CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Applications of Differentiation [2.4]
Workbook

Outline:

Learning Objectives:

- ☐ **MM34 [2.4.1]** - Find Tangents and Normal
- ☐ **MM34 [2.4.2]** - Find Minimum and Maximum
- ☐ **MM34 [2.4.3]** - Apply Newton's Method to Find the Approximation of a Root and its Limitations

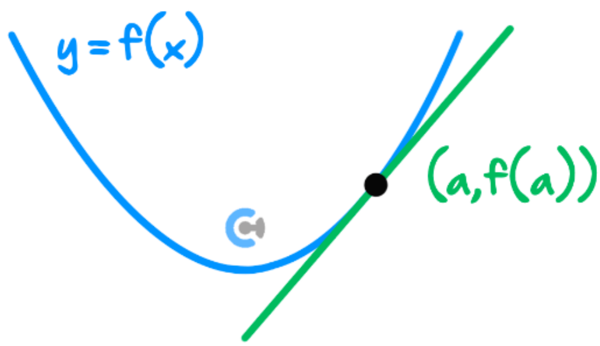
Section A: Tangents and Normals

Sub-Section: Tangents

What are tangent lines?

Tangents

- A **tangent** is a linear line which **just touches** the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$m_{tangent} = \_\_\_\_\_\_\_\_$$

Question 1 Walkthrough.

Find the equation of the tangent to $f(x) = x^2 + 4$ at $x = 2$.

Question 2

Find the equation of the tangent to $y = \sin(x)$ at $x = \frac{\pi}{6}$.

Space for Personal Notes

Question 3 Extension.

Let $f(x) = xe^x$, find the equation of the line tangent to $f(x)$ that has a gradient of $2e$.

Space for Personal Notes

How do we do this on our tech?

Calculator Commands: Finding tangents on CAS

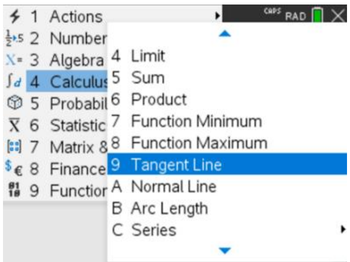
Mathematica

```
<< SuiteTools`

TangentLine[f[x], x, a]
```

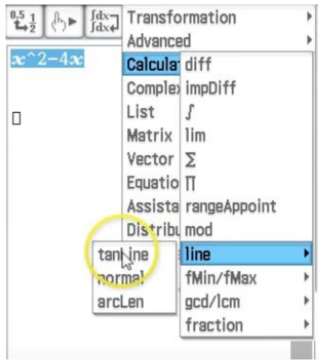
TI-Nspire

Menu 4 9



tangentLine$(f(x), x, a)$

Casio Classpad



tangentLine$(f(x), x, a)$

Question 4 Tech-Active.

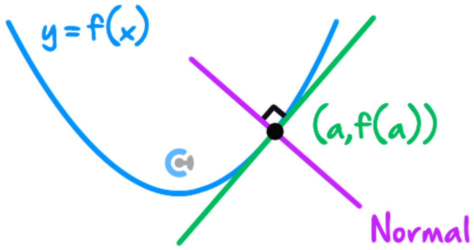
Find the equation of the tangent to $y = \sin(x)$ at $x = \frac{\pi}{6}$.

Sub-Section: Normal Lines

Discussion: Normal lines are perpendicular to tangent lines. How can we find the gradient of normal line?

Normals

- A **normal** is a linear line which is **perpendicular** to the tangent.
- The gradient of a normal line has to be equal to the **negative reciprocal** of the gradient of the curve at the intersection.



$$m_{normal} = \_\_\_\_\_\_\_\_$$

Question 5 Walkthrough.

Find the equation of the normal to $f(x) = x^3 - 3x^2 + 5$ at $x = 1$.

Question 6

Find the equation of the normal to $y = e^{2x}$ at $x = 0$.

Space for Personal Notes

Question 7 Extension.

Let $f(x) = \log_e(x+2)$. Find the equation of the normal to $f(x)$ that has a gradient of -1.

Space for Personal Notes

How do we do this on our tech?

Calculator Commands: Finding normals on CAS

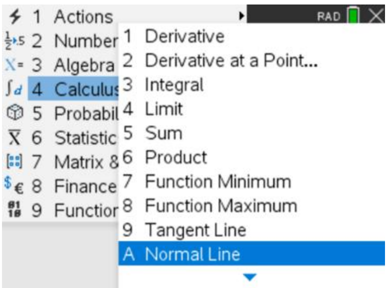
Mathematica

```
<< SuiteTools`

NormalLine[f[x], x, a]
```

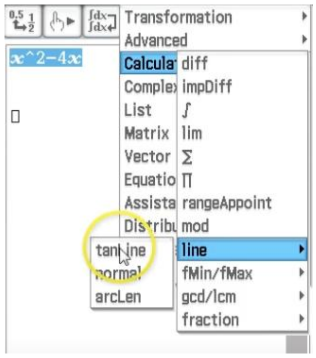
TI-Nspire

Menu 4 A



normalLine(f(x),x,a)

Casio Classpad



normalLine(f(x),x,a)

Question 8

Find the equation of the normal to $y = e^{2x}$ at $x = 0$.

Space for Personal Notes

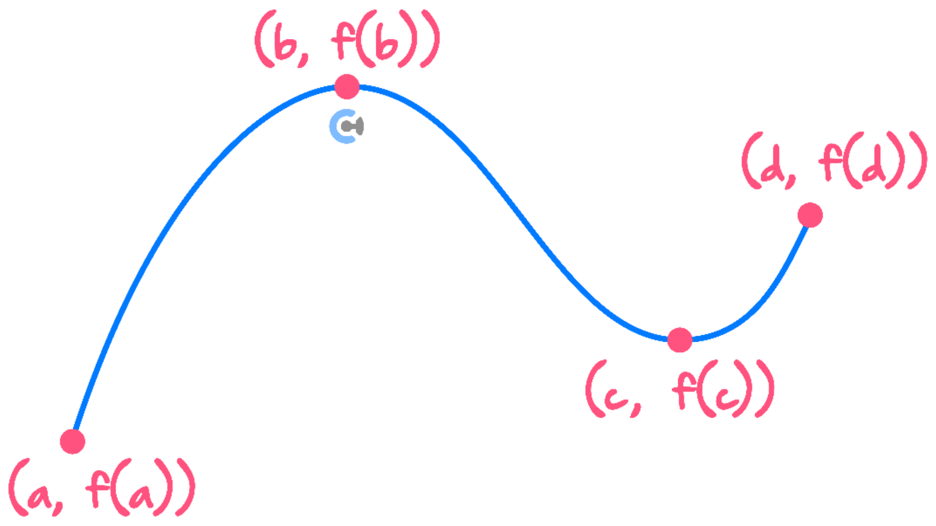
Section B: Optimisation

Sub-Section: Absolute Minimum and Maximum

Discussion: Where could the absolute maximum and minimum value lie for a function?

Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** ______________ for the given domain.
- They occur at either be an __________________, or a _________________________.



Absolute Min: $f(a)$

Absolute Max: $f(b)$

- Steps
  1. Find stationary points and endpoints.
  2. Find the largest/lowest y value for max/min.

Question 9 Walkthrough.

Find the maximum and minimum value of the function given below.

$$f: [-2,2] \to R, f(x) = x^3 - 3x + 4$$

NOTE: Find the endpoints and the turning points. Pick the largest y value for max and smallest y value for the minimum.

Active Recall: Steps for finding absolute minimum/maximum

1. Find ______________________ points and ________ points.

2. Find the ___________________ y value for max/min.

Space for Personal Notes

Question 10

Find the maximum and minimum value of the function given below.

$$f: [-1,3] \to R, f(x) = -12x - 3x^2 + 2x^3 - 1$$

Space for Personal Notes

Question 11 Extension.

Find the maximum and minimum value of the function given below.

$$f: \left(\frac{1}{2e}, e\right] \to R, f(x) = x\log_e(x)$$

Space for Personal Notes

Now on tech!

Calculator Commands: Finding Absolute Max and Min for $x \in [a, b]$

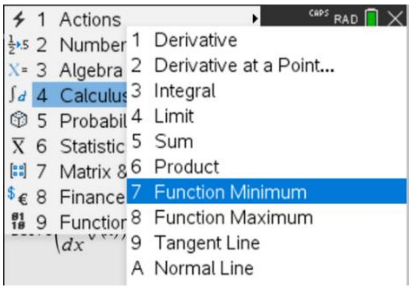
Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]

Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

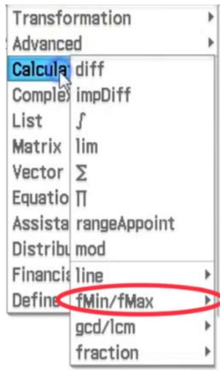
Menu 4 7 and Menu 4 8



$$\text{fMax}(f(x), x, a, b)$$

$$\text{fMin}(f(x), x, a, b)$$

Casio Classpad



$$\text{fMax}(f(x), x, a, b)$$

$$\text{fMin}(f(x), x, a, b)$$

Question 12 Tech-Active.

Find the maximum and minimum value of the function given below.

$$f: [-5,2] \to R, f(x) = 2x^3 + 3x^2 - 72x - 16$$

Sub-Section: Optimisation Problems

Let's now take a look at their applications!

Optimisation Problems

- Applying absolute maxima and minima in a real-world setting.
- **Steps**:
  1. Construct a function for the subject you want to find the maximum or minimum of.
  2. Find its domain if appropriate.
  3. Find its endpoints and turning points.
  4. Identify the maximum or minimum y value.

Question 13 Walkthrough.

Find the maximum area of a rectangle whose perimeter is equal to 20 cm.

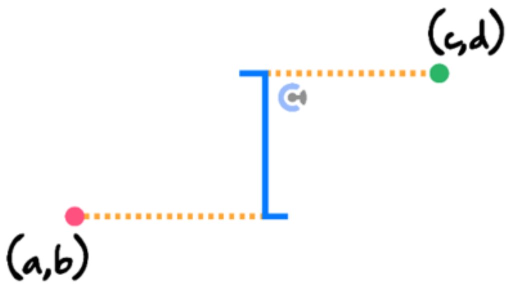
TIP: Always identify first what you want to solve the maximum of.

__Active Recall:__ **Steps for optimisation**

1. Construct a ________________ for the subject you want to find the maximum or minimum of.
2. Find its ____________ if appropriate.
3. Find its __________________ and ______________ points.
4. Identify ______________ or ______________ y value.

__REMINDER__

- Vertical distance between two functions.



- Simply find the difference between their y values.

Question 14

Find the minimum vertical distance between the functions $f(x) = x^2 + 16$ and $g(x) = x - 10$.

Question 15

Two positive variables x and y are such that $x^2y = 32$. Another variable z is given by $z = x + y$. Find the minimum value of z.

TIP: If there is no end point, simply look for the turning point!

Space for Personal Notes

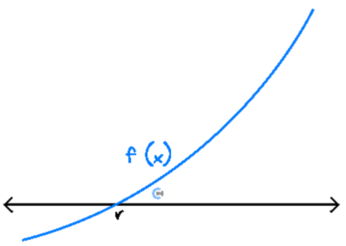
Section C: Newton's Method

Sub-Section: Introduction to Newton's Method

What is the Newton's method used for?

Context: Introduction to Newton's Method

- Consider the function:



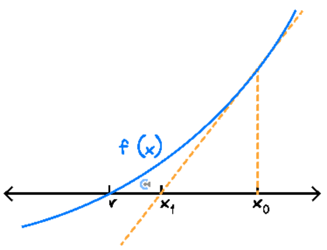
$$f(x) = x^2 - \sin(x)$$

- Is it easy to find the x-intercept? [Easy, Meh, Impossible without CAS!]

What should we do now?

- Let's first draw a tangent at $x = x_0$.
- Compare the x-intercept of $f(x)$ and the tangent.

  Are they similar[Yes, No, they are too far away.]



This is Newton's method!

- It approximates the ________________ using ________________.

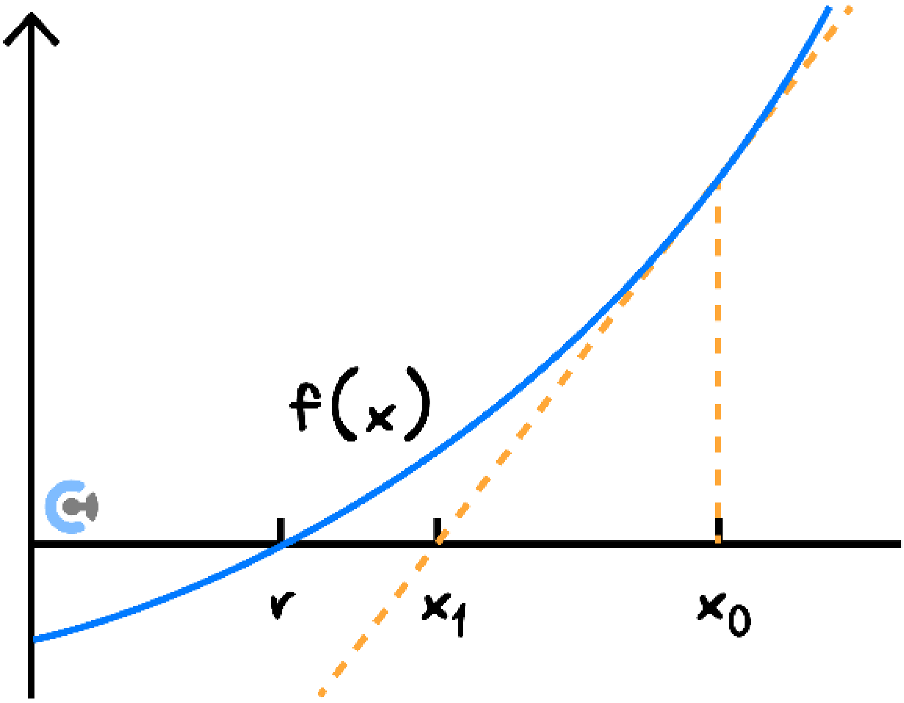
Question 16 Walkthrough. Tech-Active.

Estimate the x-intercept of $f(x) = (x-3)(x+2)$ using the tangent made at $x = 4$. Determine the error of the approximation, giving your answer in 4 decimal places.

Space for Personal Notes

Is there a way to find x-intercept of tangent quickly?

Exploration: Finding x-intercepts of a tangent made at $x = x_0$



- Say tangent is made at __________ and the x-intercept of the tangent is _______.
- Then,

$$f'(x_0) =$$

- Now rearrange for x_1 (x-intercept of the tangent);

$$x_0 - x_1 = \frac{f(x_0) - 0}{f'(x_0)}$$

$$x_1 =$$

Where x_1 is the x-intercept of the tangent made at __________.

TIP: To memorise which one is on the bottom, remember "Denominator is the derivative" ("DD").

Let's try the previous question again using the formula above!

Question 17 Tech-Active.

Estimate the x-intercept of $f(x) = (x-3)(x+2)$ using the tangent made at $x = 4$. Determine the error of the approximation, giving your answer in 4 decimal places.

Space for Personal Notes

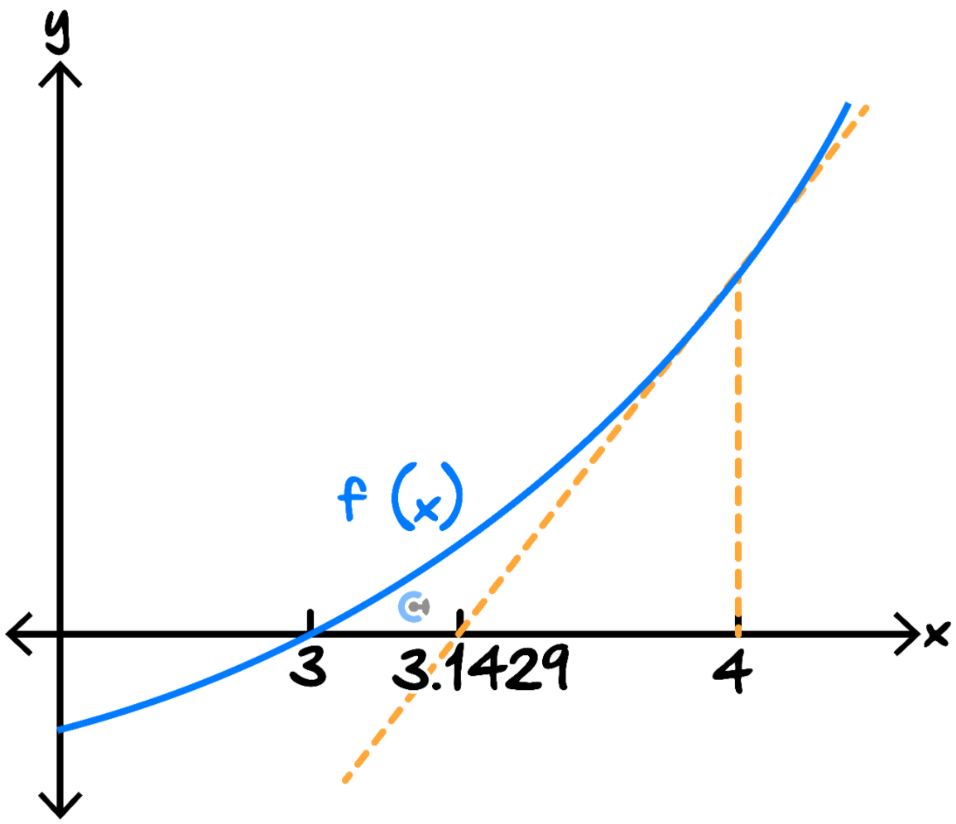
Sub-Section: Minimising Error

How do we minimise errors?

Where should the tangent be drawn for better accuracy?

Exploration: Minimise Error

- We used the x-intercept of the tangent made at $x = 4$ and the error was 0.1429.



- To minimise error should we make the tangent closer or further from 3? [Closer/Further]
- If we didn't know the actual x-intercept ($x = 3$), **at which x value** should the next tangent be drawn?

$x =$ ____________

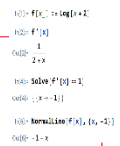
Let's try this out!

Question 18 Tech-Active.

a. Estimate the x-intercept of $f(x) = (x-3)(x+2)$ using the tangent made at $x = 3.14286$. Determine the error of the approximation, giving your answer in 4 decimal places.

b. Is it more accurate than before?

c. Estimate the x-intercept of $f(x) = (x-3)(x+2)$ using the tangent made at $x = 3.00386$. Determine the error of the approximation, giving your answer in 4 decimal places.

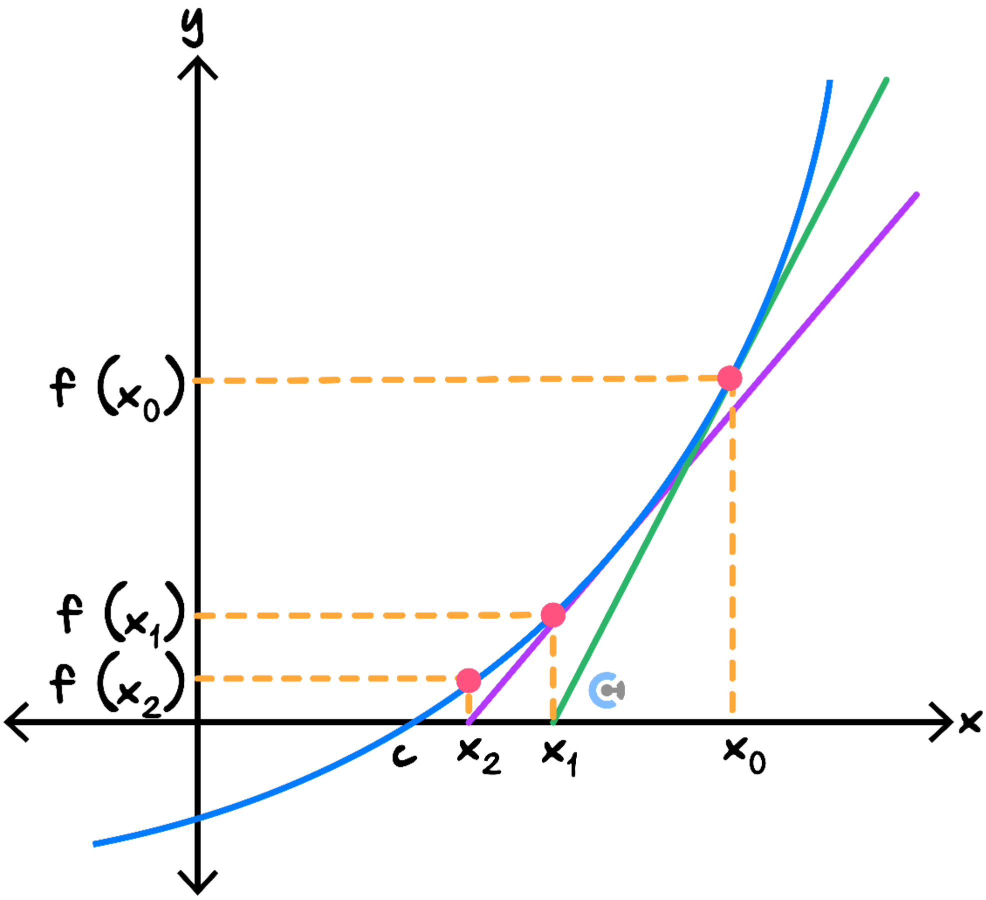


Discussion: Imagine if we repeated this process 10 times, what would happen to the error?

Performing the above process multiple times is called Newton's method!

Newton's Method

- It is a method of approximating the x-intercept using **tangents.**



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Steps**

1. Find the tangent at the x value given.
2. Find the x-intercept of the tangent using an iterative formula.
3. Find the next tangent at the $x = x$-intercept of the previous tangent.
4. Repeat until the value doesn't change by much.

Question 19 Walkthrough. Tech-Active.

Consider the function $f(x) = x^3 - 5$.

a. Show that Newton's method gives the iterative formula $x_{n+1} = \frac{2x_n^3+5}{3x_n^2}$.

b. Find the approximation of x-intercept of $f(x)$ giving your answer correct to 4 decimal places. Start $x_0 = 2$.

Space for Personal Notes

Active Recall: Newton's Formula

$$x_{n+1} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Question 20 Tech-Active.

Consider the function $f(x) = x^5 - 8$.

a. Show that Newton's method gives the iterative formula:

$$x_{n+1} = \frac{4x_n^5 + 8}{5x_n^4}$$

b. Find the approximation of x-intercept of $f(x)$ giving your answer correct to 4 decimal places. Start $x_0 = 2$.

Sub-Section: Tolerance

Could the question tell us when to stop?

Tolerance

- **Definition**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\textit{We stop when } |x_{n+1} - x_n| < \textit{Tolerance}.$$

- The question will give us the tolerance level.

NOTE: It is how much error we are willing to tolerate.

Question 21 Tech-Active.

Find the root of $f(x) = x^3 - 12$ using Newton's method with a tolerance of 0.01 and initial value of 3. Give your answer correct to two decimal places.

Question 22 Tech-Active.

Find the root of $f(x) = 1 - x^2e^{2x}$ using Newton's method with a tolerance of 0.001 and initial value of 1. Give your answer correct to three decimal places.

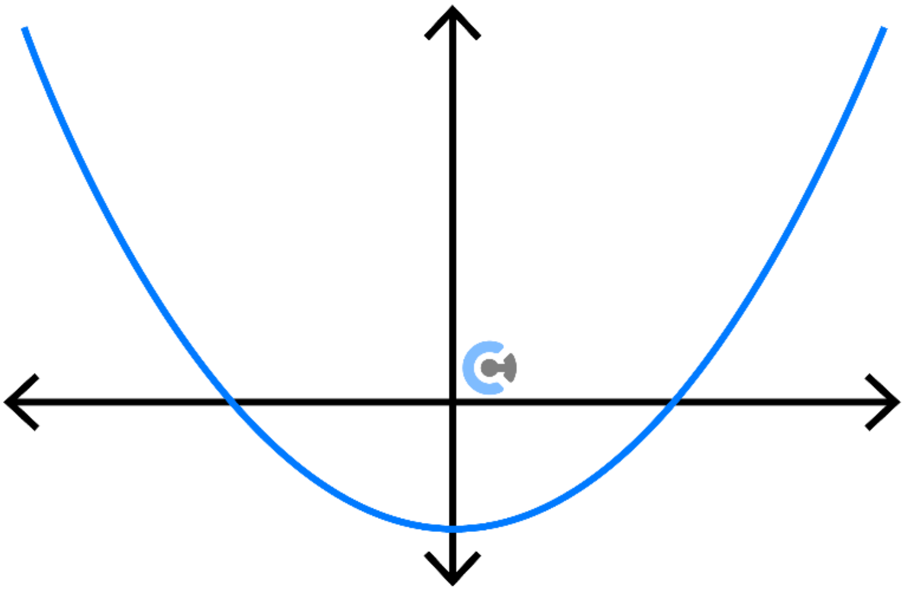
Space for Personal Notes

Sub-Section: Limitation of Newton's Method

When does Newton's method not work?

Exploration: 1. Terminating Sequence

- Consider the graph below.



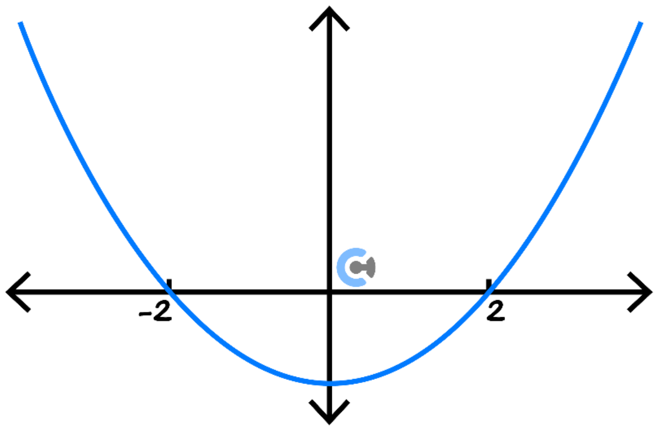
- What happens when $x_0 = 0$?

- Hence, when Newton's method reaches the x value of the stationary point, it ________________.

Space for Personal Notes

Exploration: 2. Approximating the Wrong Root

- Consider the graph below and say we wanted to approximate the x-intercept on the **right** $(x = 2)$.

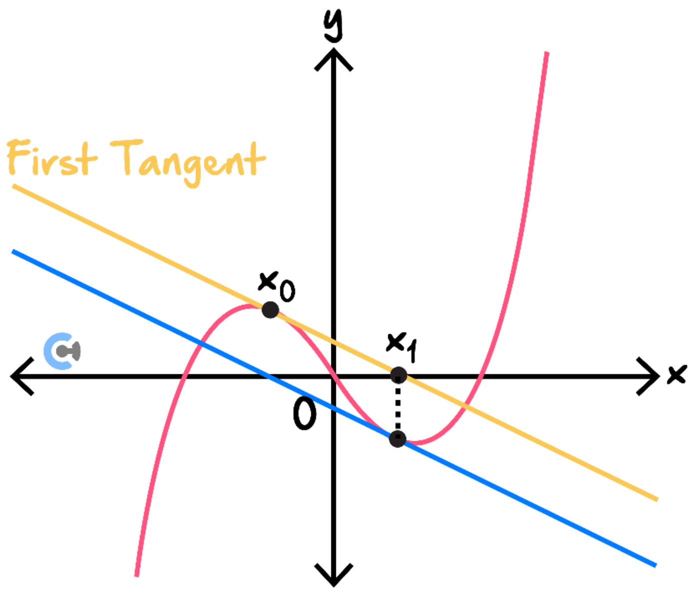


- What happens when $x_0 = -1$, do we approximate $x = -2$ or $x = 2$? $[-2/2]$

- When Newton's method starts at the wrong side of the turning point, it can approximate the ____________________.

Exploration: 3. Oscillating Sequence

- Consider the graph.



- What is the value of x_2, the next term of the sequence?

- Hence, would Newton's method approximation get closer to the actual x-intercept at $x = 0$? [Yes/No]

Definition

<u>Limitation of Newton's Method</u>

- **Terminating Sequence**: Occurs when we hit a stationary point.
- **Approximating a Wrong Root**: Occurs when we start on the wrong side.
- **Oscillating Sequence**: Occurs when we oscillate between two values without getting closer to the real root.

Question 23 Tech-Active.

Consider the function $f(x) = (x-1)^2 - 1$.

a. Write down the Newton iteration formula for x_{n+1}.

b. What limitation of Newton's method do we get if $x_0 = 1$?

Question 24 Tech-Active.

Consider the function $f(x) = \frac{x}{\sqrt{x^2+1}}$.

a. Perform three iterations of Newton's method with starting point $x = 1$.

b. What limitation of Newton's method did we get here?

Space for Personal Notes

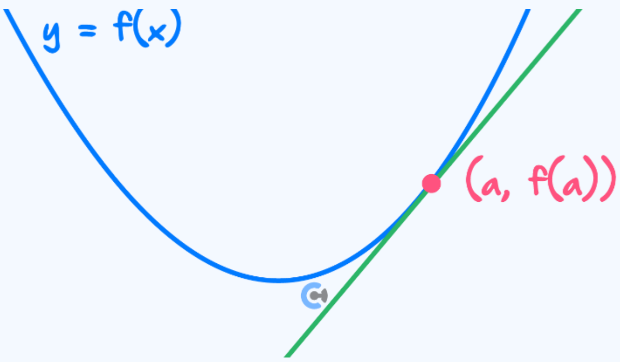
Contour Checklist

☐ Learning Objective: [2.4.1] - Find Tangents and Normals

Key Takeaways

☐ A **tangent** is a linear line which **just** ________________ the curve.

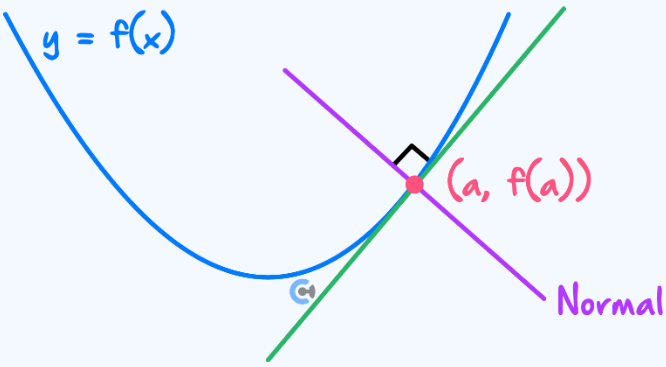
☐ The gradient of a tangent line has to be equal to the __________________ of the curve at the intersection.



$$\text{At } (a, f(a)): \quad m_{tangent} = \_\_\_\_\_\_\_\_\_\_$$

☐ Normals

- A **normal** is a linear line which is __________________ to the tangent.
- The gradient of a normal line has to equal to the ______________________ of the gradient of the curve at the intersection.



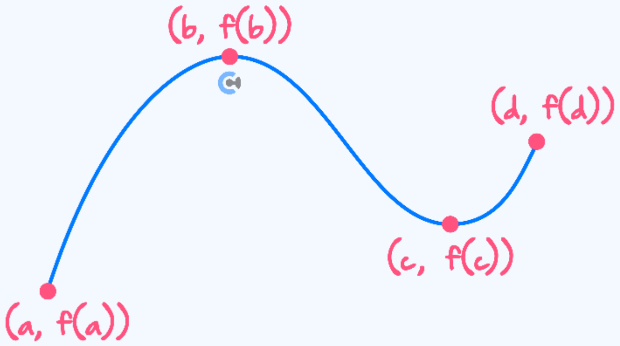
$$\text{At } (a, f(a)): \quad m_{normal} = \_\_\_\_\_\_\_\_\_\_$$

Learning Objective: [2.4.2] – Find Minimum and Maximum

Key Takeaways

Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** __________ for the given domain.
- They occur at either be an ______________, or a ____________________.



Absolute Min: ________

Absolute Max: ________

Steps

1. Find _____________ points and _______ points
2. Find the ________________ y value for max/min.

Steps for optimisation

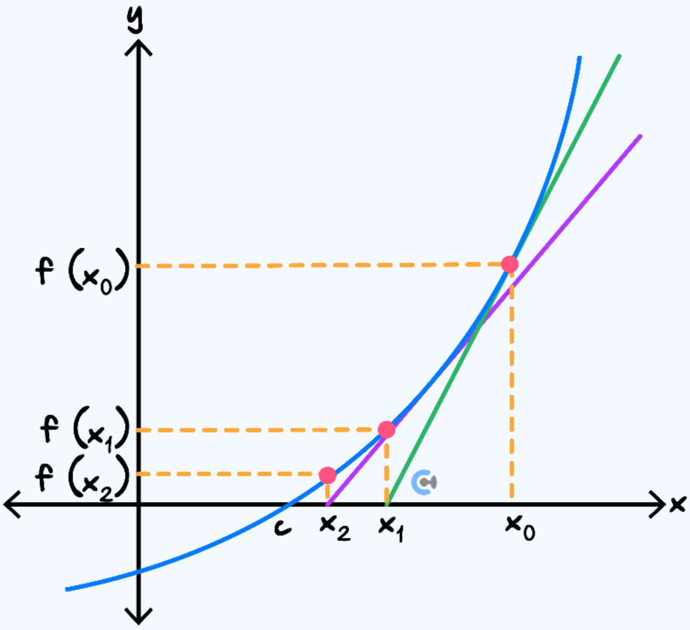
1. Construct a _______________ for the subject you want to find the maximum or minimum of.
2. Find its ____________ if appropriate.
3. Find its _________________ and _____________ points.
4. Identify _____________ or _____________ y value.

Learning Objective: [2.4.3] - Apply Newton's Method to Find the Approximation of a Root and its Limitations

Key Takeaways

Newton's Method

- It is a method of approximating the x-intercept using ______________.



$$x_{n+1} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Steps

1. Find the ______________ at the x value given.
2. Find the ______________ of the tangent using iterative formula.
3. Find the next tangent at the $x =$ ______________ of the previous tangent.
4. Repeat until the value doesn't change by much.

Tolerance: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$We\ stop\ when\ |x_{n+1} - x_n| < \_\_\_\_\_\_\_\_\_\_\_\_.$$

- Limitation of Newton's Method
  - **Terminating Sequence**: Occurs when we hit a ____________________.
  - **Approximating a Wrong Root**: Occurs when we start on the ________________.
  - **Oscillating Sequence**: Occurs when we ______________ between two values without getting closer to the real root.

CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make The Most of These (Free) Services!

- Experienced Contour tutors (45+ raw scores, 99+ ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
➤ Book via bit.ly/contour-methods-consult-2025 (or QR code below). ➤ One active booking at a time (must attend before booking the next).	➤ Message +61 440 138 726 with questions. ➤ Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726