sketch $g(x)$ **[1A]**. Calculate intersections $x = \pm\sqrt{a}$ **[1M]**. Label intersection points **[1A]**. Sketch example rectangle **[1A]**.

b. What can be said about the relationship between the x-coordinates of the vertices of the rectangles? (2 marks)

Due to symmetry of $y = x^2$ and $y = a$ about the y-axis **[1M]**.
The x-coordinates come in pairs $\pm b$ for some b $(0 < b \leq \sqrt{a})$ **[1A]**.

c. Hence, given that one of the vertices lies on $f(x)$ has an x-coordinate of b (where $0 < b \leq \sqrt{a}$), state the coordinates of the 4 vertices, in terms of a and b. (2 marks)

Vertices on $f(x)$: (b, b^2) and $(-b, b^2)$ **[1M]**.
Vertices on $g(x)$: (b, a) and $(-b, a)$ **[1A]**.

d. Hence, express the area A of the rectangle in terms of b and a. (2 marks)

Width $= b - (-b) = 2b$. Height $= a - b^2$ **[1M]**.
Area $A(b) =$ Width $\times$ Height
$= 2b(a - b^2)$ **[1A]**.

e. Hence, find the exact value of b (in terms of a) for which the area of the rectangle will be a maximum, and state this exact maximum area. (3 marks)

$A(b) = 2ab - 2b^3$.
$A'(b) = 2a - 6b^2$. Set $A'(b) = 0$ **[1M]**.
$6b^2 = 2a \implies b^2 = a/3 \implies b = \sqrt{a/3}$ (since $b > 0$) **[1A]**.
Max Area $= A(\sqrt{a/3}) = 2\sqrt{\frac{a}{3}}(a - \frac{a}{3})$
$= \frac{4a\sqrt{a}}{3\sqrt{3}} = \frac{4a\sqrt{3a}}{9}$ **[1A]**.

f. For what value(s) of b in the domain $0 < b \leq \sqrt{a}$ is the area minimised? State the minimum area. (2 marks)

Area $A(b) = 2b(a - b^2)$. Check endpoint $b = \sqrt{a}$ **[1M]**.
Minimum area is $A(\sqrt{a}) = 0$, occurring at $b = \sqrt{a}$ **[1A]**.

Space for Personal Notes

Question 3 (21 marks)

Consider the function $f(x) = x^3 - 5x + 1$, for $x \in \mathbb{R}$.

a. State the equation of the tangent to the graph of $y = f(x)$ at $x = 2$. (1 mark)

$f(2) = 2^3 - 5(2) + 1 = -1$.
$f'(x) = 3x^2 - 5 \implies f'(2) = 3(2^2) - 5 = 7$.
Tangent: $y - (-1) = 7(x - 2) \implies y = 7x - 15$ **[1A]**

b. Let $x_0 = 2$. Use Newton's method once to find x_1, and hence find the values of x_2 and x_3 using CAS. Give correct values to 4 decimal places. (3 marks)

Newton's formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
$= x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5}$ **[1M]**.

$x_1 = 2 - \frac{f(2)}{f'(2)}$
$= 2 - \frac{-1}{7} = \frac{15}{7} \approx 2.1429$ **[1A]**.

Using CAS:
$x_2 \approx 2.1286$.
$x_3 \approx 2.1284$ **[1A]**.

c. Using $x_0 = 2$, state the smallest value of n such that x_n correctly approximates the value of the positive x-intercept of $f(x)$, correct to 4 decimal places. (2 marks)

True positive root $x^* \approx 2.128419...$
Rounds to 2.1284 (4 dp) **[1M]**.

$x_1 \approx 2.1429$, $x_2 \approx 2.1286$, $x_3 \approx 2.1284$.
x_3 is the first iteration that rounds to 2.1284. So $n = 3$ **[1A]**.

A tangent line is drawn to the function $y = f(x)$ at the point where $x = a$.

d.

i. State the equation of this tangent line in terms of a. (2 marks)

$f(a) = a^3 - 5a + 1$.
$f'(a) = 3a^2 - 5$ [**1M**].

Equation: $y - (a^3 - 5a + 1) = (3a^2 - 5)(x - a)$.
$y = (3a^2 - 5)x - a(3a^2 - 5) + a^3 - 5a + 1$.
$y = (3a^2 - 5)x - 2a^3 + 1$ [**1A**].

ii. Given that this tangent line passes through the coordinate $(0, 1)$, state the possible value(s) of a. (2 marks)

Substitute $(0, 1)$ into tangent equation:
$1 = (3a^2 - 5)(0) - 2a^3 + 1$ [**1M**].

$1 = -2a^3 + 1$.
$2a^3 = 0$.
$a = 0$ [**1A**].

iii. State the possible value(s) of a such that using $x_0 = a$ in Newton's method for $f(x) = 0$ causes an oscillating sequence (e.g., $x_0, x_1, x_0, \ldots$). Give the value(s) correct to 2 decimal places. (2 marks)

Condition often related to $2a = f(a)/f'(a)$ [**1M**].

$2a(3a^2 - 5) = a^3 - 5a + 1$.
$5a^3 - 5a - 1 = 0$.

Solving using CAS gives $a \approx -0.83, a \approx -0.20, a \approx 1.03$ [**1A**].

iv. State the possible value(s) of a such that using $x_0 = a$ in Newton's method for $f(x) = 0$ terminates immediately at a root. Give value(s) correct to 2 decimal places. (2 marks)

Terminates immediately if $x_0 = a$ is a root, i.e., $f(a) = 0$ **[1M]**.

$a^3 - 5a + 1 = 0$.

Roots (via CAS) are $a \approx -2.33, a \approx 0.20, a \approx 2.13$ **[1A]**.

Let another function be $g(x) = \sqrt{x + 4}$, for $x \geq -4$.

e.

i. Determine which composite function, $f(g(x))$ or $g(f(x))$, has a domain that is a strict subset of the domain of the inner function, and state why.

Note: A strict is a subset that isn't equal to the original set. (1 mark)

Inner function $f(x)$ has domain $\mathbb{R}$.
Inner function $g(x)$ has domain $[-4, \infty)$.

$f(g(x))$ requires $x \geq -4$. Domain is $[-4, \infty)$ (same as g).

$g(f(x)) = \sqrt{x^3 - 5x + 5}$. Requires $x^3 - 5x + 5 \geq 0$.
Domain approx $[-2.627, \infty)$.

Since $[-2.627, \infty) \subset \mathbb{R}$, $g(f(x))$ has the strictly subset domain **[1A]**.

ii. For the composite function identified in **part e.i.,** state its domain correctly to 3 decimal places. (2 marks)

The function is $g(f(x))$. Need $x^3 - 5x + 5 \geq 0$ **[1M]**.

From CAS, the only real root is $x_r \approx -2.627365...$.

Domain is $[-2.627, \infty)$ (approx) **[1A]**.

iii. State the range of the composite function identified in **part e.i.** (1 mark)

$g(f(x)) = \sqrt{\text{non-negative value}}$.
Range is $[0, \infty)$ **[1A]**.

f. Using Newton's method, approximate the solution to the equation $f(x) = g(x)$. Use $x_0 = 2$ and stop when successive iterations differ by less than 10^{-4}. Give your answer correct to 4 decimal places. (3 marks)

Let $h(x) = f(x) - g(x) = x^3 - 5x + 1 - \sqrt{x+4}$. Need $h(x) = 0$.
$x_{n+1} = x_n - h(x_n)/h'(x_n)$ **[1M]**.

Using CAS with $x_0 = 2$:
$x_1 \approx 2.507586$.
$x_2 \approx 2.384763$.
$x_3 \approx 2.375343$.
$x_4 \approx 2.375289$ **[1M]**.

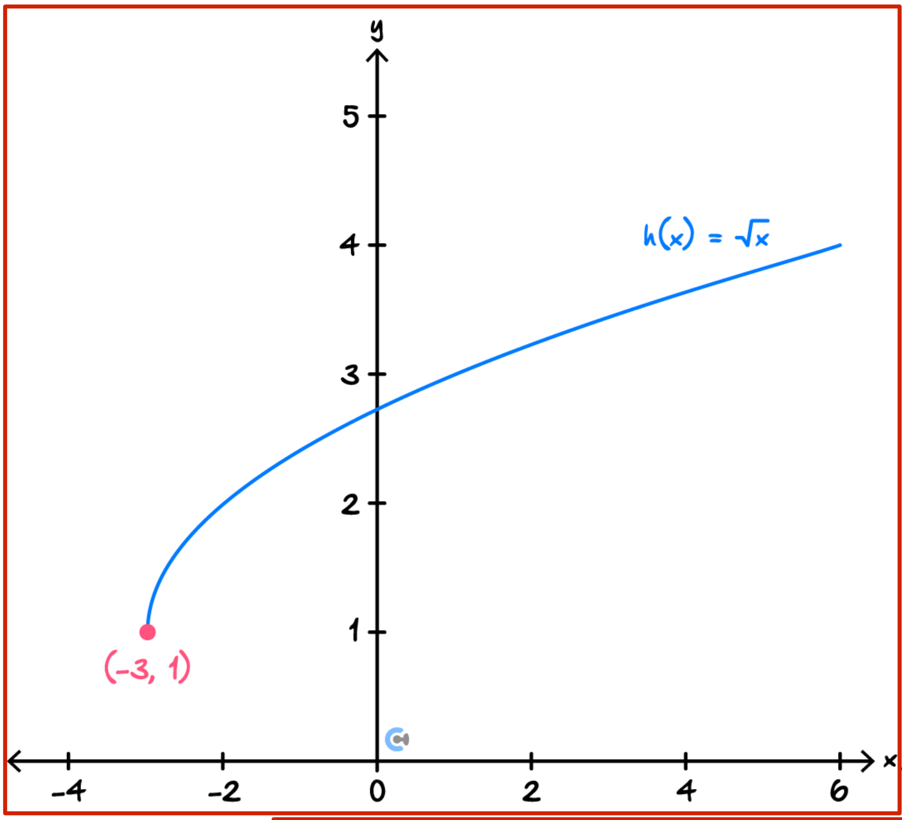
$|x_4 - x_3| \approx 0.000054 < 10^{-4}$.

Solution $x \approx 2.3753$ **[1A]**.

Question 4 (5 marks)

Let the function be $h(x) = \sqrt{x + k} + 1$.

a. Sketch the graph of $y = h(x)$ on the axes below, given that $k = 3$. Label the coordinates of the endpoint clearly. (1 mark)



Endpoint label correct **[1A]**.

b. Define the inverse function $h^{-1}(x)$, stating its domain. (2 marks)

Swap x, y in $y = \sqrt{x+k} + 1 \implies x = \sqrt{y+k} + 1$.
Solve for y: $x - 1 = \sqrt{y+k} \implies (x-1)^2 = y + k \implies y = (x-1)^2 - k$ **[1M]**.
$h^{-1}(x) = (x-1)^2 - k$.
Domain of h^{-1} = Range of $h = [1, \infty)$ **[1A]**.

c. Find the possible value(s) of k such that $h(x)$ and its inverse function $h^{-1}(x)$ have exactly one point of intersection. (2 marks)

Intersection when $h(x) = x \implies \sqrt{x+k} + 1 = x$.
Leads to quadratic $x^2 - 3x + (1-k) = 0$, for $x \geq 1, x \geq -k$ **[1M]**.
Single solution if discriminant $D = 5 + 4k = 0 \implies k = -5/4$ (tangency at $x = 1.5$).
Or if $D > 0$ ($k > -5/4$) and only one root $\geq \max(1, -k)$. This occurs when smallest root $< 1 \implies k > -1$.
Single intersection occurs for $k = -5/4$ or $k > -1$ **[1A]**.

Space for Personal Notes

CONTOUREDUCATION

Website: contoureducation.com.au | Phone: 1800 888 300 | Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make The Most of These (Free) Services!

- Experienced Contour tutors (45 + raw scores, 99 + ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After school weekdays and all day weekends.

1-on-1 Video Consults	Text-Based Support
Book via bit.ly/contour-methods-consult-2025 (or QR code below). One active booking at a time (must attend before booking the next).	Message +61 440 138 726 with questions. Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726