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VCE Mathematical Methods ¾
Family of Functions and its Exam Skills [2.6]
Homework Solutions

Admin Info & Homework Outline:

Student Name	
Questions You Need Help For	
Compulsory Questions	Pg 2 – Pg 19

Section A: Compulsory Questions

Sub-Section [2.6.1]: Applying Family of Functions

Question 1

Consider the following family of functions $f(x) = e^{ax} - 1, a > 0$.

a. Identify the "surname" (common aspect(s) of the family) and the "first name" (unique aspect(s) of the family).

Surname: All will pass through $(0,0)$, all have a horizontal asymptote at $y = -1$.
First Name: Each graph has a different dilation from the y-axis.

b. Hence, state what happens to the graph of f in terms of a transformation when the value of a increases.

As the value of a increases, the graph of f is dilated less from the y-axis, that is f moves closer to the y-axis as the value of a increases.

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Question 2

Consider the following family of functions $f(x) = (x-2)^2 + k, k \in \mathbb{R}$.

a. Show that the graph of f always has a stationary point at $x = 2$ and find the nature of this stationary point.

$f'(x) = 2(x-2) = 0$, solve for x:

$x - 2 = 0 \quad \therefore x = 2 \quad \therefore$ S.P at $x = 2$

x	1	2	3
$f'(x)$	-2	0	2
	\	—	/

$\therefore$ local minimum at $x = 2$

b. Hence, identify the "surname(s)" and "first name(s)" of the family.

Surname: Local minimum at $x = 2$, same shape.
First name: Differing vertical translations $\therefore$ different y and x-intercepts.

Question 3

Consider the family of functions $f(x) = \sin\left(kx + \frac{\pi}{2}\right), k \in \mathbb{R}$.

a. Identify the effects of k on the graph.

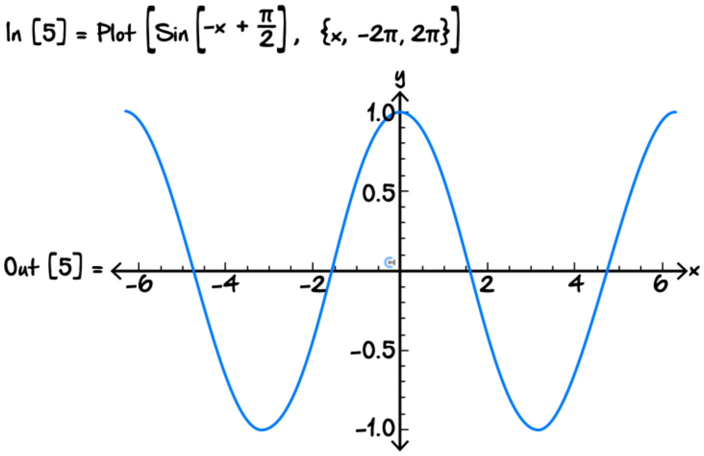
Increasing k will decrease the period of $f(x)$, and vice versa (decreasing k will increase the period of $f(x)$.

b. Hence, identify the "surname" and "first name(s)" of the family.

Surname: y-intercept at $(0,1)$ since $f(0) = \sin\left(\frac{\pi}{2}\right) = 1$ for all $k \in \mathbb{R}$, same general shape.
First name: Differing periods $\therefore$ differing dilations from the y-axis.

c. Express $f(x)$ without using sin when $k = \pm 1$.

Hint: List out the transformations and sketch the resulting graph if you get stuck!



$f(x) = \cos(x)$. Translating the graph of $\sin(x)$ $\frac{\pi}{2}$ units to the left results in $\cos(x)$, which is an even function ($f(-x) = f(x)$) therefore a reflection in the y-axis when $k = -1$ results in the same graph.

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Question 4 Tech-Active.

Consider the family of functions $f(x) = e^{ax} - ax + 1, a \in \mathbb{R}^+$.

a. State one transformation that maps the graph of $g(x) = e^x - x + 1$ onto the graph of $f(x)$.

Dilation by a factor of $\frac{1}{a}$ from the y-axis.

b. Identify a "surname" of the family.

y-intercept at $(0,2)$.

c. Describe what happens to the shape of $f(x)$ as a increases.

f dilates closer to the y-axis, which results in the graph curving upwards for positive values of x, and the graph approaches a linear shape for negative values of x.

d. By plotting $h(x) = f(x) - e^{ax}$ on the same axes as $f(x)$, state the equation of the asymptote of $f(x)$.

$y = -ax + 1$

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Sub-Section [2.6.2]: Finding Unknowns for a Certain Number of Intersections

Question 5

Find the value of a where $a \in \mathbb{R}$ such that the graph of $f(x) = e^x + a$ intersects the line $y = x$ exactly once.

$f(x) = x$ ① $f'(x) = 1$ ②

$\Rightarrow e^x + a = x$ $e^x = 1$ $\therefore x = 0$

$\Rightarrow 1 + a = 0$, $\therefore a = -1$

Question 6

Consider the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = e^x + a$ where $a \in \mathbb{R}$ and $g(x) = x - 3$.

a. Find the inverse function of $f(x)$.

$f(y) = x$ where $y = f^{-1}(x)$, solve for y:

$e^y + a = x \Rightarrow e^y = x - a \therefore y = \log_e(x - a)$

b. Find the value of a such that the graph of $f^{-1}(x)$ intersects with $g(x)$ exactly once.

$\log_e(x-a) = x-3$ ① $\frac{dy}{dx} = \frac{1}{x-a} = g'(x) = 1$ ②

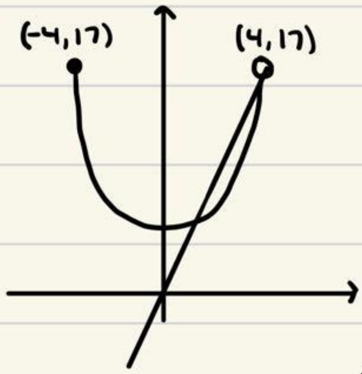
from ②: $x - a = 1$ $\therefore$ $x = a+1$, sub into ①:

$\log_e(a+1-a) = a+1-3$

$\Rightarrow \log_e(1) = a-2 \Rightarrow 0 = a-2$ $\therefore a = 2$

Question 7

Consider the functions $f: [-4,4) \to \mathbb{R}, f(x) = x^2 + 1$ and $g(x) = mx, m \in \mathbb{R}$. Find the value(s) of m where f and g intersect exactly once.



Case 1: $f(x) = g(x)$, $f'(x) = g'(x)$

$x^2 + 1 = mx$ $2x = m$

$\Rightarrow x^2 + 1 = 2x^2$

$\Rightarrow x^2 - 1 = 0, x = \pm 1$ $\therefore m = \pm 2$

Case 2: $g(4) \geq 17 \Rightarrow 4m \geq 17$ $\therefore m \geq \frac{17}{4}$

Case 3: $g(-4) > 17 \Rightarrow -4m > 17$ $\therefore m < \frac{17}{4}$

$\therefore m \geq \frac{17}{4}$ or $m < -\frac{17}{4}$ or $m = 2$ or $m = -2$

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Question 8 Tech-Active.

Consider the function $f: [k, \infty) \to \mathbb{R}, f(x) = (x-k)^2 + 4$. Find the value(s) of k such that $f(x)$ and its inverse intersect exactly once.

```
In[28]:= Solve[(y - k)^2 + 4 == x, y]
Out[28]= {{y → k - √(-4 + x)}, {y → k + √(-4 + x)}}

(*domain of f is x≥k ∴ range of f^-1 must be y≥k so take +ve solution*)

In[36]:= Solve[k + √(-4 + x) == x, x]
Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[36]= {{x → 1/2 (1 + 2 k - √(-15 + 4 k))}, {x → 1/2 (1 + 2 k + √(-15 + 4 k))}}

(*one solution must be when these points are equal*)

In[39]:= Solve[4 k - 15 == 0, k]
Out[39]= {{k → 15/4}}

In[40]:= (*there must also be only one intersection when the left solution is outside of the domain of f*)
         Reduce[1/2 (1 + 2 k - √(-15 + 4 k)) < k, k]
Out[40]= k > 4
```

$k \in (4, \infty) \cup \left\{\frac{15}{4}\right\}$ Use a slider to check your answers.

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Sub-Section [2.6.3]: Finding Unknowns for Maximums and Minimums

NOTE: This entire section can be done tech-active.

Question 9

For what value of $k \in \mathbb{R}$, will the function, $f(x) = 2x + e^{kx}$ have a minimum on the x-axis?

```
In[41]:= Solve[{2 x + E^(k x) == 0, D[2 x + E^(k x), x] == 0}, {k, x}]
Out[41]= {{k -> -2/e, x -> -e/2}}
```

$$k = -\frac{2}{e}$$

Question 10

For what value(s) of $k \in \mathbb{R}$ will the function $f(x) = (x-k)^2 \log_e(x)$ have a minimum at $x = 4$?

```
In[42]:= f[x_] := (x - k)^2 Log[x]
In[45]:= Solve[f'[4] == 0, k]
Out[45]= {{k -> 4}, {k -> 4 (1 + 2 Log[4])}}
```

Inserting these values into a slider shows a minimum when $k = 4$ but a maximum for the other solution of k $\therefore$ we must reject the other solution, leaving just $k = 4$. ALWAYS PLOT AFTER SOLVING JUST TO CHECK.

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Question 11

Find the value(s) of $k \in \mathbb{R}$ such that the minimum of the function $f: [-4,10] \to \mathbb{R}, f(x) = 3\left(\frac{x}{2} - 3k\right)^2 - 6$ occurs at $x = -4$.

```
In[72]:= Solve[3 (-2 - 3 k) == 0, k]
Out[72]= {{k → -2/3}}

(*Slider shows that minimum occurs for k≤2/3*)
```

$$\therefore k \leq -\frac{2}{3}$$

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Sub-Section: Exam 1 Questions

Question 12

Consider the function $f: [a, \infty) \to \mathbb{R}, f(x) = x^2 - 4x + k, k \in \mathbb{R}$, where a is a real constant that ensures f is a one-to-one function.

a. Find the smallest possible value of a.

$f(x) = (x-2)^2 - 4 + k$

T.P at $x = 2 \therefore a = 2$

b. Find the value of k such that $y = 4x$ is a tangent to the graph of f.

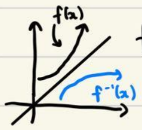
let $g(x) = 4x$

$f'(x) = 2x - 4 = g'(x) = 4$, solve for x:

$x = 4 \Rightarrow f(4) = g(4) = 16$, solve for k:

$k = 16$

c. Find the value(s) of k such that the graphs of f and f^{-1} do not intersect.



f increasing $\therefore$ must be above $y=x$ to not intersect

solve for x when $f(x)=x$:

$x^2-4x+k=x$

$\Rightarrow x^2-5x+k=0$

$\therefore x=\frac{5\pm\sqrt{25-4k}}{2}$

$\Rightarrow 25-4k<0$ for $f(x)$ to be above $y=x$

$4k>25 \;\therefore k>\frac{25}{4}$

Question 13

Consider the function $f(x)=ae^x+k$, where $a\in\mathbb{R}^+$ and $k\in\mathbb{R}$.

Let the point P be the y-intercept of the graph of $f(x)$.

a. State the coordinates of the point P in terms of a and k.

$(0, a+k)$

b. Find the gradient of f at P in terms of a.

$f'(x)=ae^x, f'(0)=a$

c. Given that the graph of $y = f(x)$ passes through the origin, express k in terms of a.

$$k = -a$$

d. Given that the graph of f also goes through the point $(1,3(e-1))$, find the values of a and k.

$$a = 3, k = -3$$

Question 14

Consider the function $f(x) = x^2 + ax + b$.

Find the value(s) of $a, b \in \mathbb{R}$ such that $f(x)$ has a turning point at $(1,4)$.

$$f'(x) = 2x + a$$
$$f'(1) = 0 \ \ [Stationary\ point\ at\ x = 1]$$
$$2(1) + a = 0, a = -2$$
$$Again\ f(1) = 4$$
$$1^2 + (-2)(1) + b = 4$$
$$b = 5$$

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Sub-Section: Exam 2 Questions

Question 15

The graph of $y = kx + 2$ intersects the graph of $y = 2x - 3x^2$ at 2 distinct points for:

A. $k = 2$

B. $k \in [2 - 2\sqrt{6}, 2 + 2\sqrt{6}]$

C. $k \in \left(-\infty, 2 - 2\sqrt{6}\right) \cup \left(2 + 2\sqrt{6}, \infty\right)$

D. $k \in (2 - 2\sqrt{6}, 2 + 2\sqrt{6})$

Question 16

The graph with rule $f(x) = x^3 - 3x^2 + c, c \in \mathbb{R}$, has 3 distinct x-intercepts.

The set of all possible values of c is:

A. $\mathbb{R}^+$

B. $[0,4]$

C. $(-4,0)$

D. $(0,4)$

Question 17

For the parabola with the equation $y = ax^2 - 2bx + c$, the equation of the axis of symmetry is:

A. $x = \frac{b}{a}$

B. $x = -\frac{b}{a}$

C. $y = c$

D. $x = \frac{2b}{a}$

$$Axis\ of\ symmtery\ at\ x = -\frac{B}{2A}\ for\ Ax^2 + Bx + C = 0$$

$$Axis\ of\ symmetry\ of\ \ ax^2 - 2bx + c = 0\ is\ at$$

$$x = -\frac{-2b}{2a} = \frac{b}{a}$$

Question 18

The largest value of a such that the function $f: (-\infty, a] \to \mathbb{R}, f(x) = x^2 - 5x + 6$, where f is one-to-one, is:

A. 2.5

B. 6

C. -2.5

D. -5

Question 19

The function $f(x) = \frac{1}{3}x^3 + mx^2 + nx + p$, for $m, n, p \in \mathbb{R}$, has turning points at $x = -3$ and $x = 1$ and passes through the point $(3,4)$.

The values of m, n and p are:

A. $m = 0, n = -\frac{7}{3}, p = 2$

B. $m = 1, n = -3, p = -5$

C. $m = -1, n = -3, p = 13$

D. $m = \frac{5}{2}, n = 6, p = -\frac{83}{4}$

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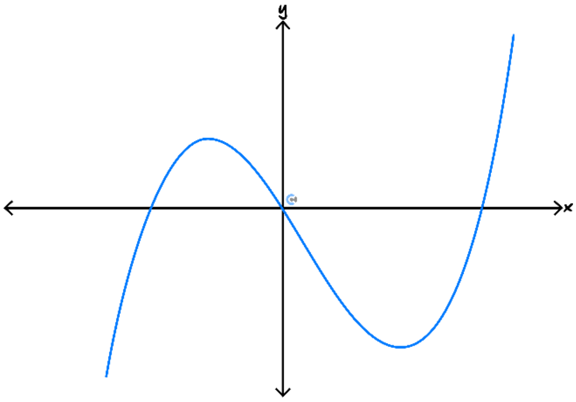
Question 20

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x(x-3)(x+2)$. Part of the graph of f is shown below:



Consider the point $A\ (a, f(a))$.

a. State the gradient of the graph of f at the point A in terms of a.

```
In[91]:= f'[a] // Expand
Out[91]= -6 - 2 a + 3 a^2
```

b. Hence, find the equation of the tangent to the graph of f at point A in the form $y = mx + c$.

```
In[95]:= Solve[y - f[a] == f'[a] (x - a), y]
Out[95]= {{y → a^2 - 2 a^3 - 6 x - 2 a x + 3 a^2 x}}
```

$$y = (3a^2 - 2a - 6)x - 2a^3 + a^2$$

c. Find the value(s) of a where the tangent to f at A intersects the graph of f once.

```
In[99]:= Solve[f''[a] == 0, a]
Out[99]= {{a → 1/3}}
```

d. Let $h: \mathbb{R} \to \mathbb{R}, h(x) = (x-a)(x-b)^2$, where $h(x) = f(x) + k$ and $a, b, k \in \mathbb{R}$.

Find the possible values of a and b.

```
In[105]:= (x - a) (x - b)^2 // Expand
Out[105]= -a b^2 + 2 a b x + b^2 x - a x^2 - 2 b x^2 + x^3

In[107]:= f[x] + k // Expand
Out[107]= k - 6 x - x^2 + x^3

          (*equating coefficients, we get*)
In[108]:= Solve[{-a - 2 b == -1, b^2 + 2 a b == -6}, {a, b}]
Out[108]= {{a -> 1/3 (1 + 2 Sqrt[19]), b -> 1/3 (1 - Sqrt[19])}, {a -> 1/3 (1 - 2 Sqrt[19]), b -> 1/3 (1 + Sqrt[19])}}
```

Must split both sets into separate cases.

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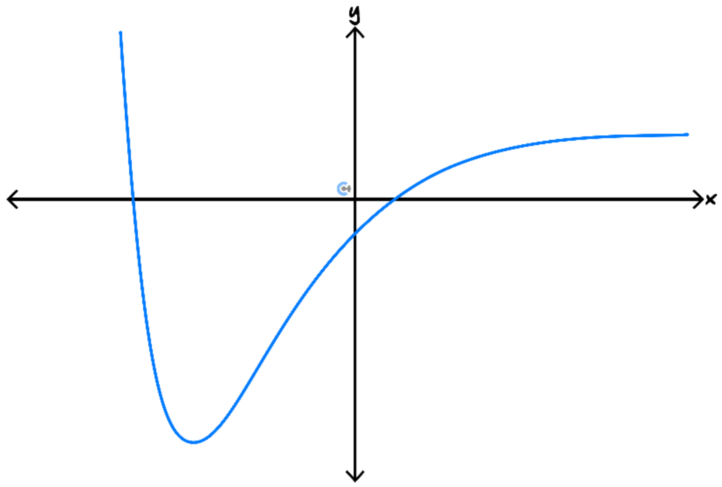
Question 21

Consider the function $p(x) = e^{-3x} - 4e^{-2x} + 2$.

a. Explain why $p(x)$ is not a one-to-one function.

Some y-values have multiple corresponding x-values/fails the horizontal line test/minimum turning point.

The diagram below shows part of the graph of p.



b. Find the gradient of the tangent to the graph of p at $x = a$.

```
p'[a]
```

$-3e^{-3a} + 8e^{-2a}$

c. Find the value(s) of a for which the tangent to f at $x = a$ intersects p only at $x = a$.

```
Reduce[{p''[a] > 0, p'[a] ≤ 0}, a, Reals] // N
```

$a \leq -0.980829$

Slider tells us that this happens when the gradient is negative or 0 and the graph is concave up.

d. Find the smallest value of $q \in \mathbb{R}$, such that $f: [q, \infty) \to \mathbb{R}, f(x) = p(x)$ is a one-to-one function.

```
Solve[D[E^(-3 x) - 4 E^(-2 x) + 2, x] == 0, x, Reals] // N
{{x → -0.980829}}
```

e. Consider the function $g: [-2,2] \to \mathbb{R}, g(x) = p(x + k)$.

i. Find the value(s) of k such that the minimum of p occurs at $x = -1$.

```
In[129]:= Solve[p'[k - 1] == 0, Reals]
Out[129]= {{k → 1 - 3 Log[2] + Log[3]}}
```

$\therefore k = 1 - 3\log_e(2) + \log_e(3)$

ii. Find the value(s) of k such that the minimum of p occurs at $x = -2$.

```
In[130]:= Solve[p'[k - 2] == 0, Reals]
Out[130]= {{k → 2 - 3 Log[2] + Log[3]}}
```

Using a slider shows that increasing the value of k beyond this moves the turning point outside of the domain thus the minimum will still be at the endpoint $\therefore k \geq 2 - 3\log_e(2) + \log_e(3)$.

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