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VCE Mathematical Methods ¾
Family of Functions & its Exam Skills [2.6]
Workbook

Outline:

Learning Objectives:

- ☐ MM34 [2.6.1] - Find Unknowns for General Requirements
- ☐ MM34 [2.6.2] - Find Unknowns for Number of Solutions
- ☐ MM34 [2.6.3] - Find Unknowns for Minimum and Maximum

Section A: Families of Functions Exam Skills

Sub-Section: Introduction to Family of Functions

What is family of functions?

Exploration: Understanding Family of Functions

- Consider the following functions:

$$f(x)+2$$

$$f(x)+3$$

$$f(x)-5$$

- What do all the functions have in common?

- What are the differences between the functions?

Families of Functions

Functions with an unknown.

- They involve understanding and using ________________________.
- They involve the use of ______________________________ on CAS/technology.

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Question 1

Which one of the following is **not** a part of the same family of functions as the rest?

A. $2f(x)+1$

B. $3f(x)+1$

C. $-2f(x)+1$

D. $f(x)-2$

E. $f(x)+1$

<u>Discussion:</u> How do we approach questions involving families of functions?

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Question 2 Walkthrough.

Consider the family of functions $f(x) = e^{kx} + 1$ where $k > 0$.

Identify the "last name" (the common aspect in the function family) and the "first name" (the unique aspects in the function family).

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Question 3

Consider the family of functions $f(x) = (x-2)^2 + k$.

Identify the "last name" (the common aspect in the function family) and the "first name" (the unique aspects in the function family).

What do we do when the "first and last name" of the family of functions are difficult to identify?

Definition

<u>Using Sliders/Manipulate on Technology</u>

1. **Understanding the effect of the unknown on the graph.**

   As often this is not obvious from transformations.

2. **Checking your answer.**

   When finding the value(s) of an unknown, check the value smaller and larger than the value obtained to see which side satisfies the condition.

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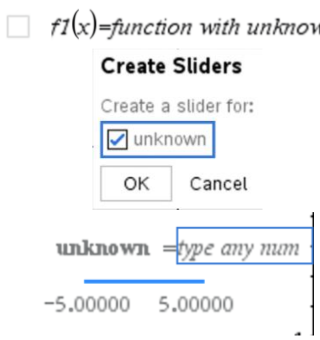
Calculator Commands: Using Sliders/Manipulate on CAS

Mathematica

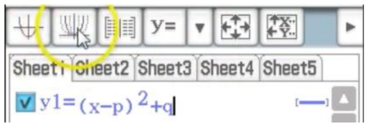
```
Manipulate[Plot[function, {x, xmin, xmax}],
  {unknown, lowerbound, upperbound}]
```

- **NOTE:** The function **must** be typed out instead of using its saved name.

TI



Casio



Question 4 Tech-Active.

Sketch the function $y = (x - k)^2 + 4$ using technology, and identify the effects of k on the graph.

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What type of questions could you get for the family of functions?

Analogy: Picking the Right Family Member

- Consider a family of functions below:

- They all have a different value for their ______________________.

  We can think of their unknowns to be like their_______________________________________.

- If VCAA asked you to pick the member of the family that is shorter than 150 cm, how would you do that?

$$\boldsymbol{VCAA: Pick\ a\ member\ shorter\ than\ 150\ cm!}$$

$$\boldsymbol{Us: Give\ the\ unknown\ value\ for\ the\ member\ that\ are\ shorter\ than\ 150\ cm.}$$

- More mathematically:

$$\boldsymbol{VCAA: Find\ a\ function\ that\ intersects\ with\ y = x\ once!}$$

$$\boldsymbol{Us: Give\ the\ unknown\ value\ for\ the\ function\ that\ intersects\ once.}$$

- In summary.

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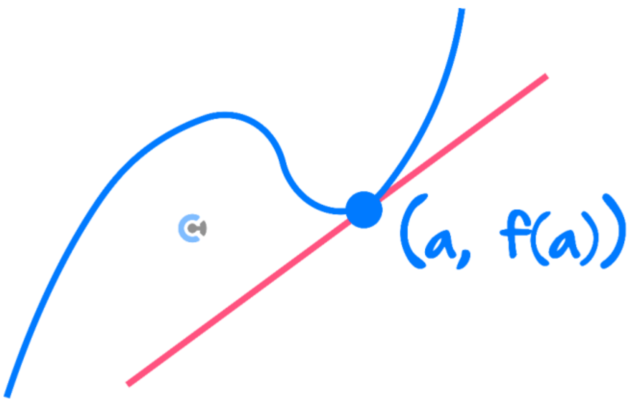
Sub-Section: Number of Intersections

Discussion: How can any function meet with a line as a tangent?

Exploration: Tangential Relationships

- Consider a line.



- For them to intersect, what must be the same?

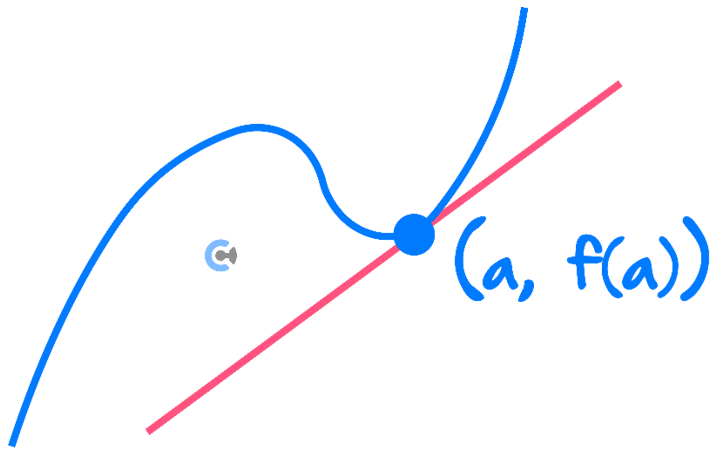
- For them to intersect as a tangent, what must be the same?

  HINT: Think about how we found the gradient of a tangent.

- We call this relationship ________________________.

  Function hits a line as a tangent.

Tangent to a Family of Functions



- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

  - With the same gradient.

$$f'(a) = m$$

- We just solve these simultaneously.

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Question 5 Walkthrough.

Find the value of k such that the function $f(x) = e^x + k$ hits the line $y = x$ only once.

<u>Active Recall</u>

- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

  - With the same gradient.

$$f'(a) = \_\_\_\_\_\_$$

- We solve these ____________________.

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Question 6

Find the value(s) of k such that the function $f(x) = 2e^{x-1} + k$ hits the line $y = 2x$ twice. Check your answer using sliders/manipulate.

NOTE: The wording "value(s)" always implies that there is more than one value for the solution!

<u>Discussion:</u> **How can we use what we learnt above to find the number of intersections with the inverse function?**

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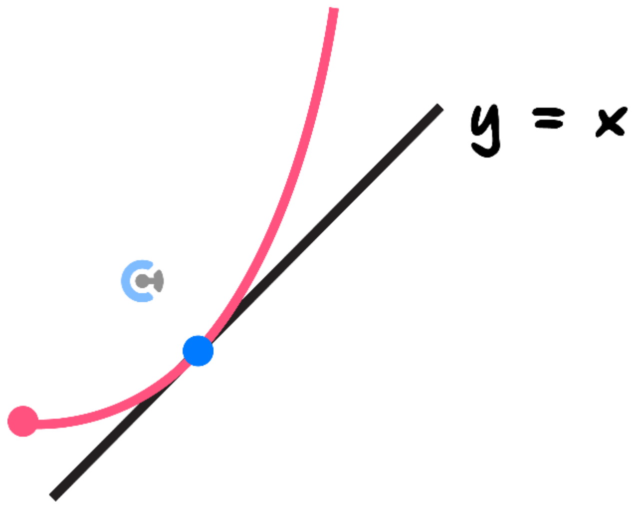
Definition

Family of Functions and Inverse



- For a function to "touch" $y = x$ as a tangent:
  - They intersect.

$$f(a) = a$$

  - With the same gradient.

$$f'(a) = 1$$

- We just solve these simultaneously.

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Question 7 Walkthrough.

Consider the function $f(x) = (x - k)^2 + 4, x \geq k$.

Find the value(s) of k such that $f(x)$ and its inverse never intersects. Check your answer using sliders.

TIP: Always sketch the function.

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Question 8

Consider the function $f(x) = \sqrt{x+k} + 1$.

Find the value(s) of k such that $f(x)$ and its inverse intersects once. Check your answer using sliders.

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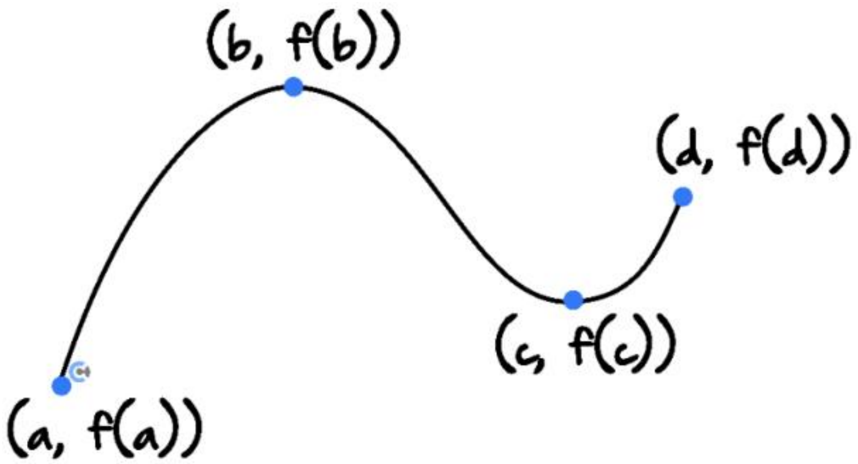
Sub-Section: Maximum and Minimum

REMINDER: Absolute Maximum and Minimum



Absolute Min: $f(a)$

Absolute Max: $f(b)$

- Absolute maximum and minimum can happen at either an endpoint or a stationary point.

Discussion: How can we make sure a minimum or a maximum occurs at a point that is not an endpoint?

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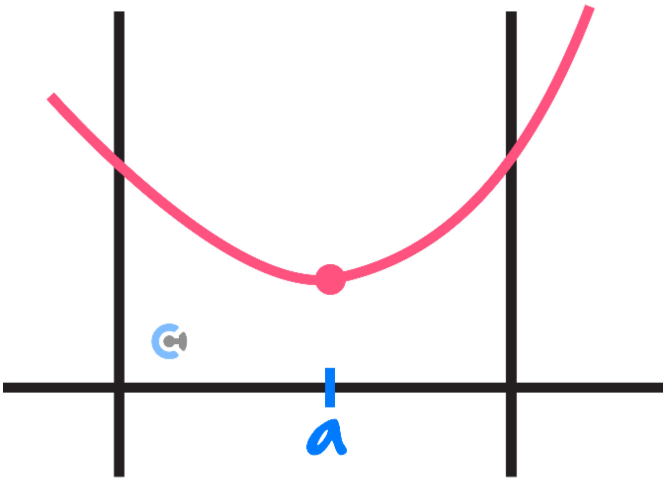
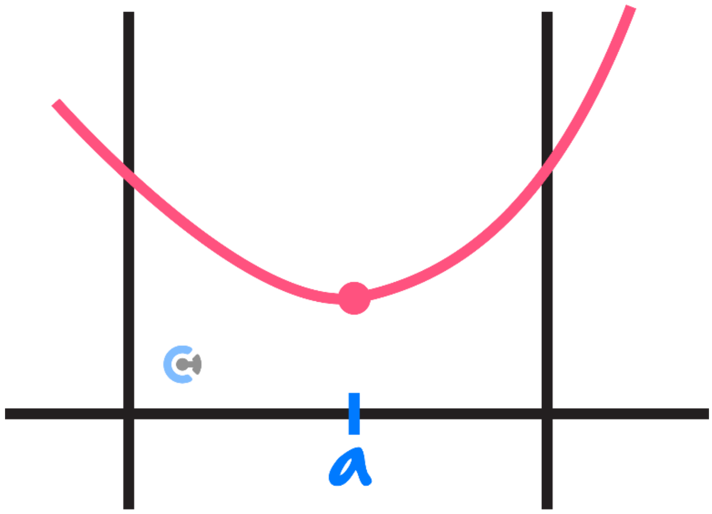
Exploration: Minimum/Maximum at $x = a$ where $x = a$ is not an endpoint

- Consider a function below:



- What would be necessary for the minimum to occur at $x = a$?

Minimum/Maximum at a Turning Point



- To achieve minimum/maximum at $x = a$.

$$f'(a) = 0$$

- This is only when $x = a$ is not an ____________________.

Question 9 Walkthrough.

Consider the function $y = e^{(x+k)^2}$. Find the value of k such that the function has a minimum at $x = 10$.

NOTE: The minimum/maximum is not at an endpoint. Hence, it must be on the turning point.

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Question 10

Consider the function $y = (x - k)\log_e(x - k)$. Find the value of k such that the function has a minimum at $x = 20$.

What about if $x = a$ is it an endpoint?

<u>Discussion:</u> For the endpoint to be a maximum/minimum, does it need to be a turning point?

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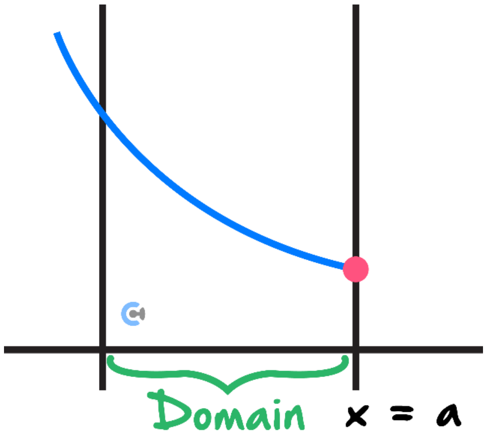
<u>Exploration</u>: **Minimum/Maximum at $x = a$ where $x = a$ is an endpoint**

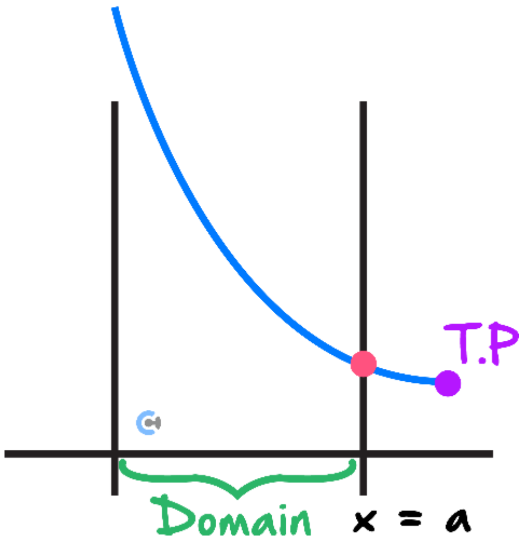
- Consider a function below:



- How can we find the unknown value such that the turning point occurs at $x = a$?

- Is that the only way for the $x = a$ to be the minimum?

- Consider the graph that has been moved to the right from the previous graph.



- Is the minimum still at $x = a$?

-

- Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

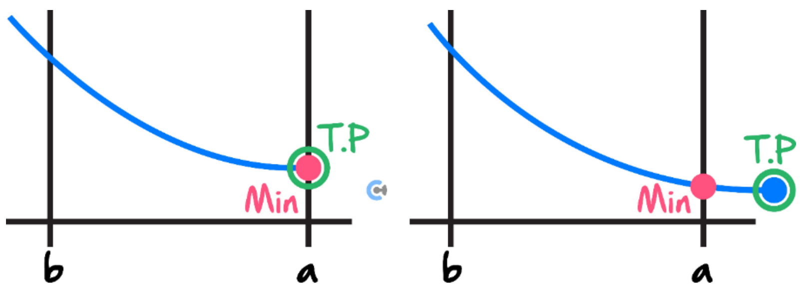
- Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.

- We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = 0$$

Discussion: **For the above graph, should we find the unknown value such that the turning point occurs after $x = a$ or before?**

Minimum/Maximum at an End Point



- Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

- Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.

  We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = 0$$

Question 11 Walkthrough.

Consider the family of functions $f(x) = (2x - k)^2 + 4, x \in [5, 15]$. Solve for the value(s) of k such that the minimum value of the function occurs at $x = 5$. Check your answer using sliders.

TIP: Find the k value such that the turning point is after the endpoint. Hence, we can check whether the k value can be bigger or smaller.

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Question 12

Consider the family of functions $f(x) = e^{2(x-k)^2} + 1, x \in [1, 10]$.

Solve for the value(s) of k such that the minimum value of the function given by $y = f(x)$ occurs at $x = 10$. Check your answer using sliders.

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Section B: Exam 1 (17 Marks)

INSTRUCTION: 17 Marks. 25 Minutes Writing.

Question 13 (6 marks)

Let $f: [a, \infty) \to \mathbb{R}, f(x) = 2x \log_e(x) + k$, for $k \in \mathbb{R}$, be a one-to-one function.

a. Find the smallest possible value of a. (2 marks)

b. Find the value of k for which $y = 2x$ is a tangent to the graph of f. (1 mark)

c. Find all values of k for which the graph of f and f^{-1} do not intersect. (3 marks)

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Question 14 (4 marks)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = xe^{kx^2}$, where $k \in \mathbb{R}$.

a. Show that $f'(x) = e^{kx^2}(1 + 2kx^2)$. (1 mark)

b. Find the values of k for which the graphs of $y = f(x)$ and $y = f'(x)$ do not intersect each other. (3 marks)

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Question 15 (7 marks)

Let $g: \mathbb{R} \to \mathbb{R}, g(x) = \sqrt{x+k} + m$, where $k > 0$ and $m \in \mathbb{R}$.

Let the point P be the y-intercept of the graph of $y = g(x)$.

a. Find the coordinates of P, in terms of k and m. (1 mark)

b. Find the gradient of g at P, in terms of k. (2 marks)

c. Given that the graph of $y = g(x)$ passes through the origin, express k in terms of m and give a restriction on the values that m can take. (1 mark)

d. Let the point Q be a point on the graph of g, such that the gradient of g at P is three times the size of the gradient of g at the point Q.

Given that the graph of $y = g(x)$ passes through the origin, find the coordinates of Q in terms of m. (3 marks)

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Section C: Tech Active Exam Skills

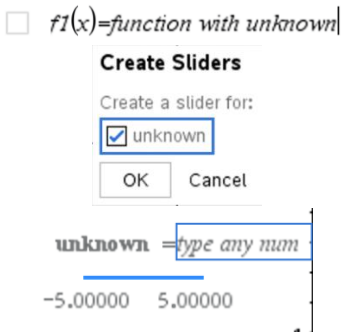
Calculator Commands: Using Sliders/Manipulate on CAS

- **Mathematica**

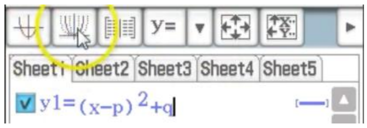
```
Manipulate[Plot[function, {x, xmin, xmax}],
   {unknown, lowerbound, upperbound}]
```

- **NOTE:** The function **must** be typed out instead of using its saved name.

- **TI**



- **Casio**



Calculator Commands: Finding k so that f and f^{-1} intersect once

- Step 1: Plot the functions with sliders.
- Step 2: Solve the equations $f(a) = a$ and $f'(a) = 1$ simultaneously on our CAS.
- Step 3: Check that your answer makes sense by using your sliders.
- **Example** Consider the function $f(x) = e^{kx}$, where $k > 0$. Find the exact value of k for which f and f^{-1}, have exactly one point of intersection.
- **Mathematica:**

```
In[29]:= f[x_] := Exp[k x]
In[33]:= Solve[f[x] == x && f'[x] == 1]
```

Out[33]= $\left\{\left\{k \to \frac{1}{e},\ x \to e\right\}\right\}$

- **TI:**

Define $f(x)=e^{k\cdot x}$	*Done*
Define $df(x)=k\cdot e^{k\cdot x}$	*Done*
⚠ solve$(f(x)=x$ and $df(x)=1,k,x)$	$k=0.367879$ and $x=2.71828$
solve$(f(e)=e,k)$	$k=e^{-1}$
solve$(k\cdot x=\ln(x)$ and $df(x)=1,k,x)$	$k=e^{-1}$ and $x=e$

- **Casio:**

$$\begin{cases}\exp(k*x)=x \\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{e^{k\cdot x}-x=0,\ k\cdot e^{k\cdot x}-1=0\}$$

$$\begin{cases}k*x = \ln(x) \\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{x=e,\ k=e^{-1}\}$$

NOTE: Sometimes for tricker equations we will not immediately get a solution for the system of equations or it will not be exact, but often altering the equations with some simple algebra will then allow the CAS to solve it correctly.

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Section D: Exam 2 (28 Marks)

INSTRUCTION: 28 Marks. 40 Minutes Writing.

Question 16 (1 mark)

The graph of $y = kx - 3$ intersects the graph of $y = x^2 + 4x$ at two distinct points for:

A. $k = 8$

B. $k > 4 + 2\sqrt{3}$ or $k < 4 - 2\sqrt{3}$

C. $4 \leq k \leq 8$

D. $4 - 2\sqrt{3} \leq k \leq 4 + 2\sqrt{3}$

Question 17 (1 mark)

Consider the function $g(x) = e^{x^2-2kx+k^2}$. The value of k for which g has a local minimum at $x = 2$ is:

A. $k = 1$

B. $k = 2$

C. $k = 3$

D. $k = 4$

Question 18 (1 mark)

The tangent to the graph of $y = x^3 - ax^2 + 4$ at $x = 1$ passes through the origin. The value of a is:

A. -2

B. -1

C. 1

D. 2

Question 19 (1 mark)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = (x-1)(2x+1)(3x-2)$ and $g: (0, \infty) \to \mathbb{R}, g(x) = x\log_e(x)$. The maximum number of solutions for the equation $f(x-k) = g(x)$, where $k \in \mathbb{R}$, is:

A. 0

B. 1

C. 2

D. 3

Question 20 (1 mark)

For the function $p(x) = \frac{1}{k}e^{-2kx^2}$, where $x \in \mathbb{R}$ and $k > 0$, the value of k for which $p(x)$ has a local maximum at $(0, 5)$ is:

A. 2

B. 5

C. $\frac{1}{5}$

D. $\frac{1}{2}$

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Question 21 (14 marks)

Contour students' stress levels were measured between their Contour Chemistry Mock Exam and their Contour Methods Mock Exam, which were 1 day (24 hours) apart.

$$S(t) = \frac{1}{t+1} + \frac{a}{25-t} \text{ where } 0 \le t \le 24, a \in (0, 1000)$$

The Chemistry Mock Exam was first, and occurred at $t = 0$. The Methods Exam followed, at $t = 24$.

a. Vivian's stress level can be modelled with $a = \frac{3}{5}$.

i. Find her stress level at the start of the Chemistry Exam. (1 mark)

ii. Find her stress level at the Methods Exam. Hence, find which subject she is more comfortable with. (1 mark)

iii. Find the value of t for which she was least stressed between the Chemistry and Methods Exams, and find this minimum stress level. Give your answer correct to two decimal places. (3 marks)

iv. If her stress level is above 0.4, Vivian goes into her *"tryhardmode"*. How long was she in her *"tryhardmode"*, in hours, during the two days? Give your answer correct to two decimal places. (2 marks)

b. Let's consider the general case for Year 12 students.

i. For what value(s) of a would students be less stressed for Chemistry than Methods? (1 mark)

ii. What value(s) of a would explain students having minimum stress levels at the Chemistry Exam? (3 marks)

iii. What value(s) of a would explain students having minimum stress level at the Methods Exam? (3 marks)

Space for Personal Notes

Question 22 (9 marks)

Let $f: [0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$ and $g(x) = f(x) \times (f(x) - 2)$.

Let $h: [s, \infty) \to \mathbb{R}, h(x) = g(x)$.

a. Find the minimum value of s for the inverse function h^{-1} to exist. (1 mark)

b. The inverse function h^{-1} can be written as $h^{-1}(x) = \left(\sqrt{x + a} + a\right)^2$, where $a \in R$. Find the value of a. (2 marks)

c. State the domain and range of the inverse function h^{-1}. (1 mark)

d. Let $d(x) = h^{-1}(x) - h(x)$. Find the maximal domain and range of d. (2 marks)

e. Hence, show that graphs of $y = h(x)$ and $y = h^{-1}(x)$ do **not** intersect. (2 marks)

f. Let $q(x) = h(x) + c$ where c $\in \mathbb{R}$. For what value(s) of c will the equation $q(x) = h^{-1}(x)$ have exactly one solution? (1 mark)

____

____

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Contour Check

Learning Objective: [2.6.1] – Find unknowns for general requirements

Key Takeaways

- **Families of Functions:** Functions with an unknown.
- They involve understanding and using ________________________.
- They involve the use of ______________________________ on CAS/technology.

Learning Objective: [2.6.2] – Find unknowns for number of solutions

Key Takeaways

- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

  - With the same gradient.

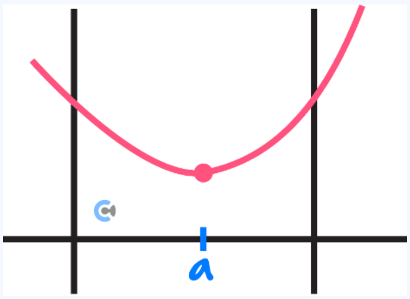
$$f'(a) = \_\_\_\_\_\_$$

- We solve these __________________________.

Learning Objective: [2.6.3] - Find unknowns for minimum and maximum

Key Takeaways

Minimum/Maximum at a turning point:

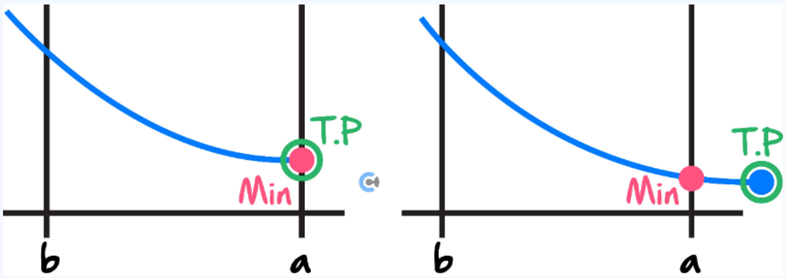


- To achieve minimum/maximum at $x = a$.

$$f'(a) = 0$$

- This is only when $x = a$ is not an _______________________.

Minimum/Maximum at an endpoint:



- Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

- Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.

  We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = \_\_\_\_\_\_$$

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