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VCE Mathematical Methods ¾
Applications of Differentiation Exam Skills [2.5]
Workbook

Outline:

Learning Objectives:

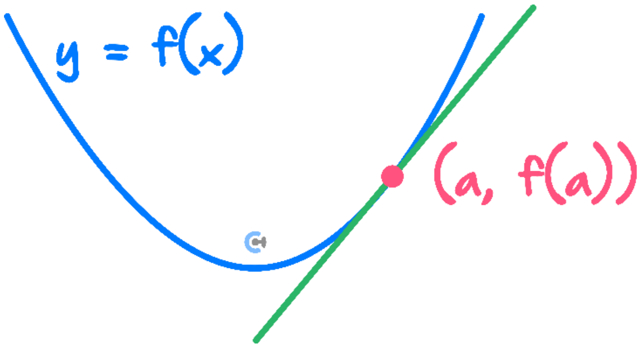
- ☐ **MM34 [2.5.1]** - Advanced Tangents and Normal Questions
- ☐ **MM34 [2.5.2]** - Advanced Maximum/Minimum Questions

Section A: Recap

If you were here last week skip to Section B Warmup Test.

Tangents

- A **tangent** is a linear line which **just touches** the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$m_{tangent} = f'(a)$$

Question 1

Find the equation of the tangent to $f(x) = x^2 + 4$ at $x = 2$.

Calculator Commands: Finding tangents on CAS

➤ **Mathematica**

```
<< SuiteTools`
TangentLine[f[x], x, a]
```

➤ **TI-Nspire**

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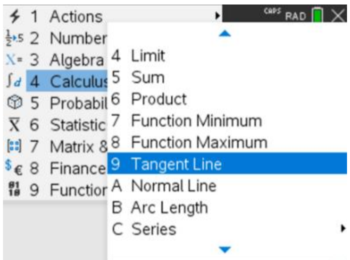
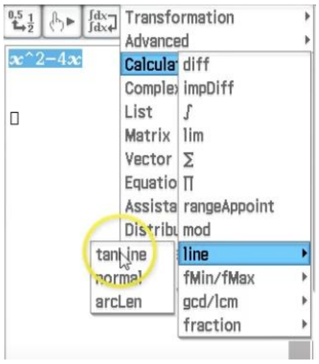


tangentLine$(f(x), x, a)$

➤ **Casio Classpad**



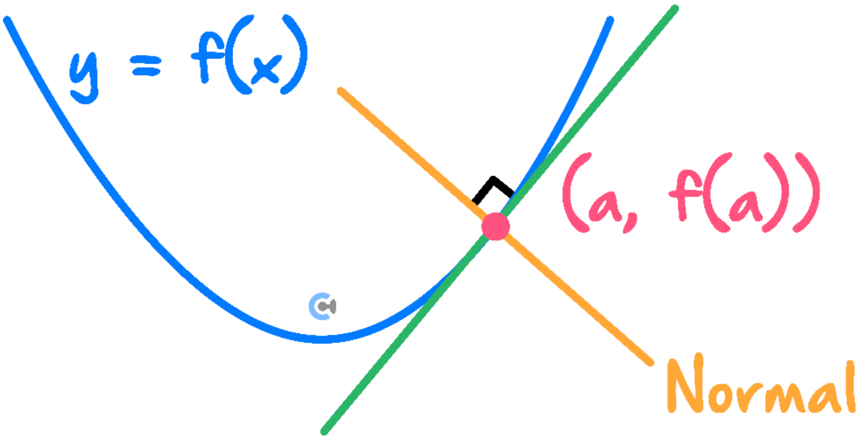
tangentLine$(f(x), x, a)$

Question 2 Tech-Active.

Find the equation of the tangent to $y = \sin(x)$ at $x = \frac{\pi}{6}$.

Normals

- A **normal** is a linear line which is **perpendicular** to the tangent.
- The gradient of a normal line has to be equal to the **negative reciprocal** of the gradient of the curve at the intersection.



$$m_{normal} = -\frac{1}{f'(a)}$$

Question 3

Find the equation of the normal to $f(x) = x^3 - 3x^2 + 5$ at $x = 1$.

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Calculator Commands: Finding normals on CAS

Mathematica

```
<< SuiteTools`

NormalLine[f[x], x, a]
```

TI-Nspire

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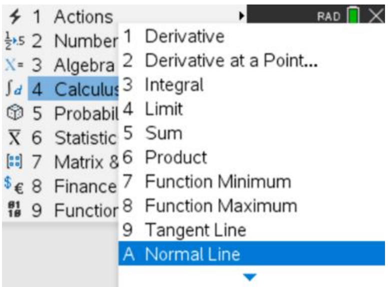
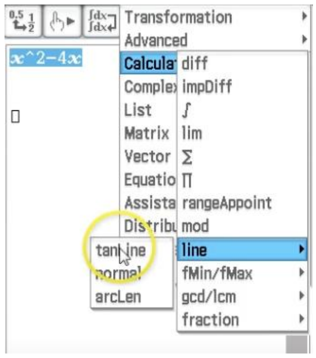


normalLine$(f(x),x,a)$

Casio Classpad



normalLine$(f(x),x,a)$

Question 4

Find the equation of the normal to $y = e^{2x}$ at $x = 0$.

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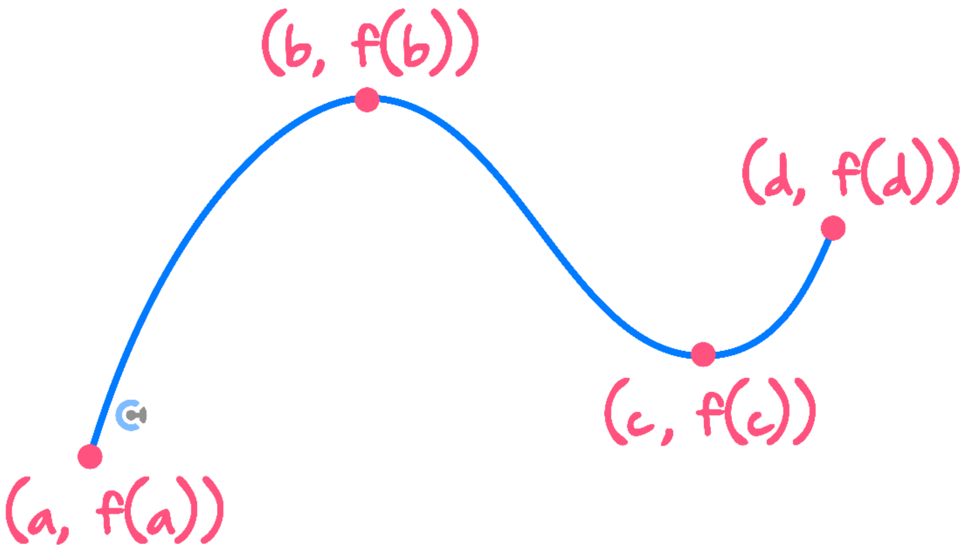
Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** y-values for the given domain.
- **They occur at either an endpoint** or a **turning point.**



Absolute Min: $f(a)$

Absolute Max: $f(b)$

- **Steps**

1. Find stationary points and endpoints.
2. Find the largest/lowest y-value for max/min.

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Question 5

Find the maximum and minimum value of the function given below.

$$f: [-2,2] \to R, f(x) = x^3 - 3x + 4$$

NOTE: Find the endpoints and the turning points. Pick the largest y value for max and the smallest y-value for the min.

Calculator Commands: Finding Absolute Max and Min for $x \in [a, b]$

Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]
```

```
Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

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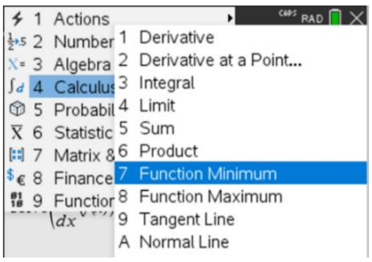
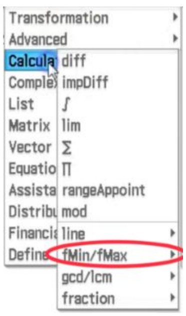


$\text{fMax}(f(x), x, a, b)$

$\text{fMin}(f(x), x, a, b)$

Casio Classpad



$\text{fMax}(f(x), x, a, b)$

$\text{fMin}(f(x), x, a, b)$

Question 6 Tech-Active.

Find the maximum and minimum value of the function given below.

$$f: [-5,2] \to R, f(x) = 2x^3 + 3x^2 - 72x - 16$$

Definition

Optimisation Problems

- Applying absolute maxima and minima in a real-world setting.
- Steps
  1. Construct a function for the subject you want to find the maximum or minimum of.
  2. Find its domain if appropriate.
  3. Find its endpoints and turning points.
  4. Identify the maximum or minimum y-value.

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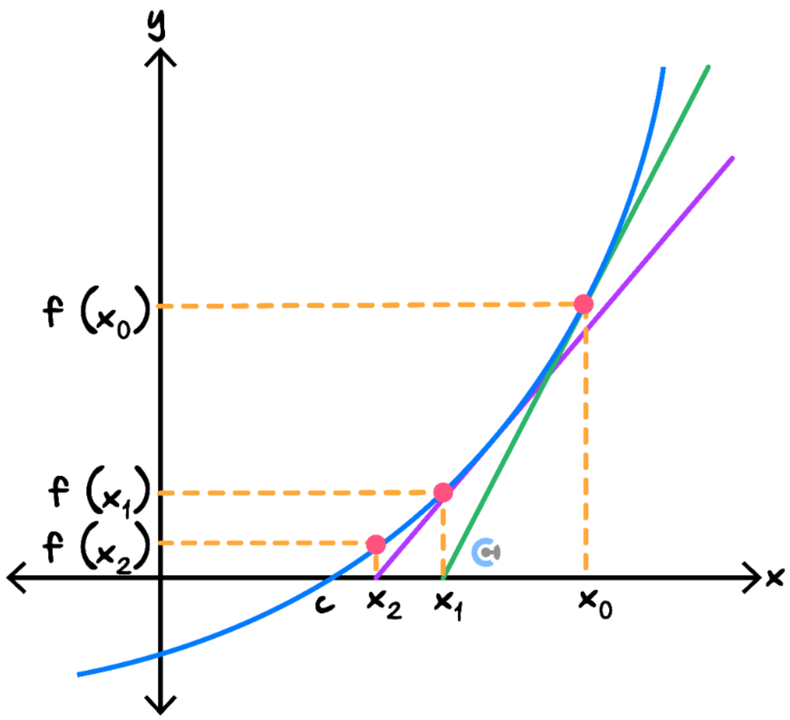
Question 7

Find the minimum vertical distance between the functions $f(x) = x^2 + 16$ and $g(x) = x - 10$.

Newton's Method

- It is a method of approximating the x-intercept using **tangents.**



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Steps

1. Find the tangent at the x-value given.
2. Find the x-intercept of the tangent using an iterative formula.
3. Find the next tangent at the $x = x$-intercept of the previous tangent.
4. Repeat until the value doesn't change by much.

Question 8 Walkthrough. Tech-Active.

Consider the function $f(x) = x^3 - 5$.

a. Show that Newton's method gives the iterative formula $x_{n+1} = \frac{2x_n^3+5}{3x_n^2}$.

b. Find the approximation of x-intercept of $f(x)$ giving your answer correct to 4 decimal places. Start $x_0 = 2$.

Definition

Tolerance

- **Definition**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\text{We stop when } |x_{n+1} - x_n| < \text{Tolerance}.$$

- The question will give us the tolerance level.

Question 9 Tech-Active.

Find the root of $f(x) = x^3 - 12$ using Newton's method with a tolerance of 0.01 and initial value of 3. Give your answer correct to two decimal places.

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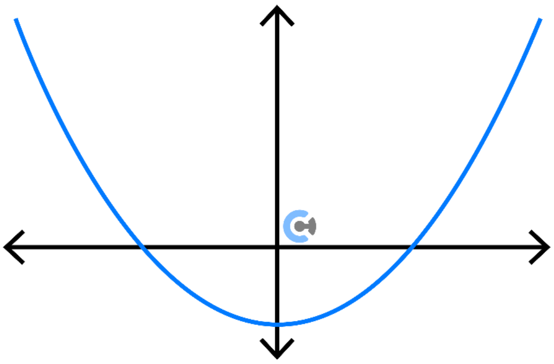
When does Newton's method not work?

Exploration: 1. Terminating Sequence

- Consider the graph below.

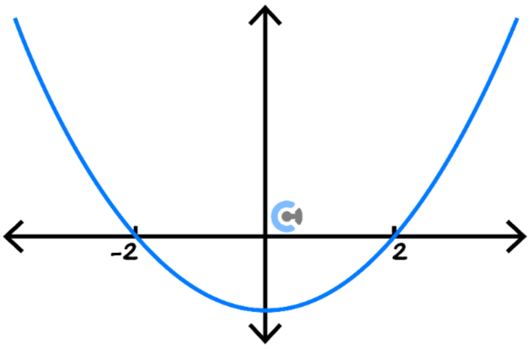


- What happens when $x_0 = 0$?

- Hence, when Newton's method reaches the x-value of the stationary point, it ________________.

Exploration: 2. Approximating the Wrong Root

- Consider the graph below and say we wanted to approximate the x-intercept on the **right** ($x = 2$).

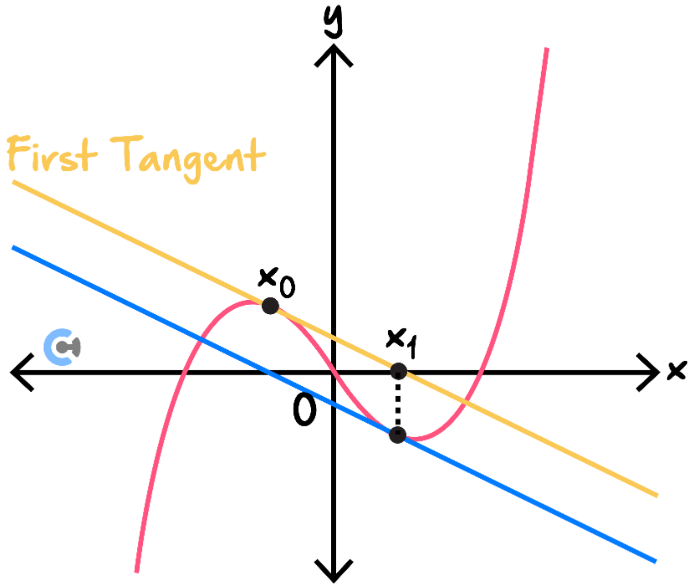


- What happens when $x_0 = -1$, do we approximate $x = -2$ or $x = 2$? [$-2/2$]

- When Newton's method starts at the wrong side of the turning point, it can approximate the ____________________.

Exploration: 3. Oscillating Sequence

- Consider the graph.



- What is the value of x_2, the next term of the sequence?

- Hence, would Newton's method approximation get closer to the actual x-intercept at $x = 0$? [Yes/No]

Limitation of Newton's Method

- **Terminating Sequence**: Occurs when we hit a stationary point.

- **Approximating a Wrong Root**: Occurs when we start on the wrong side.

- **Oscillating Sequence**: Occurs when we oscillate between two values without getting closer to the real root.

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Question 10 Tech-Active.

Consider the function $f(x) = \frac{x}{\sqrt{x^2+1}}$.

a. Perform three iterations of Newton's method with starting point $x = 1$.

b. What limitation of Newton's method did we get here?

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Section B: Warm Up Test (15 Marks)

INSTRUCTION: 15 Marks.15 Minutes Writing.

Question 11 (2 marks)

Find the equation of the line that is normal to $y = x^3 - 4x^2 + 6x$ at $x = 1$.

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Question 12 (3 marks)

Consider the function $f: (1, \infty) \to R, f(x) = \frac{4}{x-1}$. Find the equation of the line tangent to f that is parallel to $y = 8 - x$.

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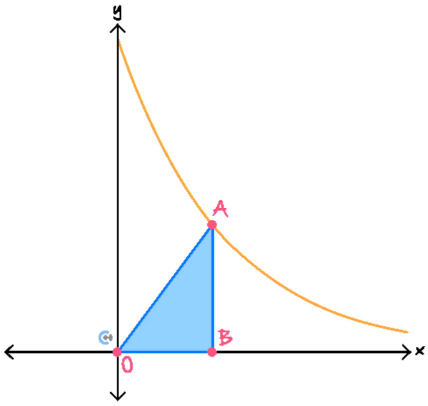
Question 13 (4 marks)

Consider the function $f(x) = e^{-3x}, x > 0$.



A triangle has vertices OAB, as shown in the diagram below, with coordinates $O(0,0)$, $A(a, f(a))$, and $B(a, 0)$, where $a > 0$.

a. Find the area A of triangle OAB in terms of a. (1 mark)

b. Find the value of a for which A is maximised. (2 marks)

c. Hence, find the maximum area of OAB. (1 mark)

Question 14 (6 marks)

Consider the function $f(x) = x^3 - 3x$. Newton's method is used to approximate a root of this function.

a. Using Newton's method, an expression for x_{n+1} is:

$$x_{n+1} = \frac{ax_n^3}{b(x_n^2 - 1)}$$

Find the values of a and b. (3 marks)

b. Explain why $x_0 = 1$ will be a bad starting point. (1 mark)

c. Find x_1 if $x_0 = 2$. (1 mark)

d. Write down the surd that Newton's method approximates when used on the function f with $x_0 = 2$. (1 mark)

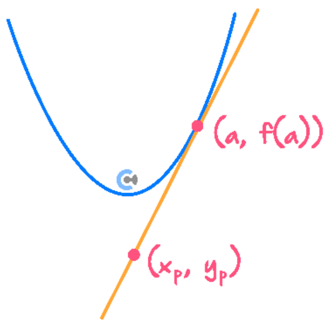
Section C: Application of Differentiation Exam Skills

Sub-Section: Finding Tangents/Normals Which Passes Through a Point

Discussion: If the question didn't give us the point where the tangent/normal was made but rather a random point it passes through, how do we solve for the tangent/normal?

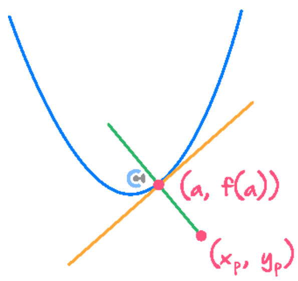
Finding Tangents/Normals to Functions, which also pass through a given point

- Tangent of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$f'(a) = \frac{f(a) - y_p}{a - x_p}$$

- Normal of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$-\frac{1}{f'(a)} = \frac{f(a) - y_p}{a - x_p}$$

NOTE: We simply equate the gradient (rise/run) with the gradient of tangents/normals.

Question 15 Walkthrough.

Find the tangent(s) of $f(x) = x^2 + 4$ that pass(es) through $(0,0)$.

Question 16

Find the equation of the tangent to $f(x) = e^x$ which passes through $(0,0)$.

Question 17

Find the equation of the normal to $f(x) = \log_e(x - 1)$ which passes through $(0,2)$.

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Sub-Section: Finding Tangents/Normals with Coordinate Geometry

REMINDER: Parallel and Perpendicular Lines

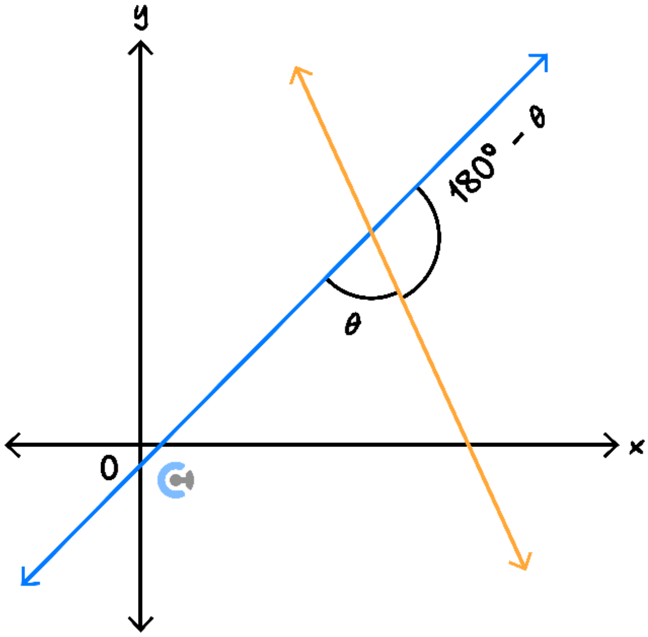
$$m_1 = m_2$$

- Parallel lines have the same gradient.

$$m_1 = -\frac{1}{m_2}$$

- Gradients of perpendicular lines are negative reciprocals of each other.

REMINDER: Acute Angle Between Two Lines



$$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)|$$

- **Alternatively:**

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Definition

Finding Tangents/Normals with Coordinate Geometry

1. Using derivative = gradient of the tangent, find the point where the tangent is made.
2. Find the tangent.

Question 18 Walkthrough.

Find the tangent of $f(x) = x^2 + 4x + 6$ that is parallel to the line $y = 2x + 2$.

NOTE: Gradient of the tangent is always equal to the derivative.

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Question 19

Find the tangent of $f(x) = -x^2 + 3x + 2$ that is perpendicular to the line $y = \frac{1}{3}x + 3$.

Question 20

Find the tangent of $f(x) = \frac{1}{2}x^2 + 2x + 3$ that makes an angle of 60 degrees with the positive x-axis.

Question 21 Extension. Tech Active.

Find the equation of the tangent to $f(x) = -\frac{1}{2}x^2 + 4x + 2$ that makes an angle of 60 degrees with the line $y = x + 3$ and has a gradient less than -1. Give your answer correct to two decimal places.

Space for Personal Notes

Sub-Section: Finding Maximum/Minimum Instantaneous Rate of Change

Definition

Finding Maximum/Minimum Instantaneous Rate of Change

- Find the turning point of the ______________________________ function.

Question 22 Walkthrough.

Find the maximum instantaneous rate of change of $f(x) = -x^3 + 3x^2 + 4x + 3$.

Question 23

A hill is modelled by the function $f: [0, 3] \to R, f(x) = x^3 - 6x^2 + 8x + 4$. Find the value of x for which the gradient of the hill is smallest and state this smallest gradient.

Section D: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 25 Minutes Writing.

Question 24 (3 marks)

Newton's method is used to estimate the x-intercept of the function $f(x) = \frac{1}{3}x^3 + 4x + 6$.

a. Verify that $f(-1) > 0$ and $f(-2) < 0$. (1 mark)

b. Using an initial estimate of $x_0 = -1$, find the value of x_1. (2 marks)

Space for Personal Notes

Question 25 (5 marks)

Let P be the point on the straight line $y = 2x - 6$ such that the length of OP, the line segment from the origin O to P, is a minimum.

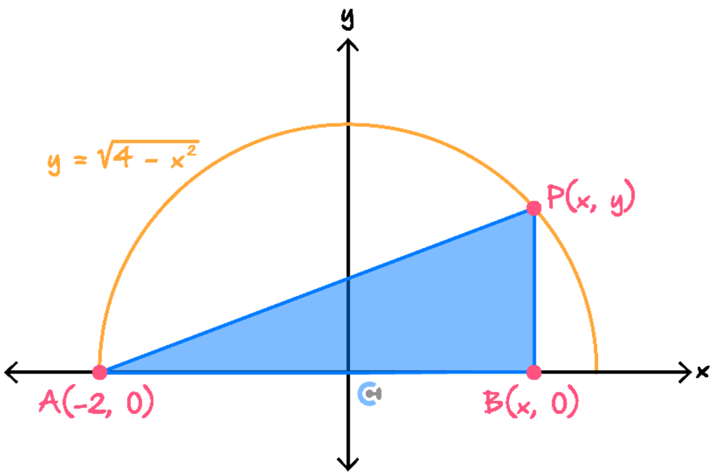
a. Find the coordinates of P. (3 marks)

b. Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers. (2 marks)

Space for Personal Notes

Question 26 (4 marks)

The graph of the relation $y = \sqrt{4 - x^2}$ is shown on the axes below. P is a point on the graph of this relation, A is the point $(-2, 0)$ and B is the point $(x, 0)$.

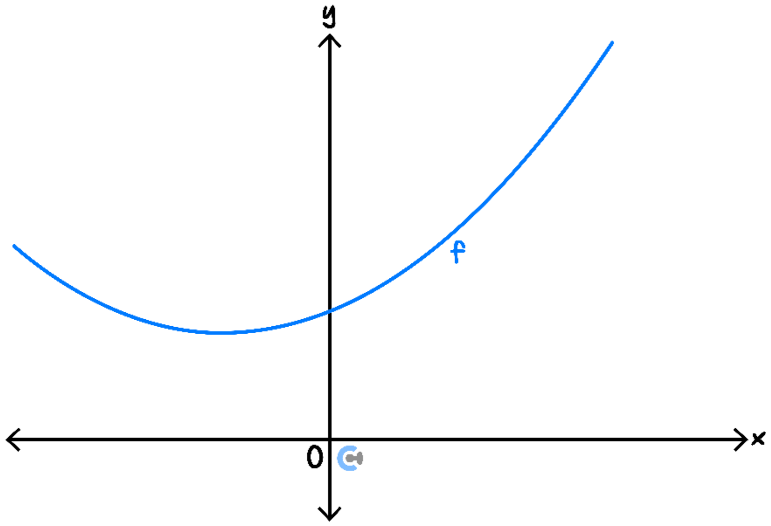


a. Find an expression for the length PB in terms of x only. (1 mark)

b. Find the maximum area of the triangle ABP. (3 marks)

Question 27 (8 marks)

Consider the function $f(x) = x^2 + 2x + 6$ and the point $P(1, 0)$. Part of the graph of $y = f(x)$ is shown below.



a. Show that point P is not on the graph of $y = f(x)$. (1 mark)

b. Consider a point $Q\big(a, f(a)\big)$ to be a point on the graph of f.

i. Find the slope of the line connecting points P and Q in terms of a. (1 mark)

ii. Find the slope of the tangent to the graph of f at point Q in terms of a. (1 mark)

iii. Let the tangent to the graph of f at $x = a$ pass through point P. Find the values of a. (2 marks)

iv. Give the equation of the line with a positive gradient and passing through point P that is tangent to the graph of f. (1 mark)

c. Find the value, k, that gives the shortest possible distance between the graph of the function $y = f(x - k)$ and point P. Also, state this minimum distance. (2 marks)

Section E: Tech-Active Exam Skills

Sub-Section: Finding Tangents/Normals

Calculator Commands: Finding tangents on CAS

Mathematica

```
<< SuiteTools`
TangentLine[f[x], x, a]
```

TI-Nspire

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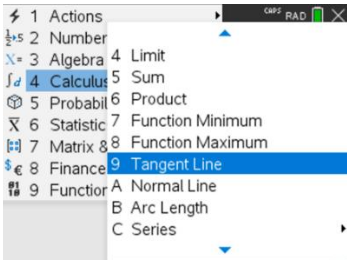
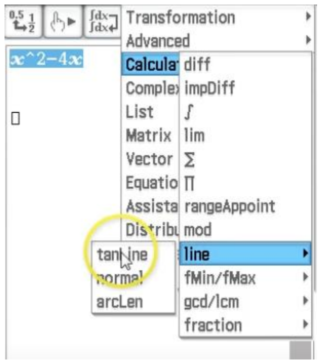


tangentLine$(f(x),x,a)$

Casio Classpad



tangentLine$(f(x),x,a)$

Calculator Commands: Finding normals on CAS

Mathematica

```
<< SuiteTools`
NormalLine[f[x], x, a]
```

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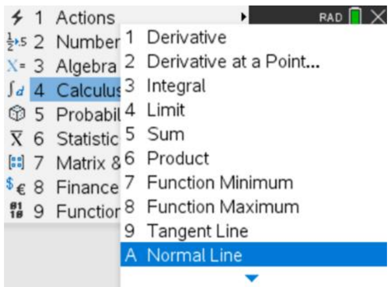

normalLine$(f(x),x,a)$

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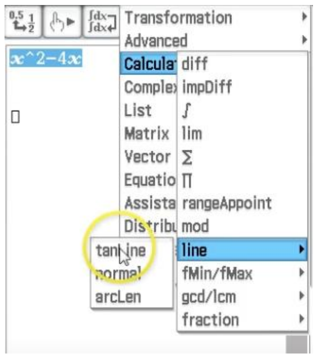

normalLine$(f(x),x,a)$

Question 28

Find the tangent to $f(x) = x^3 + 2x - 3$ when $x = 1$.

Space for Personal Notes

Sub-Section: Finding Minimum/Maximum

Calculator Commands: Finding Absolute Max and Min for $x \in [a, b]$

Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]

Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

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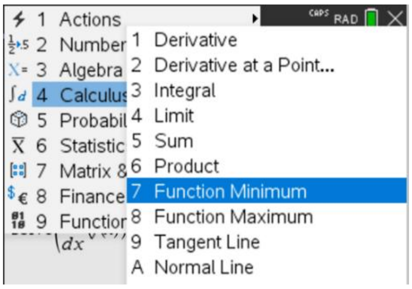
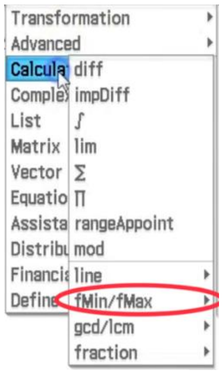


$$\text{fMax}(f(x), x, a, b)$$

$$\text{fMin}(f(x), x, a, b)$$

Casio Classpad



$$\text{fMax}(f(x), x, a, b)$$

$$\text{fMin}(f(x), x, a, b)$$

Space for Personal Notes

Question 29

Consider the function $f: [-2, 3] \to R, f(x) = x^3 - 4x^2 - 6x + 4$.

a. Find the maximum and minimum values of f.

b. Find the maximum and minimum gradient of f.

Space for Personal Notes

Sub-Section: Newton's Method

Calculator Commands: Newton's Method on Technology

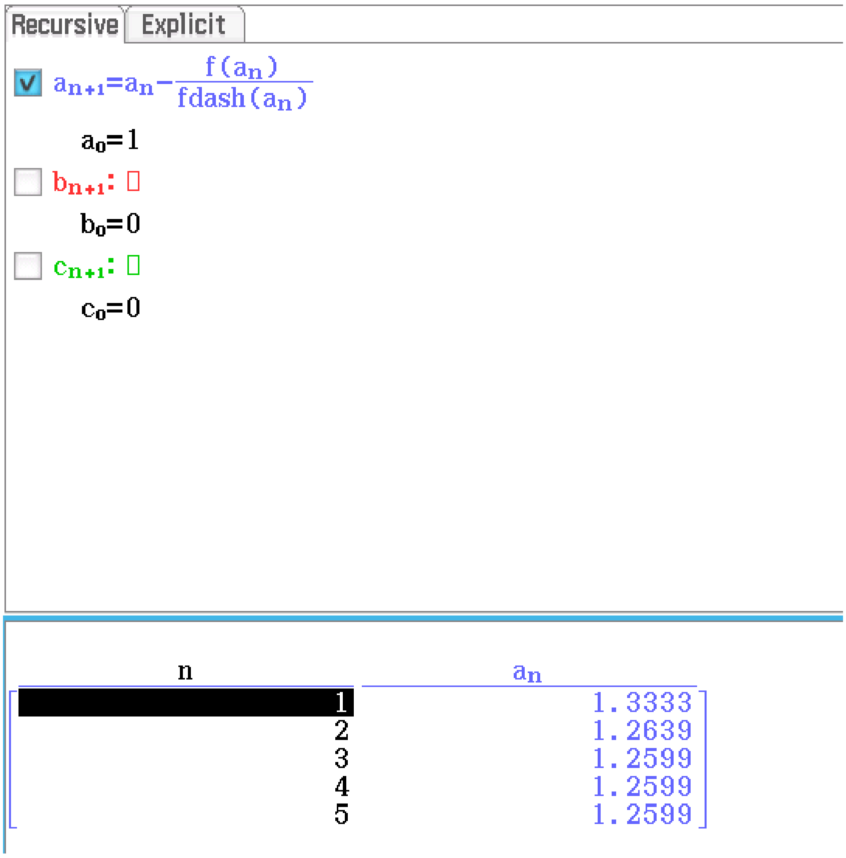
- Consider finding a root to $f(x) = x^3 - 2$ with initial value $x_0 = 1$.
- **Mathematica.**

```
In[531]:= f[x_] := x^3 - 2

In[533]:= n[x_] := x - f[x]/f'[x]

In[534]:= n[x] // Together

Out[534]= 2 (1 + x^3)/(3 x^2)

In[537]:= For[i = 1; x = 1, i < 5, i++, x = 2 (1.0 + x^3)/(3 x^2); Print[x]]

1.33333
1.26389
1.25993
1.25992
```

- **TI.** Define the $n(x)$ function then keep iterating by putting your previous value back into $n(x)$.

Define $f(x)=x^3-2$	*Done*
$x-\dfrac{f(x)}{\frac{d}{dx}(f(x))}$	$\dfrac{2\cdot(x^3+1)}{3\cdot x^2}$
Define $n(x)=\dfrac{2\cdot(x^3+1)}{3\cdot x^2}$	*Done*
$n(1)$	1.33333
$n(1.3333333333333)$	1.26389
$n(1.2638888888889)$	1.25993

- **Classpad.**
  - Under Sequences.



Question 30

Perform two iterations of Newton's method to approximate a root of $f(x) = x^3 - 7$ with the initial value $x_0 = 2$.

Sub-Section: Finding Tangents/Normals Which Passes Through a Point

Calculator Commands: Suppose we want to find the equation of a tangent/normal to the graph of $f(x)$ that passes through the point $P(x_1, y_1)$.

- Steps:
  1. Find the equation of the tangent to $f(x)$ at arbitrary point $x = a$.
  2. Let this tangent line be $t(x)$.
  3. Solve the equation $t(x_1) = y_1$ to find possible value(s) of a.
  4. Find the equation of the tangent at $x = a$.
- Similar procedure for the normal line.
- **Example:** Find the equation of a tangent to $f(x) = x^3 - 2x$ that passes through the point $(0,2)$.

```
In[564]:= f[x_] := x^3 - 2 x

In[565]:= TangentLine[f[x], {x, a}]
Out[565]= -2 a^3 + (-2 + 3 a^2) x

In[566]:= t[x_] := -2 a^3 + (-2 + 3 a^2) x

In[568]:= Solve[t[0] == 2, a, Reals]
Out[568]= {{a → -1}}

In[570]:= t[x] /. a → -1
Out[570]= 2 + x

In[571]:= TangentLine[f[x], {x, -1}]
Out[571]= 2 + x
```

Space for Personal Notes

Question 31

Find the equation of a tangent to $f(x) = x^3 - 2x$ that passes through the point $(2,0)$ and has y-intercept between $y = -3$ and $y = 0$.

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Sub-Section: Finding x_0 Values for an Oscillating Sequence

Newton's Method: Finding x_0 values which result in an oscillating sequence.

- **Step 1:** Define $n(x) = x - \frac{f(x)}{f'(x)}$
- **Step 2:** Solve $n(n(x)) = x$
- **Step 3:** Reject the roots of the function.

Question 32 Walkthrough (Tech Active)

Find the x_0 values that give an oscillating sequence when trying to find a root to the function $f(x) = x^3 - 5x$ using Newton's method.

Space for Personal Notes

Question 33 (Tech Active)

Find the x_0 values that give an oscillating sequence when trying to find a root to the function $f(x) = x^3 - 12x$ using Newton's method.

Space for Personal Notes

Section F: Exam 2 (25 Marks)

INSTRUCTION: 25 Marks. 30 Minutes Writing.

Question 34 (1 mark)

Find the equation of the tangent to $f(x) = x^2 + 3x + 1$, which makes an angle of $45°$ with the positive x-axis.

A. $y = x + 1$

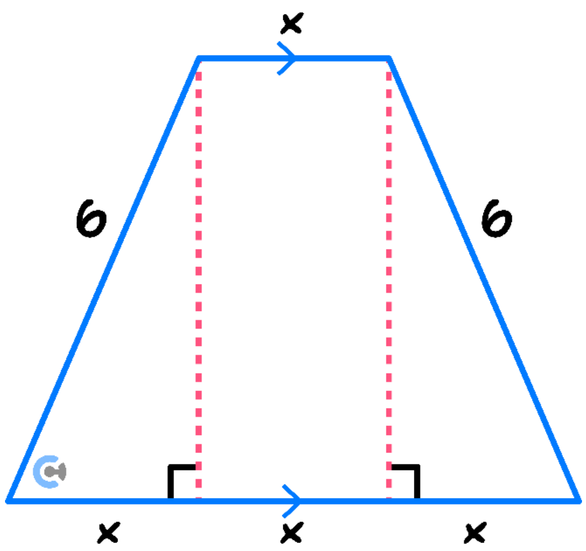
B. $y = x$

C. $y = -x - 1$

D. $y = 2x + 1$

Question 35 (1 mark)

Find the value of x that maximises the area of the trapezium below.



A. $3\sqrt{2}$

B. 3

C. 6

D. $4\sqrt{2}$

Question 36 (1 mark)

The tangent to the graph of $y = x^3 - ax^2 + 3$ at $x = 1$ passes through $(-2, 0)$. The value of a is:

A. $\frac{1}{2}$.

B. 1.

C. $\frac{3}{2}$.

D. 2.

Question 37 (1 mark)

The maximum instantaneous rate of change of the function $f : [-1, 2] \to \mathbb{R}, f(x) = -x^4 + 3x^2 + 2x$, correct to two decimal places is:

A. 0.70.

B. 2.53.

C. 4.83.

D. 4.84.

Question 38 (1 mark)

Newton's method is used to find a root of the function $f(x) = x^3 - 9x$. Which of the following values of x_0 will lead to an oscillating sequence?

A. $x_0 = 1$

B. $x_0 = \frac{3}{\sqrt{5}}$

C. $x_0 = \frac{2}{\sqrt{3}}$

D. $x_0 = \frac{5}{\sqrt{3}}$

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Question 39 (1 mark)

Extension. A box is formed from a rectangular sheet of cardboard, which has a width of $2a$ units and a length of b units, by first cutting out squares of side length x units from each corner and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when x is equal to:

A. $\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$.

B. $\frac{2a+b+\sqrt{4a^2-2ab+b^2}}{6}$.

C. $\frac{2a-b-\sqrt{a^2-ab+b^2}}{6}$.

D. $\frac{2a+b-\sqrt{4a^2-2ab+b^2}}{6}$.

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Question 40 (10 marks)

Let $h : \mathbb{R} \to \mathbb{R}, h(x) = 4^x - x^4$.

a. Determine the number of inflection points that h has. (1 mark)

b. Find the largest interval of x values for which h is strictly decreasing. Give your answer correct to two decimal places. (1 mark)

c. Apply Newton's method, with an initial estimate of $x_0 = 0$, to find an approximate x-intercept of h. Write the estimates x_1, x_2 and x_3 in the table below, correct to three decimal places. (2 marks)

x_0	0
x_1	
x_2	
x_3	

d. For the function h, explain why a solution to the equation $4^x \log_e(4) - 4x^3 = 0$ should not be used as an initial estimate for x_0 in Newton's method. (1 mark)

e. There is a positive real number n for which the function $f(x) = n^x - x^n$ has a local minimum on the x-axis.

Find this value of n. (2 marks)

f. **Extension.** Newton's method is used to approximate a root to the function $g(x) = x^3 - 7x$.

Find the initial values of x_0 that will result in an oscillating sequence. (3 marks)

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Question 41 (9 marks)

Let $f(x) = \log_e(x^2 - 4) - \log_e(2 - x)$.

a. State the maximal domain and the range of f. (2 marks)

b.

i. Find the equation of the tangent to the graph of f when $x = -3$. (1 mark)

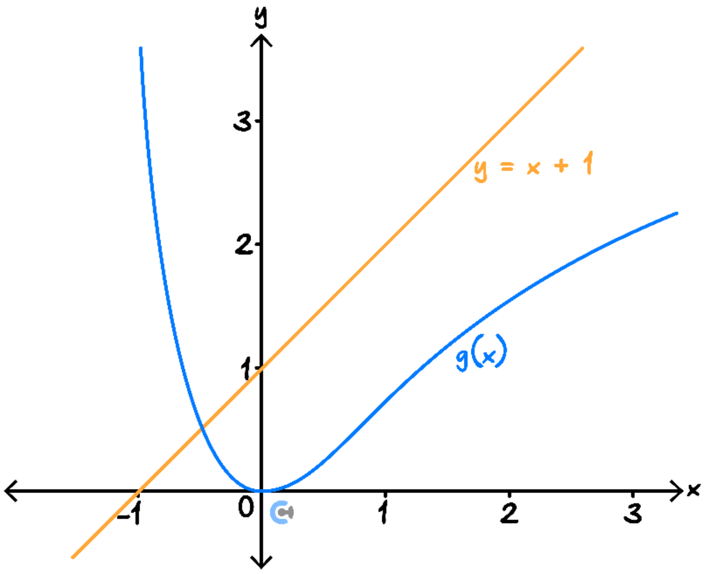
ii. Find the equation of the line that is perpendicular to the graph of f when $x = -3$ and passes through the point $(-3, 0)$. (1 mark)

Let $g(x) = e^{-2x} - 4e^{-x/2} + 3$.

c. Explain why g is not a one-to-one function. (1 mark)

d. Find the gradient of the tangent to the graph of g at $x = a$. (1 mark)

The diagram below shows parts of the graph of g and the line $y = x + 1$.



The line $y = x + 1$ and the tangent to the graph of g at $x = a$ intersect with an acute angle of θ between them.

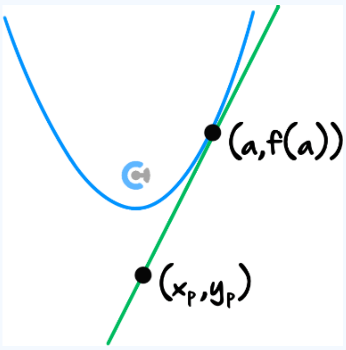
e. Find the value(s) of a for which $\theta = 60°$. Give your answer(s) correct to two decimal places. (3 marks)

Contour Checklist

☐ Learning Objective: [2.5.1] - Advanced Tangents and Normal Questions

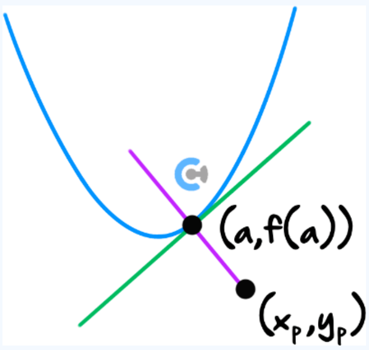
Key Takeaways

- ☐ Finding tangents/normals to functions, which also pass through a given point
- ☐ Tangent of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$f'(a) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- ☐ Normal of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$-\frac{1}{f'(a)} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Learning Objective: [2.5.2] - Advanced Maximum/Minimum Questions.

Key Takeaways

- To find the maximum/minimum instantaneous rate of change, we find the turning point of the ______________________________ function.

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