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VCE Mathematical Methods ¾
Applications of Differentiation [2.4]
Homework Solutions

Homework Outline:

Section A: Compulsory Questions

Sub-Section [2.4.1]: Find Tangents and Normals

Question 1

Find the equation of the tangent to the graph of $f(x) = 2x^2 + 4x + 3$ at the point $x = 1$.

$f'(x) = 4x + 4 \implies f'(1) = 8$ and $f(1) = 2 + 4 + 3 = 9$. Thus the equation of the tangent is

$$y - 9 = 8(x - 1)$$
$$y = 8x + 1$$

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Question 2

Let $f: \left(-\infty, -\frac{1}{2}\right] \to \mathbb{R},\ f(x) = (3x+2)^3$. Find the equation of the tangent to the graph of $y = f(x)$ which has a gradient of 9.

$f'(x) = 9(3x+2)^2$. We solve $9(3x+2)^2 = 9 \implies (3x+2)^2 = 1 \implies x = -1, -\frac{1}{3}$. Only $x = -1$ is in the domain. $f(-1) = -1$. Thus

$$y + 1 = 9(x+1)$$
$$y = 9x + 8$$

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Question 3

Find the equation of the tangent to the graph of $f : (0,5) \to \mathbb{R}, f(x) = x^2 - 4x$ at the point $x = a$. Hence, obtain the equation of the tangent that passes through the point $(6,3)$.

We have $f'(x) = 2x - 4 \implies f'(a) = 2a - 4$ and $f(a) = a^2 - 4a$. Thus the tangent at $x = a$ is

$$y = (2a-4)(x-a) + a^2 - 4a$$
$$y = (2a-4)x - a^2$$

Furthermore, this tangent passes through the points $(a, a^2 - 4a)$ and $6, 3$ and therefore has gradient

$$m = \frac{a^2 - 4a - 3}{a - 6}$$

We equate

$$\frac{a^2 - 4a - 3}{a - 6} = 2a - 4$$
$$\implies a = 3, 9$$

Reject $a = 9$ since outside the domain.
The tangent is $y = 2x - 9$, which occurs when $x = 3$.

Question 4 Tech-Active.

Find the equation of the line tangent to $y = 2\log_e(2x - 1)$ when $x = 1$.

$y = 4x - 4$

TI:	Mathematica:	Casio:

tangentLine(2·ln(2·x−1),x,1) 4·x−4	In[445]:= f[x_] := 2 Log[2 x - 1]
	In[447]:= y == f'[1] (x - 1) + f[1] // Expand
	Out[447]= y == -4 + 4 x
		tanLine(2ln(2x−1), x, 1)
		4·x−4

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Sub-Section [2.4.2]: Find Minimum and Maximum

Question 5

Find the maximum and minimum values of the function $f(x) = x^3 + 3x^2 - 9x - 10$ with domain $x \in [-3,2]$.

The global extrema occur either where $f'(x) = 0$ or at an endpoint.
We have $f'(x) = 3x^2 + 6x - 9 = 0 \implies x = -3, 1$.
$f(-3) = 17$ and $f(1) = -15$. Also note the second endpoint $f(2) = -8$.
Thus maximum is 17 and minimum is -15.

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Question 6

A farmer is building a rectangular pen for his pigs using 40 metres of fencing.

a. Write down a function $A(x)$ which gives the area of the pen.

Let the pen have side lengths x and y. Note that $2x + 2y = 40 \implies 2y = 40 - 2x \implies y = 20 - x$.
Thus $A(x) = x(20 - x) = 20x - x^2$

b. Hence, determine the side lengths of the pen to maximise the area and give this maximum area.

$A'(x) = 20 - 2x = 0 \implies x = 10$.
Therefore maximum area is when the pen is a square with all sides lengths 10 m and has an area of 100m^2.

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Question 7

Find the maximum positive rate of change for the function $f(x) = -x^3 + 9x^2 + 6x - 10$

The rate of change of $f(x)$ is given by $f'(x) = -3x^2 + 18x + 6$ to find stationary points of f' we solve

$$f''(x) = -6x + 18 = 0 \implies x = 3$$

Then $f'(3) = -27 + 54 + 6 = 33$. Note that f' is a negative quadratic. Thus the maximum positive rate of change of f is $f'(3) = 33$.

Question 8 Tech-Active.

Find the maximum and minimum values of the function $f(x) = -x^3 + 6x^2 + 4x - 10$ for $x \in [-2, 7]$.

Max $\frac{2}{9}\left(64\sqrt{3} + 63\right)$, when $x = 2 + \frac{4}{\sqrt{3}}$

Min -31 when $x = 7$.

TI:	**Mathematica:**	**Casio:**

Define $f(x)=-x^3+6\cdot x^2+4\cdot x-10$ Done
fMax$(f(x),x,-2,7)$ $x=\frac{2\cdot(2\cdot\sqrt{3}+3)}{3}$
$f\left(\frac{2\cdot(2\cdot\sqrt{3}+3)}{3}\right)$ $\frac{128\cdot\sqrt{3}}{9}+14$
fMin$(f(x),x,-2,7)$ $x=7$
$f(7)$ -31
	`f[x_] := -x^3 + 6 x^2 + 4 x - 10`
	`Maximize[{f[x], -2 ≤ x ≤ 7}, x] // FullSimplify`
	$\left\{\frac{2}{9}(63+64\sqrt{3}), \left\{x \to 2+\frac{4}{\sqrt{3}}\right\}\right\}$
	`Minimize[{f[x], -2 ≤ x ≤ 7}, x] // FullSimplify`
	$\{-31, \{x \to 7\}\}$
		define f(x) = -x^3+6x^2+4x-10
		done
		fmin(f(x),x,-2,7)
		{MinValue=-31, x=7}
		fmax(f(x),x,-2,7)
		$\left\{\text{MaxValue}=\frac{128\cdot\sqrt{3}}{9}+14, x=\frac{4\cdot\sqrt{3}}{3}+2\right\}$

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Sub-Section [2.4.3]: Apply Newton's Method to Find the Approximation of a Root and its Limitations

Question 9

Approximate the root of the equation $x^3 - 5x + 3$ using Newton's method with an initial value of $x_0 = 1$ and a tolerance level of 0.01. Leave your answer correct to two decimal places.

$x = 0.66$

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	1.0000000	0.5000000	0.5000000
1	0.5000000	0.647059	0.147059
2	0.647059	0.656573	0.009514

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Question 10

Approximate a solution of the equation $e^x = \sin(x)$ using Newton's method with an initial value of $x_0 = -2$. Use a tolerance level of 0.01 and give your answer correct to two decimal places.

We approximate a root of $f(x) = e^x - \sin(x)$. We find this to be approximately $x = -3.18$

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	-2.0000000	-3.894228	1.894228
1	-3.894228	-3.010248	0.883980
2	-3.010248	-3.183451	0.173204
3	-3.183451	-3.18306	0.00039

Question 11

Consider the function $f(x) = x^3 - 6x$. Determine two possible initial values which are not suitable to use in Newton's method to approximate the roots of f.

Initial value is unsuitable if it at a the x-value of a stationary point.
$f'(x) = 3x^2 - 6 = 0 \implies x^2 = 2 \implies x = \pm\sqrt{2}$.
Thus $x = \pm\sqrt{2}$ are not suitable initial values.

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Sub-Section: The 'Final Boss'

Question 12

A piece of wire is 16 metres long. The wire is cut into two pieces and used to form two squares.

a. If one piece of wire has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}x^2 - 2x + 16 = \frac{1}{8}(x^2 - 16x + 128)$.

First square has perimeter x and thus area $\left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$.
Second square has perimeter $16 - x$ and thus has area

$$\left(\frac{16-x}{4}\right)^2 = \frac{1}{16}(x^2 - 32x + 16^2) = \frac{1}{16}x^2 - 2x + 16.$$

Add the two areas to get

$$A = \frac{1}{8}x^2 - 2x + 16 = \frac{1}{8}(x^2 - 16x + 128)$$

b. Find $\frac{dA}{dx}$.

$$\frac{dA}{dx} = \frac{1}{4}x - 2$$

c. Find the value of x that minimises A and gives this minimum value.

Solve $\frac{dA}{dx} = 0 \implies x = 8$ then $A = 8 - 16 + 16 = 8$.

d. Find the maximum possible area of the two squares if $x \in [1, 12]$.

Check the endpoints $A(1) = \frac{113}{8}$ and $A(12) = 10$.
Thus max area of $\frac{113}{8}$ when $x = 1$

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Section B: Supplementary Questions

Sub-Section [2.4.1]: Find Tangents and Normals

Question 13

Find the equation of the normal to the graph of $f(x) = \cos(5x)$ at the point $x = \frac{\pi}{4}$.

We first calculate the derivative $f'(x) = -5\sin(5x)$. Thus, the gradient of the tangent is $f'\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$ Hence, the normal has gradient equal to $-\frac{2}{5\sqrt{2}}$. Furthermore, $f\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. Therefore, the equation of the normal is

$$\begin{aligned} y + \frac{\sqrt{2}}{2} &= -\frac{2}{5\sqrt{2}}\left(x - \frac{\pi}{4}\right) \\ \implies y &= -\frac{2}{5\sqrt{2}}x + \frac{\pi}{10\sqrt{2}} - \frac{\sqrt{2}}{2}. \\ \implies y &= -\frac{\sqrt{2}}{5}x + \frac{\sqrt{2}\cdot\pi}{20} - \frac{\sqrt{2}}{2} \end{aligned}$$

Question 14

Find the equation of the normal to the graph of $f(x) = x^2 - 3x - 1$ which has a gradient of $-\frac{1}{5}$.

Firstly, we calculate the derivative $f'(x) = 2x - 3$. If the gradient of the normal is $-\frac{1}{5}$, then the gradient of the tangent at the same point will be equal to 5. Therefore, we solve $2x - 3 = 5$, which gives $x = 4$. Furthermore, $f(4) = 3$. Therefore, the equation of the normal is

$$\begin{aligned} y - 3 &= -\frac{1}{5}(x - 4) \\ \implies y &= -\frac{1}{5}x + \frac{19}{5}. \end{aligned}$$

Question 15

Find the equation of the normal to the graph of $f: (2, \infty) \to \mathbb{R},\ f(x) = x^2 - 2x$ at the point $x = a$. Hence by using a CAS, obtain the equation of the normal that passes through the point $(-1, 4)$.

The derivative is $f'(x) = 2x - 2$. The gradient of the tangent is given by $2a - 2$ and furthermore, the function passes through the point $(a, a^2 - 2a)$. Thus, the equation of the normal is

$$\begin{aligned} y - (a^2 - 2a) &= -\frac{1}{2a-2}(x - a) \\ \implies y - (a^2 - 2a) &= -\frac{1}{2a-2}x + \frac{a}{2a-2} \\ \implies y &= -\frac{1}{2a-2}x + \frac{a}{2a-2} + a^2 - 2a \end{aligned}$$

Furthermore, the normal is known to contain the point $(-1, 4)$. Therefore,

$$4 = \frac{1}{2a-2} + \frac{a}{2a-2} + a^2 - 2a$$

Using a CAS, we solve to find that $a = 3$. We reject the other roots as they are not within the specified domain. Therefore, the equation of the normal is

$$y = -\frac{1}{4}x + \frac{15}{4}$$

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Question 16

Consider the function given by $f(x) = e^{x^2} - \cos(x)$.

a. Find the equation of the tangent to the graph of $f(x)$ at the point $x = 1$.

We first calculate $f'(x) = 2xe^{x^2} + \sin(x)$ and thus $f'(1) = 2e + \sin(1)$. Furthermore, $f(1) = e - \cos(1)$. Therefore, we can conclude that the equation of the tangent is given by the rule

$$\begin{aligned} y - e + \cos(1) &= (2e + \sin(1))(x - 1) \\ \implies y &= (2e + \sin(1))x - e - \sin(1) - \cos(1) \end{aligned}$$

b. Without needing to do any further differentiation/solving, find the equation of the normal that passes through the point $x = -1$.

As $f(x)$ is actually an even funtion, we may simply relect the tangent from the above part to obtain the tangent at $x = -1$, i.e. with $(x, y) \to (-x, y)$, we deduce $y = -(2e + \sin(1))x - e - \sin(1) - \cos(1)$.

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Sub-Section [2.4.2]: Find Minimum and Maximum

Question 17

Find the maximum and minimum values of the function $f(x) = x^3 - 3x^2 - 24x + 15$ with domain $x \in [0,5]$.

The global extrema occur either where $f'(x) = 0$ or at an endpoint. Firstly, we consider $f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x-4)(x+2)$. Thus, $f'(x) = 0$ for $x = -2$ or $x = 4$. However, we only consider $x = 4$ as $x = -2$ is not in the domain. Hence, the possible values of x where the global extrema occur are $x = 0, 4$ and 5. Indeed, the values of $f(x)$ at these points are $f(0) = 15$, $f(4) = -65$ and $f(5) = -55$. Thus, the global maximum is 15 and the global minimum is -65

Question 18

Find the maximum area of a rectangle with a perimeter equal to 18 m.

Let x denote the width of the rectangle, then the height of the rectangle must be $9 - x$. Hence, we see the rectangle has area given by $A(x) = x(9-x)$ with $x \in (0, 9)$. Solving $A'(x) = 0$ gives us the value $x = 9/2$. As a result, the maximum area of the rectangle is $81/4 \text{ m}^2$.

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Question 19

Find the maximum rate of change of the function $f(x) = -x^3 + 6x^2 + 10x - 5$.

The rate of change of $f(x)$ is given by $f'(x) = -3x^2 + 12x + 10$. To find the minimum rate of change, we should solve for where $f''(x) = 0$. Indeed, $f''(x) = -6x + 12$ and $f''(x) = 0$ if $x = 2$. By inspection, $f'(x)$ is a quadratic with negative leading coefficient, so $x = 2$ corresponds to a maximum. The maximum rate of change is $f'(2) = 22$.

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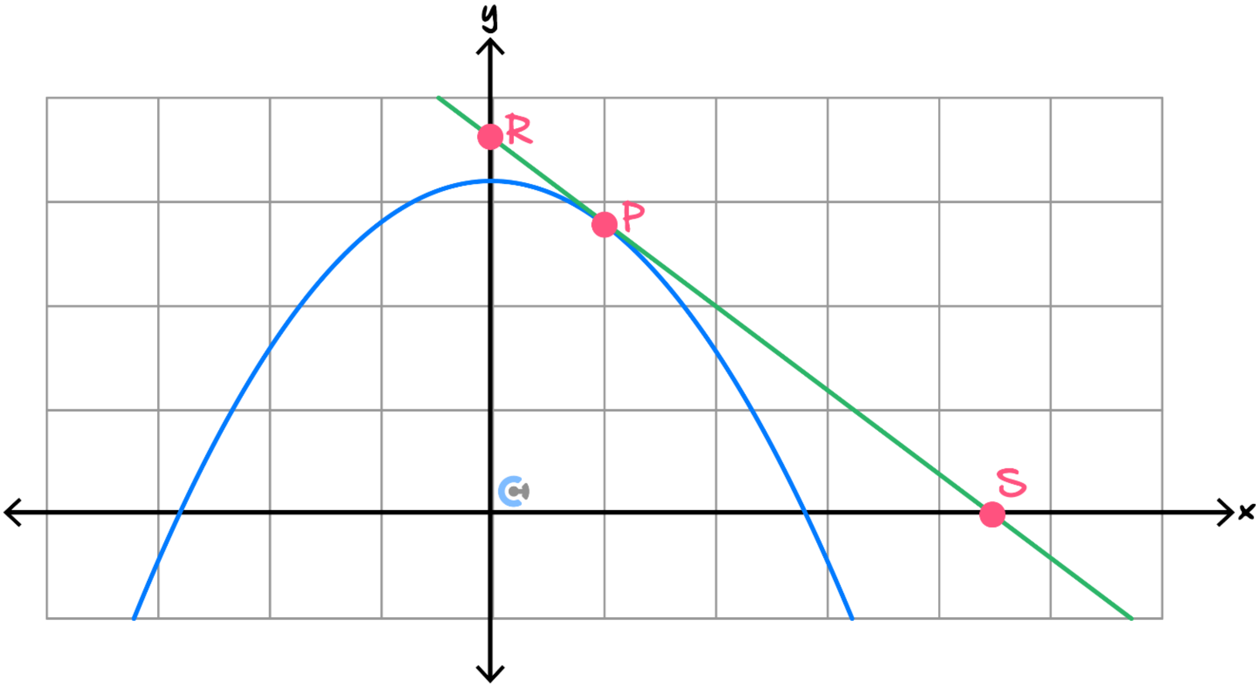
Question 20 Tech-Active.

The diagram below shows the graph of the function $f(x) = 16 - 2x^2$.



The graph of the tangent to the curve at the point $P(p. f(p))$, where $p \in \left[\frac{1}{2}, \frac{5}{2}\right]$ is also shown.

Determine the equation of the tangent line in terms of p.

$$f'(p) = -4p$$

$$y - f(p) = -4p(x - p)$$
$$y - (16 - 2p^2) = -4p(x - p)$$
$$y = -4px + 2p^2 + 16$$

Which is the required equation of tangent.

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Sub-Section [2.4.3]: Apply Newton's Method to Find the Approximation of a Root and its Limitations

Question 21

Approximate the root of the equation $x^3 - 2x^2 + 5x - 6$ using Newton's method with an initial value of $x_0 = 1.2$ and a tolerance level of 0.01. Leave your answer correct to two decimal places.

Begin the algorithm with $x_0 = 1.2 \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.4549$. Note that $|x_1 - x_0| > 0.01$, so we must perform another iteration of Newton's method. Thus, $x_2 = 1.4549 - \frac{f(1.4549)}{f'(1.4549)} = 1.4331$. Again, $|x_2 - x_1| > 0.01$, so we must perform another iteration. Now, $x_3 = 1.4331 - \frac{f(1.4331)}{f'(1.4331)} = 1.4329$. Now, $|x_3 - x_2| < 0.01$, thus we may end Newton's method and approximate the root to be $x = 1.4329$.

Hence, the approximated root is: 1.43

Question 22

Approximate a solution of the equation $e^x = \cos(2x - 1)$ using Newton's method with an initial value of $x_0 = -2$. Use only one iteration for your approximation.

We want to approximate a solution to $e^x = \cos(2x - 1)$. We can equivalently approximate a root of $f(x) = e^x - \cos(2x - 1)$. Thus, by noting that $f'(x) = e^x + 2\sin(2x - 1)$, we obtain that $x_1 = (-2) - \frac{e^{-2} - \cos(2 \cdot -2 - 1)}{e^{-2} + 2\sin(2 \cdot -2 - 1)} = -1.93$.

For interest, the root is actually approximately $x = -1.92909$, which you can check on a CAS.

Question 23

Consider the function $f(x) = \sin(x) - e^{2x}$. Explain why it would be unsuitable to choose an initial value that solves the equation $\cos(x) - 2e^{2x}$.

If x_0 solves $\cos(x_0) - 2e^{2x_0} = 0$, then $f'(x_0)$ is undefined, so x_1 would be undefined as well. *Notice also that the tangent to the graph would become parallel to the x-axis, and therefore would not have a (unique) x-intercept, meaning we would not know the value of the next iteration.*

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Question 24

An issue that can arise when using Newton's method is that the derivative may not be easy to calculate.

a. Explain why Newton's method is impractical for approximating the roots of $f(x) = \sin^{-1}(x^2 - \frac{\pi}{2})$ within the context of VCE Mathematical Methods Units 3 and 4.

We are not taught how to differentiate $f(x) = \sin^{-1}(x)$ in Methods 3/4. Hence, we would not be able to find the rule for $f'(x)$ needed in the formula.

b. Nevertheless, we can try to use a similar method known as the secant method with an initial guess of $x_0 = 1.5$. Approximate the tangent to $x_0 = 1.5$ by finding the equation of secant (i.e. the straight line) passing through the points $(1.5, 0.7467)$ and $(1.51, 0.7885)$. Your answer should be given to two decimal places.

Firstly, we calculate the gradient is $m = \frac{0.7885-0.7467}{1.51-1.5} = 4.18$. Therefore, $y - 0.7647 = 4.18(x - 1.5)$ gives $y = 4.18x - 5.5233$. The final answer is $y = 4.18x - 5.51$

c. Hence, obtain an approximation for a root of $f(x) = \sin^{-1}(x^2 - \frac{\pi}{2})$ based on the line obtained above.

Note that this method approximates the root which is $x = \sqrt{\pi/2} \approx 1.2533$.

We should calculate the x-intercept of the above secant, which is $x = 1.32$.

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