Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Differentiation Exam Skills [2.3]
Workbook

Outline:

Learning Objectives:

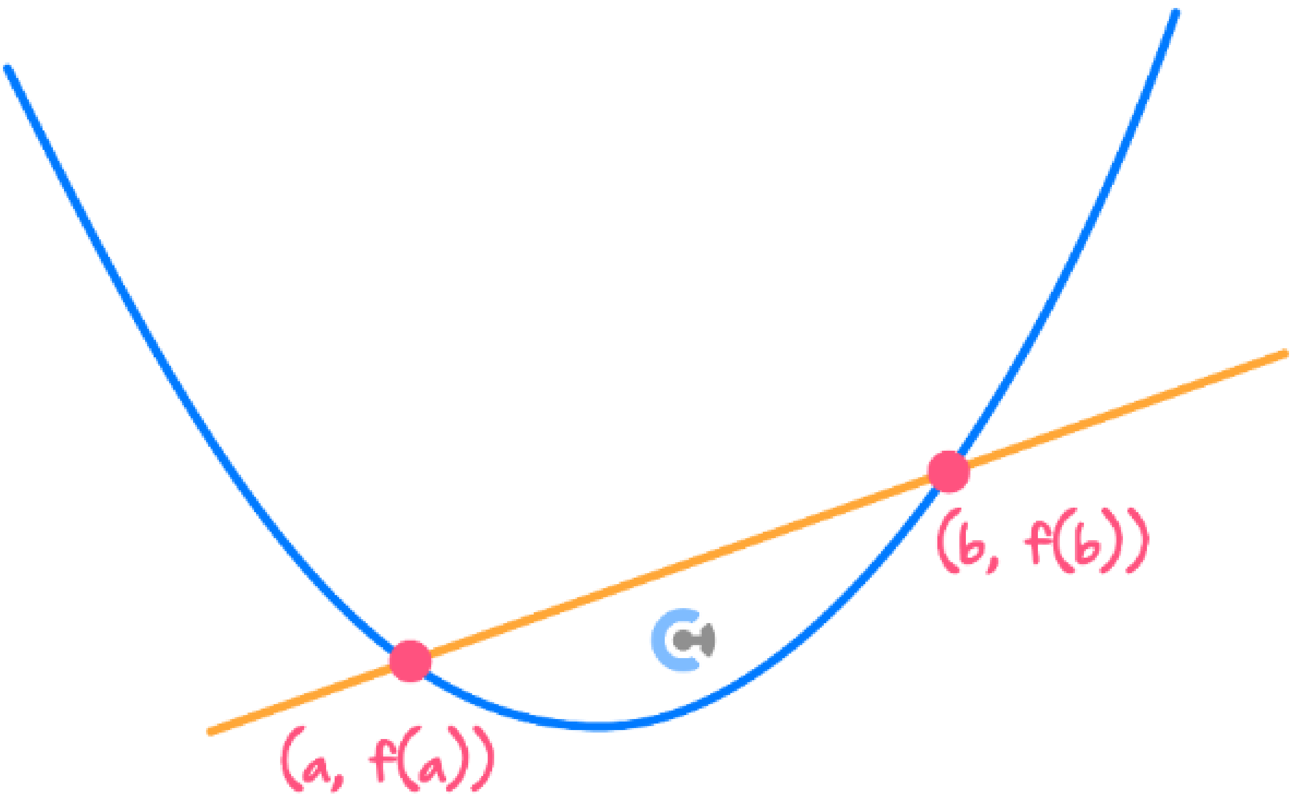
- ☐ **MM34 [2.3.1]** – Find derivatives with functional notation
- ☐ **MM34 [2.3.2]** – Apply differentiability to join two functions smoothly

Section A: Recap

If you were here last week, skip to Section B - Warmup Test.

Average and Instantaneous Rate of Change

Average Rate of Change



- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textbf{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- It is the gradient of the line joining the two points.

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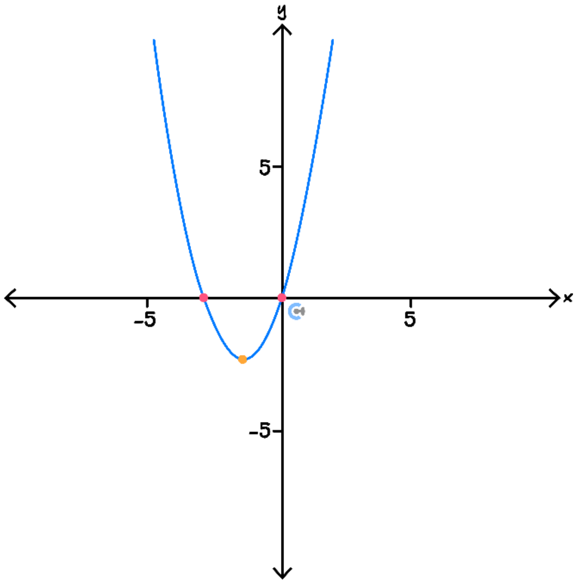
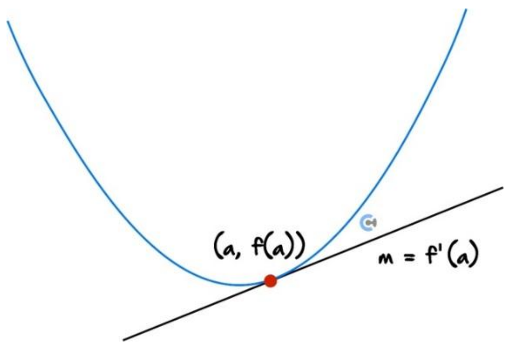
Question 1

Find the average rate of change of $y = x^2 + 3x$ over the interval $x \in [-1, 2]$.



<u>Instantaneous Rate of Change</u>

Definition



- **Instantaneous Rate of Change** is a gradient of a graph at a single point/moment.

$$\textit{Instantaneous Rate of Change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.

Definition

Alternative Notation for Derivative

$$f'(x) = \frac{dy}{dx}$$

How do we find derivative functions?

Definition

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \times x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$

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Question 2

Consider the function $f(x) = x^3 - 4x$.

Find the gradient of the function at $x = 2$.

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Question 3

Consider the function $f(x) = 2e^{x} - 4$.

Find the gradient of the function at $x = 3$.

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Question 4

Consider the function $f(x) = 2\log_e(x)$.

Find the gradient of the function at $x = 2e$.

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Question 5

Consider the function $f(x) = \cos(x) + \sin(x)$.

Find the gradient of the function at $x = \frac{\pi}{4}$.

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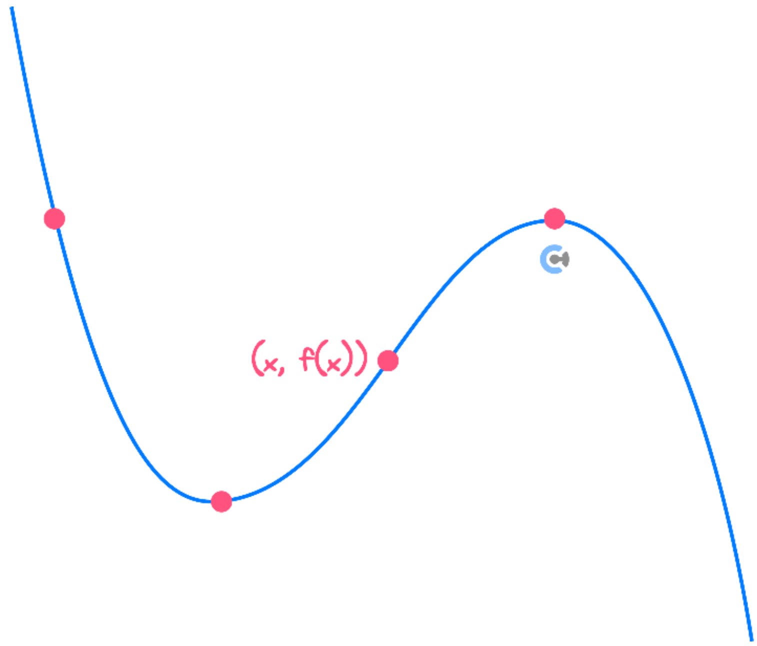
Sub-Section: First Principle

Where do all the derivative rules come from?

First Principle



$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

- The fundamental method of differentiation.

Sub-Section: Product Rule

How do we find the derivative when two functions are multiplied?
For example: $x^2 \sin(x)$.

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

NOTE: We never differentiate two functions at once!

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Question 6

Find the derivatives of:

a. $f(x) = x^2 e^x$

b. $y = 3\sin(x)\cos(x)$

c. $g(x) = \log_e(x) \cdot x$

Sub-Section: Quotient Rule

Definition

The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Always differentiate the top function first.

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Question 7

Find the derivatives of:

a. $\frac{e^x}{4x^3}$

b. $\frac{\log_e(x)}{x}$

c. $g(x) = \frac{\sin(x)}{\cos(x)}$

NOTE: The last question is a derivative of $\tan$.

How does the quotient rule work? (Extended)

Sub-Section: Chain Rule

The Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

- The process for finding derivatives of **composite functions**.

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Question 8

Find the derivatives of:

a. $e^{x^2+\frac{1}{2}x}$

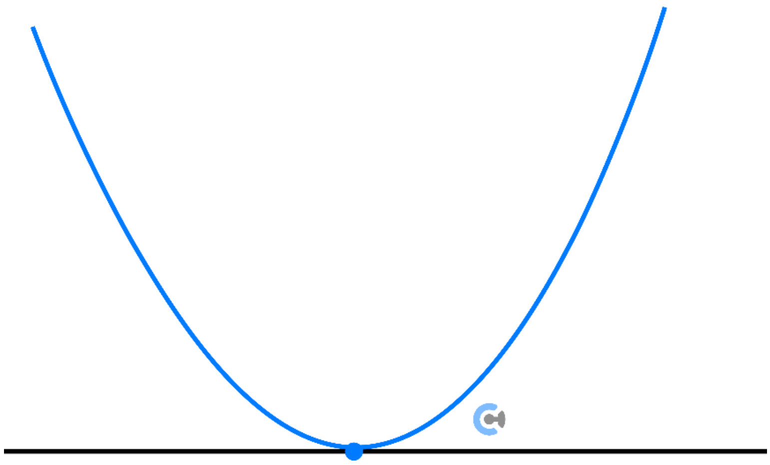
b. $\left(4x+\frac{1}{x}\right)^3$

c. $\log_e(x^2-4)$

Sub-Section: Stationary Points

What would be the gradient of a point that is neither increasing nor decreasing?

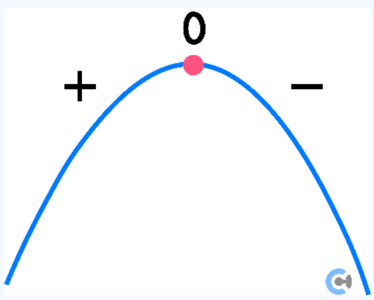
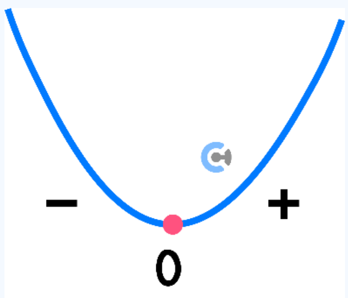
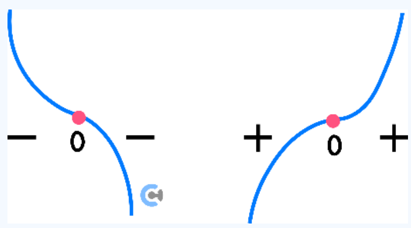
Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

- Sign Test

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing Curve	Stationary Point	∪ - Increasing Curve

- Find the gradient of the neighbouring points.

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Question 9

Find and identify the nature of the stationary points of $y = -e^{x^2+4}$.

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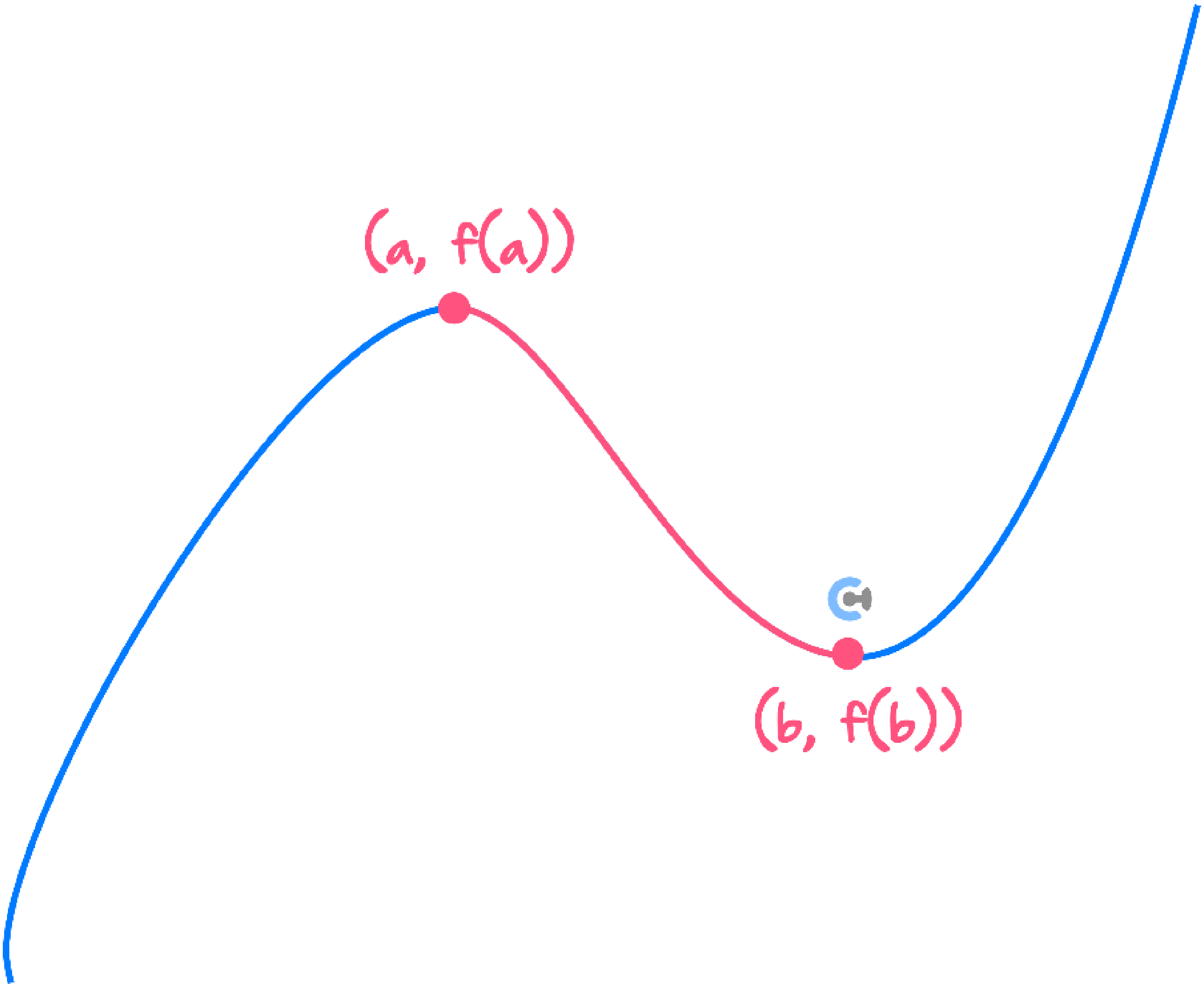
Sub-Section: Strictly Increasing and Decreasing

Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

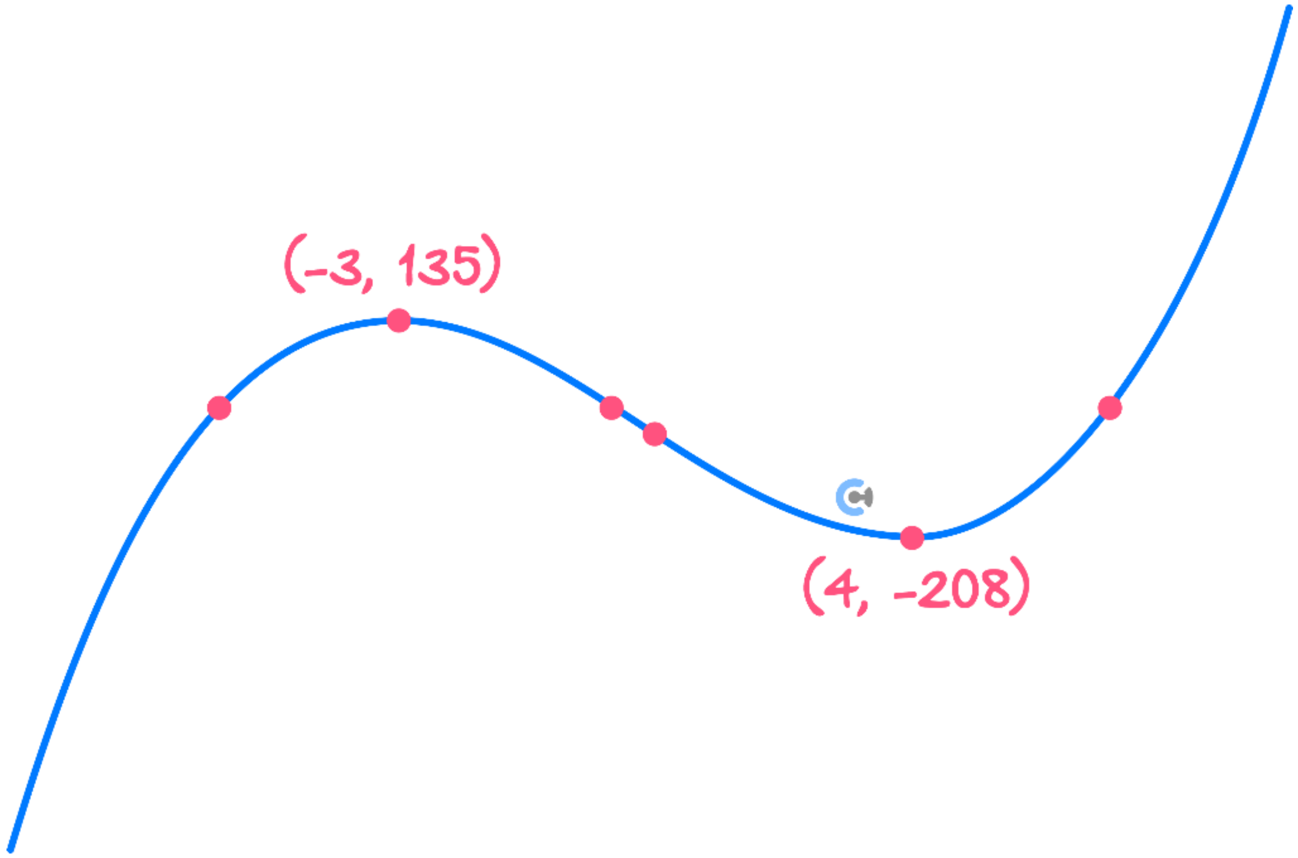
- Steps:

1. Find the turning points.
2. Consider the sign of the derivative between/outside the turning points.

Question 10

State the value(s) of x for the function below for which it is strictly increasing and strictly decreasing.

$$y = -72x - 3x^2 + 2x^3$$



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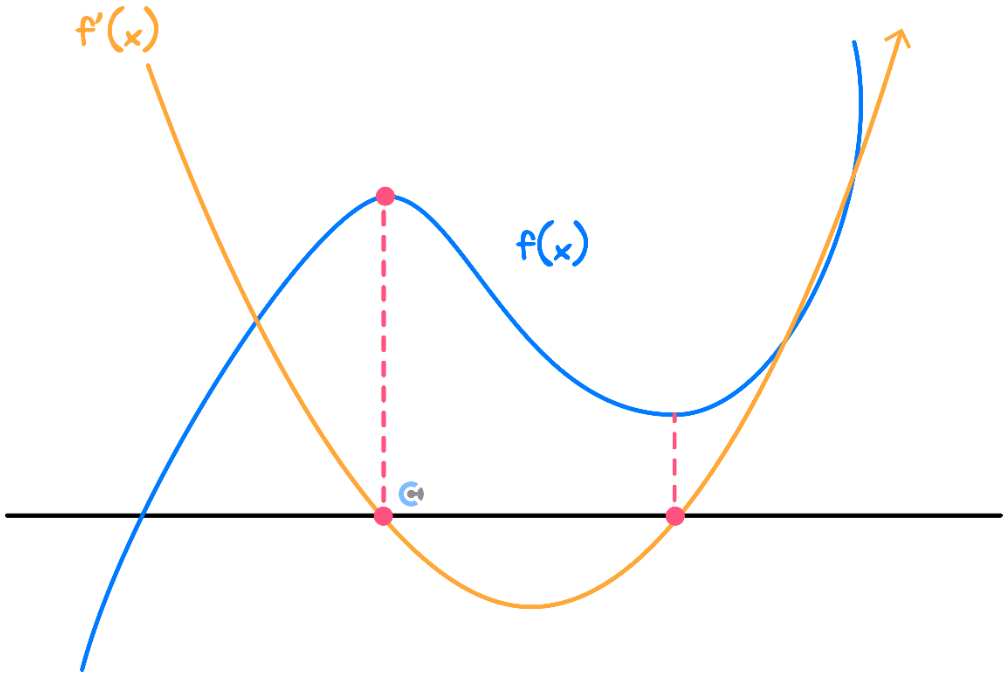
Sub-Section: Graphs of Derivative Function

Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	x-intercept
Increasing	Positive
Decreasing	Negative

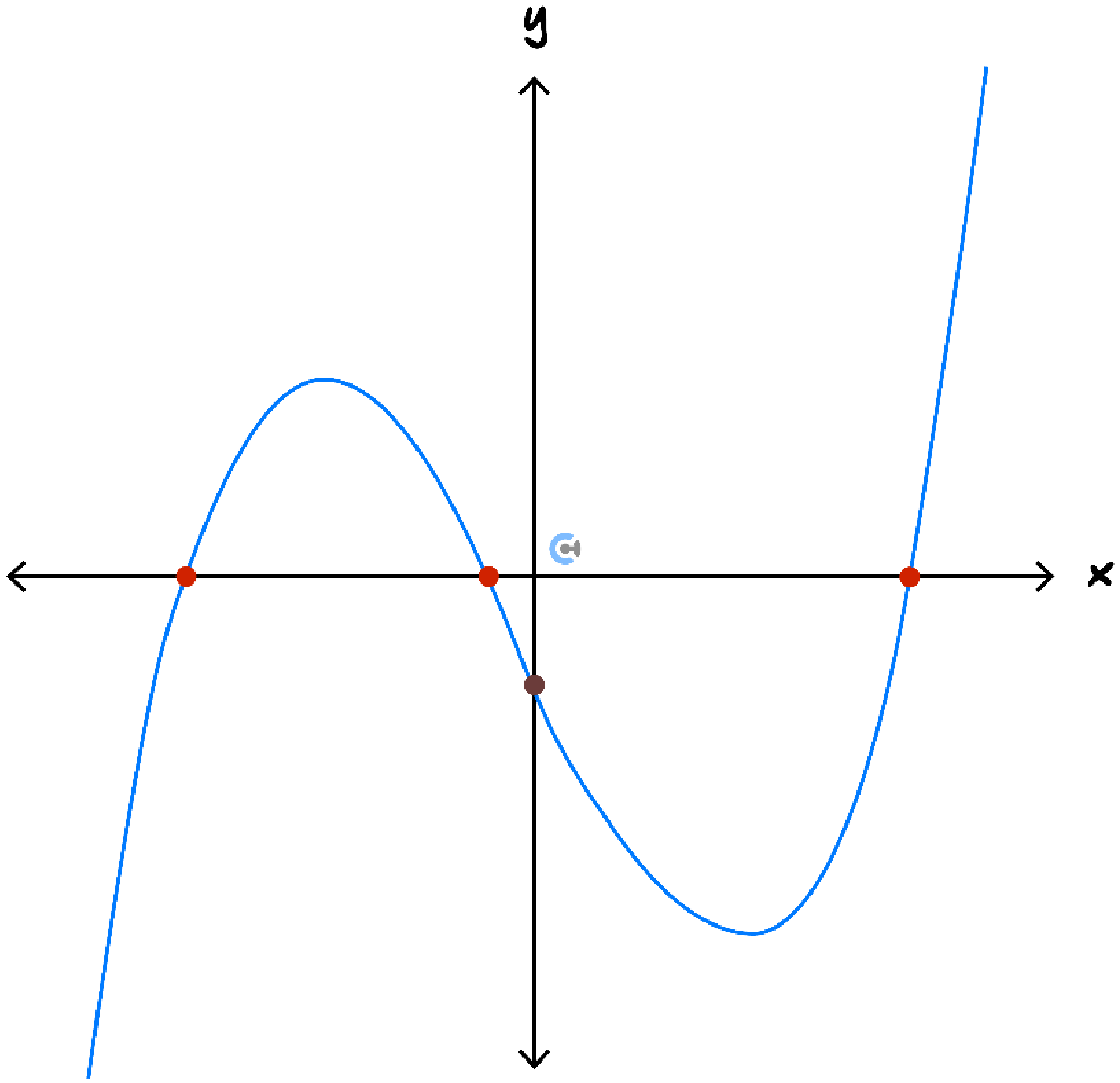
$$y \text{ value of } f'(x) = \text{Gradient of } f(x)$$

- **Steps**

1. Plot x-intercept at the same x value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.

Question 11

Sketch the derivative graph of the function shown below, on the same set of axes.



Limits

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a."

- Limit is the value that a function (y-value) approaches as the x-value approaches a value.

Question 12

Evaluate the following limit:

$$\lim_{x \to 1} \left(3 + \frac{1}{x^2}\right)$$

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Question 13

Evaluate $\lim_{x \to -2} \left(\frac{-1}{(x+2)^2} + 4 \right)$.

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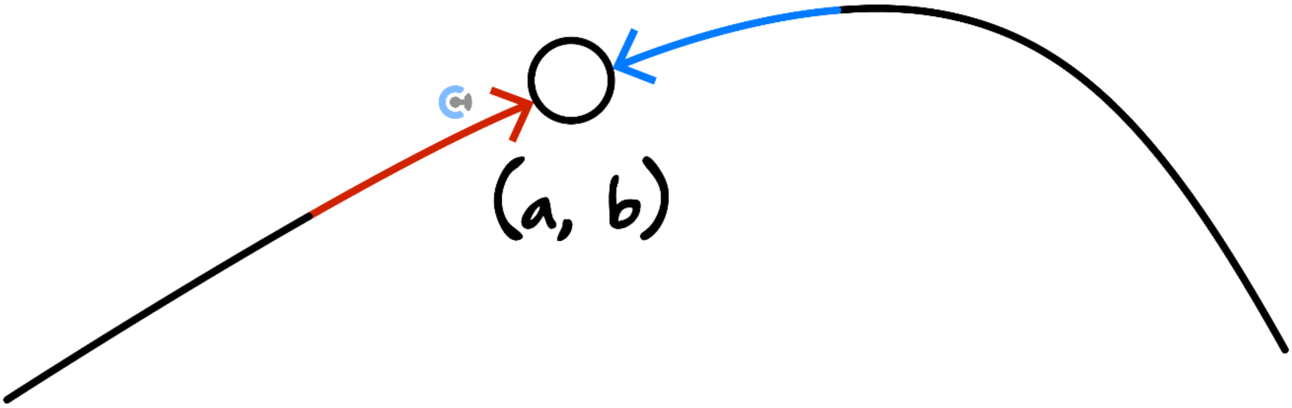
TIP: Sketch the function and see the y-value that the function approaches.

Validity of Limits



$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

- Limit is defined when the left limit equals the right limit.

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Question 14

Consider $f(x) = \frac{1}{x-2} + 4$.

Evaluate the left and right limits of $f(x)$ for $x = 2$, and hence, state whether the limit is defined.

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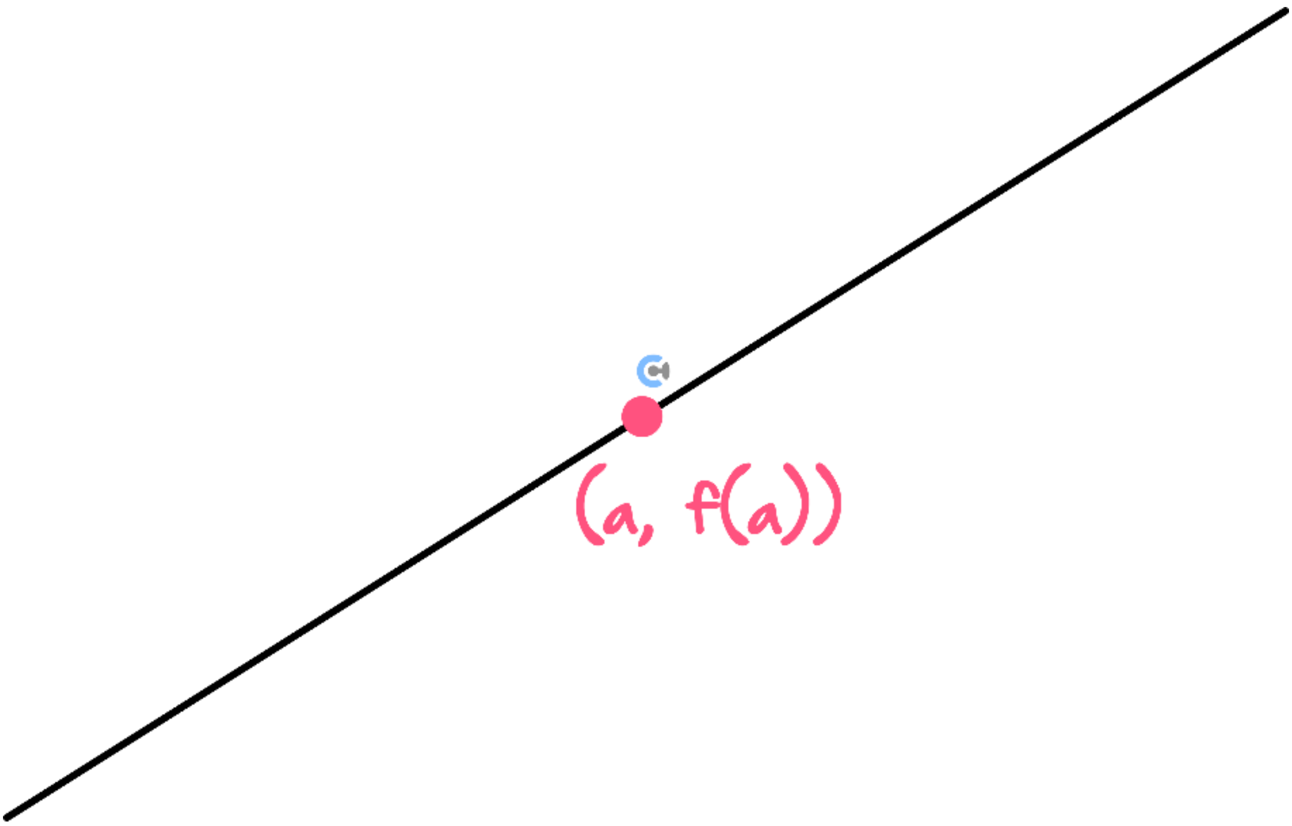
Continuity



- A function f is said to be continuous at a point $x = a$ if:

   1. $f(x)$ is defined at $x = a$.

   2. $\lim\limits_{x \to a} f(x)$ exists.

   3. $\lim\limits_{x \to a} f(x) = f(a)$.

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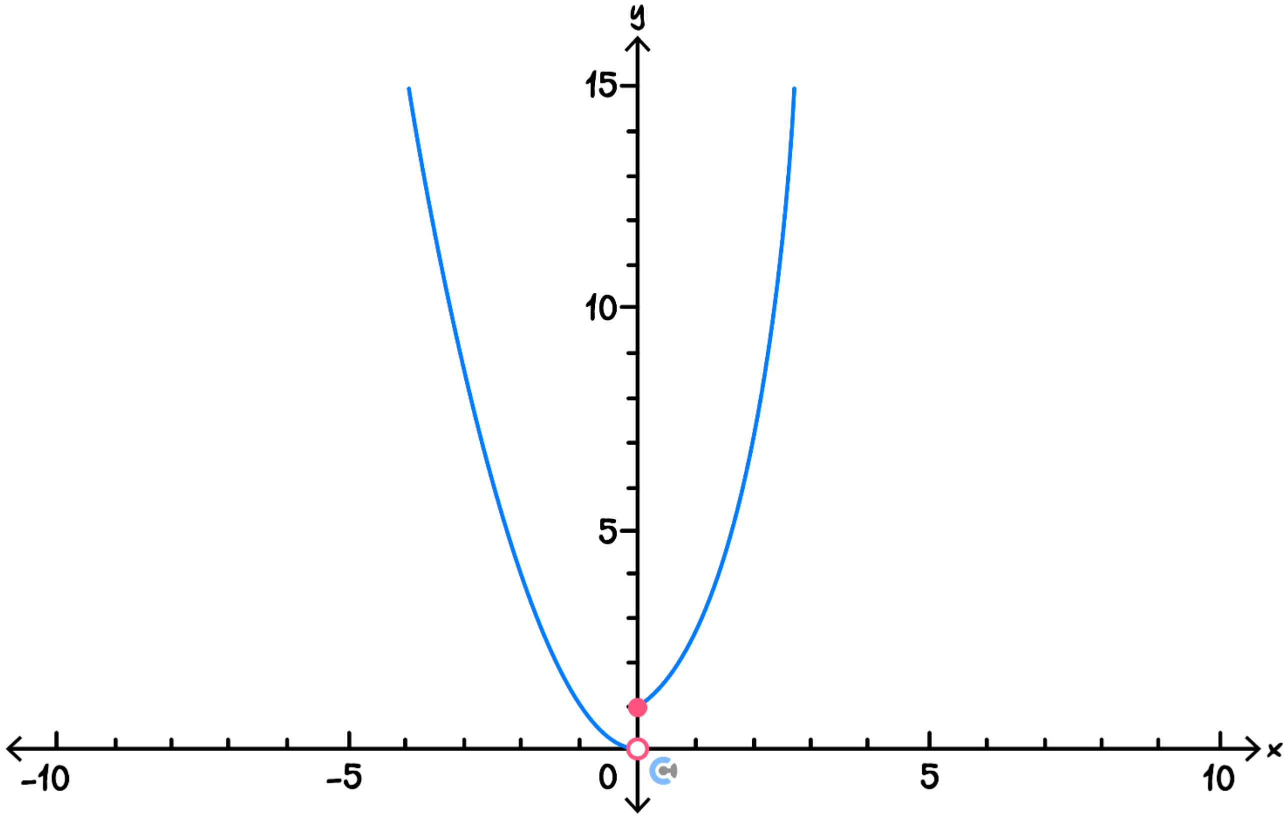
Question 15

Find the values of x for which the following functions have a discontinuity, and state the reason.

$$f(x) = \begin{cases} x^2, & x \in (-\infty, 0) \\ e^x, & x \in [0, \infty) \end{cases}$$



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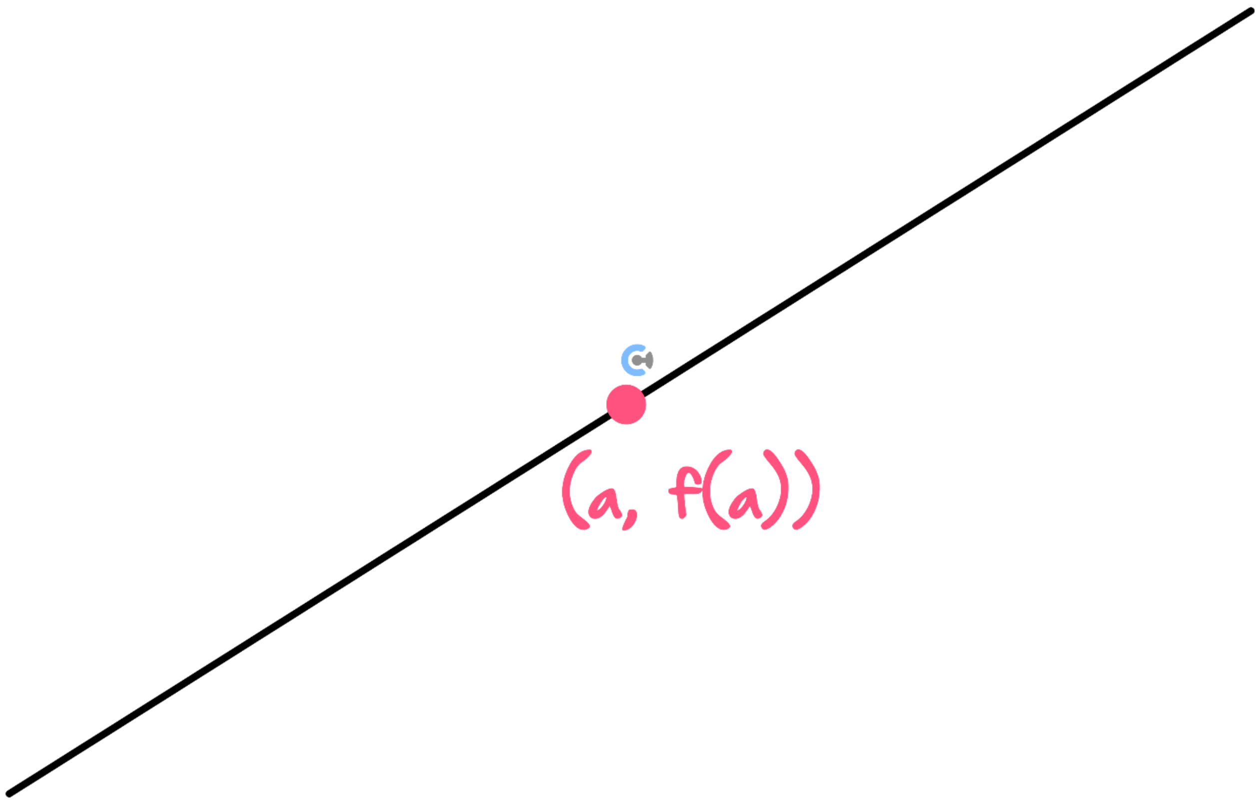
<u>Differentiability</u>



- **A function f is said to be differentiable at a point $x = a$ if:**

  1. **$f(x)$ is continuous at $x = a$.**

  2. **$\lim_{x \to a} f'(x)$ exists.**

     - Limit exists when the left and right limits are the same.

     - Gradient on the ______________________ must be the same.

- We **cannot** differentiate:

  1. Discontinuous Points

  2. Sharp Points

  3. Endpoints

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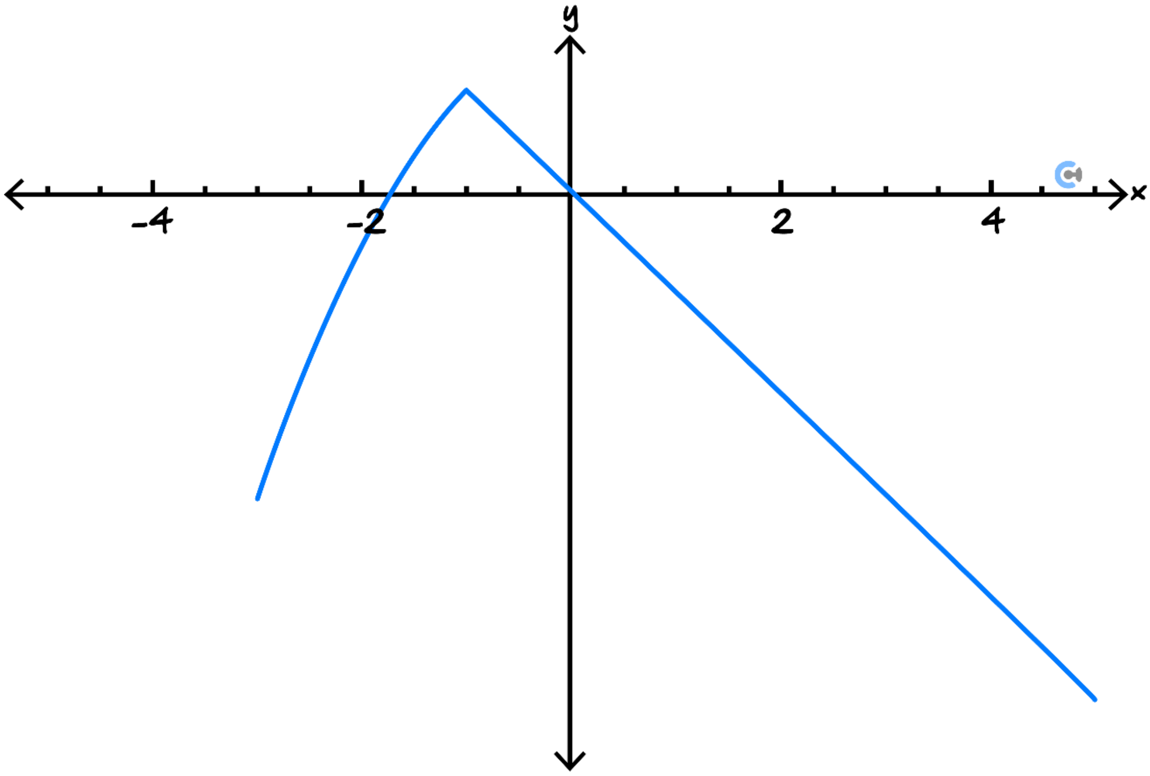
Question 16

Consider the function below:



$$f(x) = \begin{cases} -x^2 + 3, & -3 \leq x \leq -1 \\ -2x, & x > -1 \end{cases}$$

State the points that are not differentiable and state the reason.

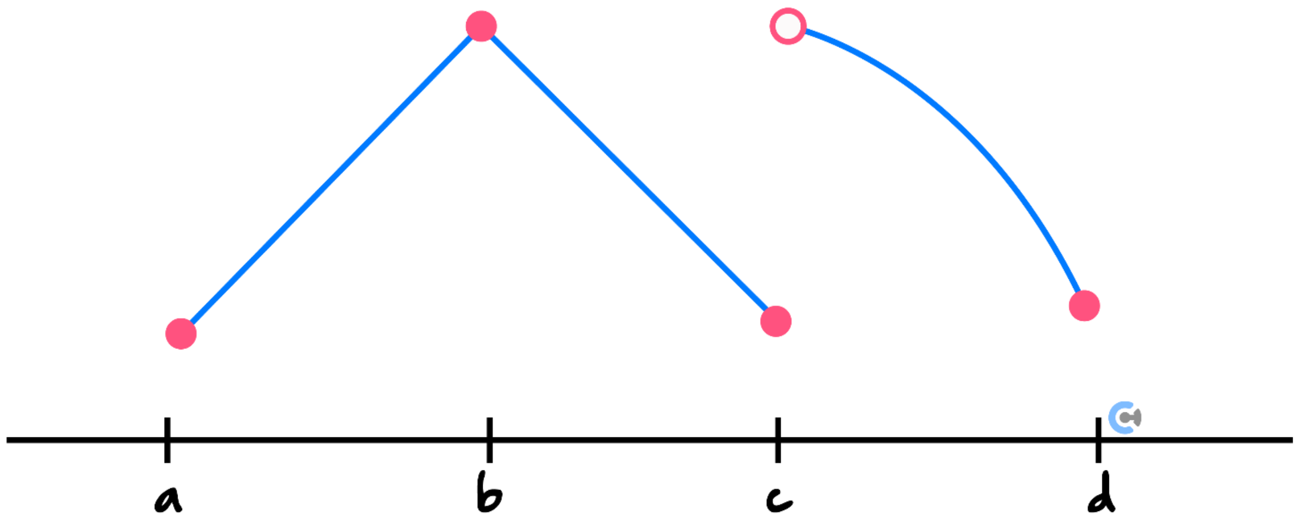
NOTE: Left and right limit of the gradient is simply the gradient from the LHS and RHS.

ALSO NOTE: We call this a sharp point!

NOTE: We cannot differentiate endpoints as they only have a left or right limit.

Question 17

Find the domain of the derivative function for the function shown below.



NOTE: Endpoints, sharp points, and points of discontinuity need to be taken out.

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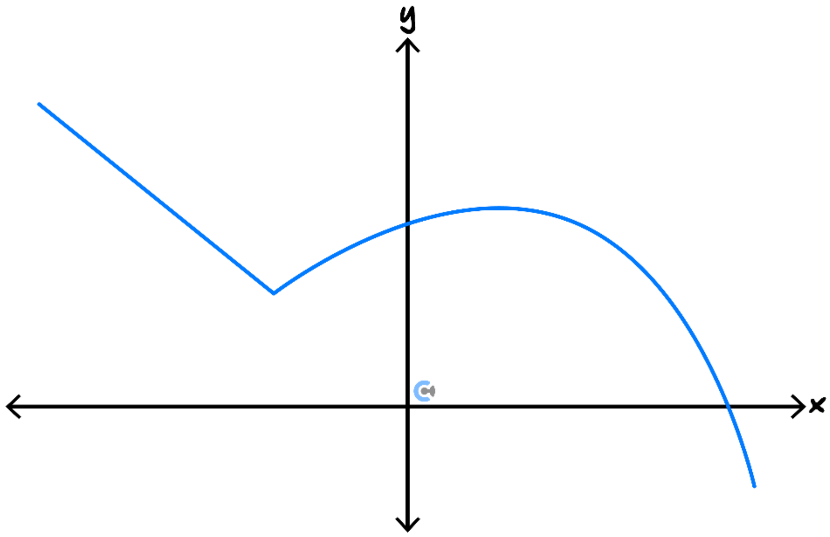
Sub-Section: Defining Derivative Functions

Finding the Derivative of Hybrid Functions

Definition



1. Simply derive each function.

2. Reject the values for x that are not differentiable from the domain.

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Question 18

For the following function, define the derivative function.

$$f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 3x - 4, & x > 0 \end{cases}$$

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Definition

Second Derivatives

- The derivative of the derivative.
- To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

Concavity

- **Concave up is when the gradient is increasing.**

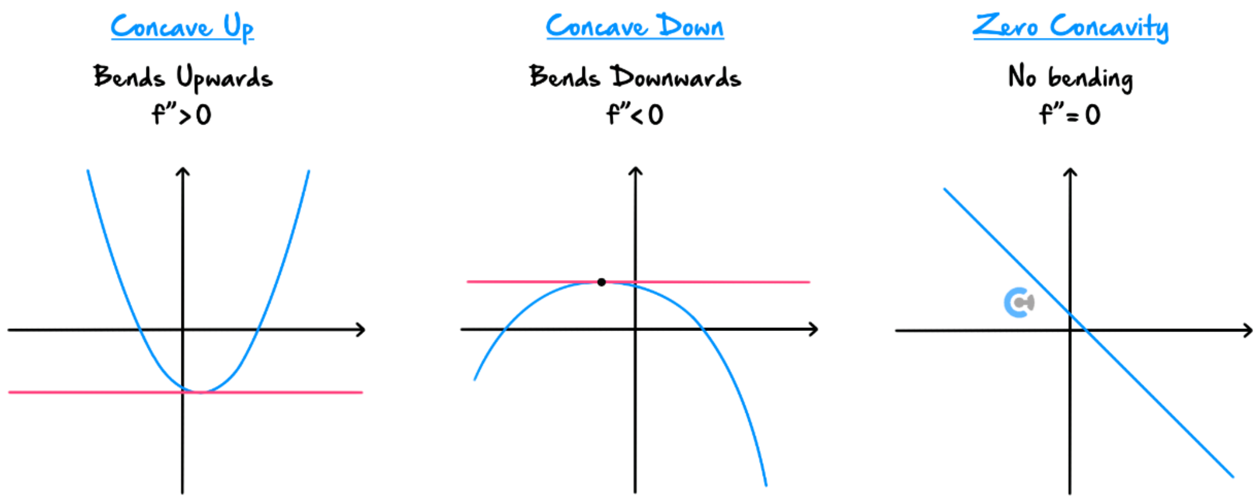
$$f''(x) > 0 \rightarrow \text{Concave Up}$$

- **Concave down is when the gradient is decreasing.**

$$f''(x) < 0 \rightarrow \text{Concave Down}$$

- **Zero concavity is when the gradient is neither increasing nor decreasing.**

$$f''(x) = 0 \rightarrow \text{Zero Concavity}$$



Concavity is also linked to how the curve is bent.

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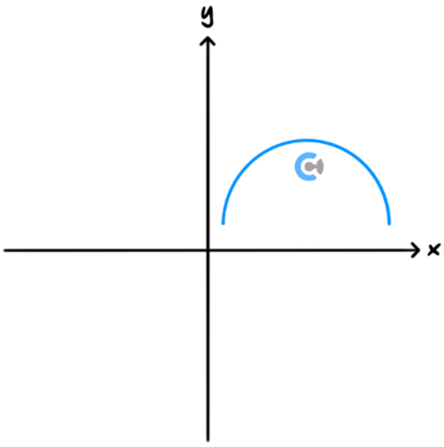
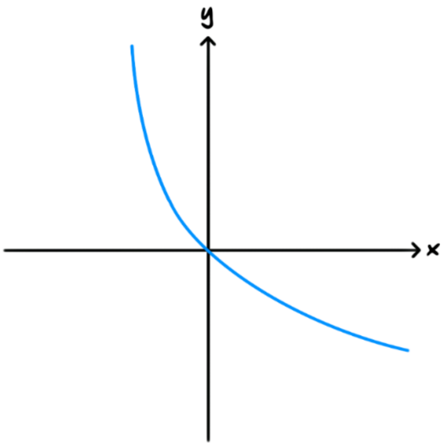
Question 19

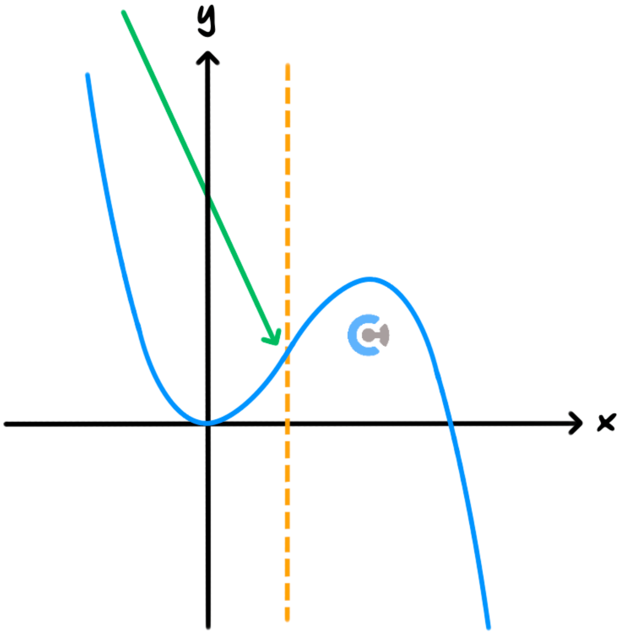
Classify the following curves as concave up or down.





Points of Inflection

- A point at which a curve **changes concavity** is called a **point of inflection**.

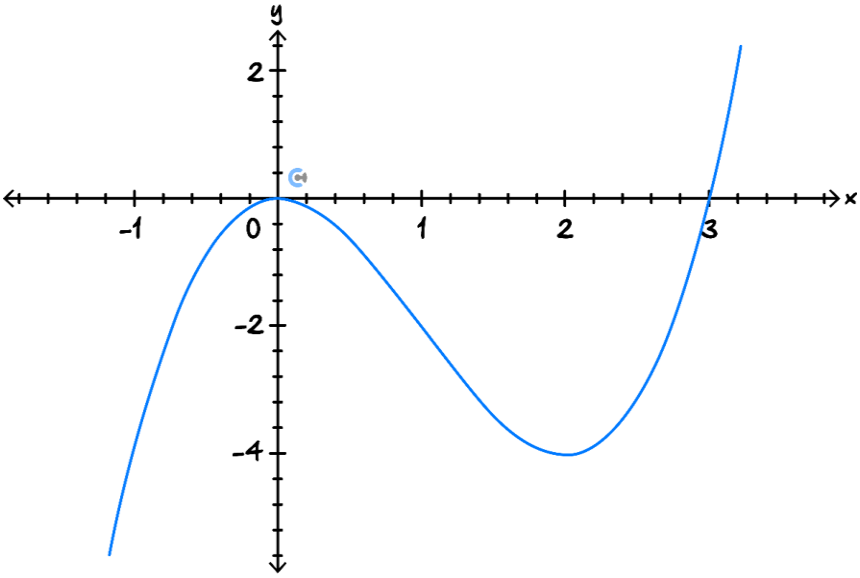


$$f''(x) = 0$$

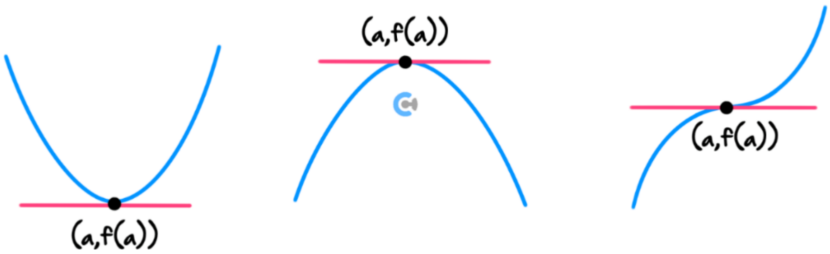
Simply, it is when the bending changes.

Question 20

Circle the point of inflection on the graph below.



The Second Derivative Test



- Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
  - Concave up gives us a local minimum.

$$f''(x) > 0 \rightarrow \text{Local Minimum}$$

  - Concave down gives us a local maximum.

$$f''(x) < 0 \rightarrow \text{Local Maximum}$$

  - Zero concavity gives us a stationary point of inflection.

$$f''(x) = 0 \rightarrow \text{Stationary Point of Inflection}$$

Question 21

Consider the function $f(x) = \log_e(x^2 + 4)$.

Find the stationary point and identify its nature by using the second derivative test.

NOTE: This is much faster than using the table (comparing neighbouring gradients) from [2.1].

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Section B: Warmup Test (14 Marks)

INSTRUCTION: 14 Marks. 11 Minutes Writing.

Question 22 (3 marks)

a. Let $y = x^3 \cos(2x)$.

Find $\frac{dy}{dx}$. (1 mark)

b. Evaluate $f'(3)$, where $f: R \to R$, $f(x) = e^{-x^2+4x}$. (2 marks)

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Question 23 (4 marks)

Consider $f(x) = xe^{2x}$.

a. Find the stationary point and its nature. (3 marks)

b. Hence, state the value(s) of x where the function is strictly increasing. (1 mark)

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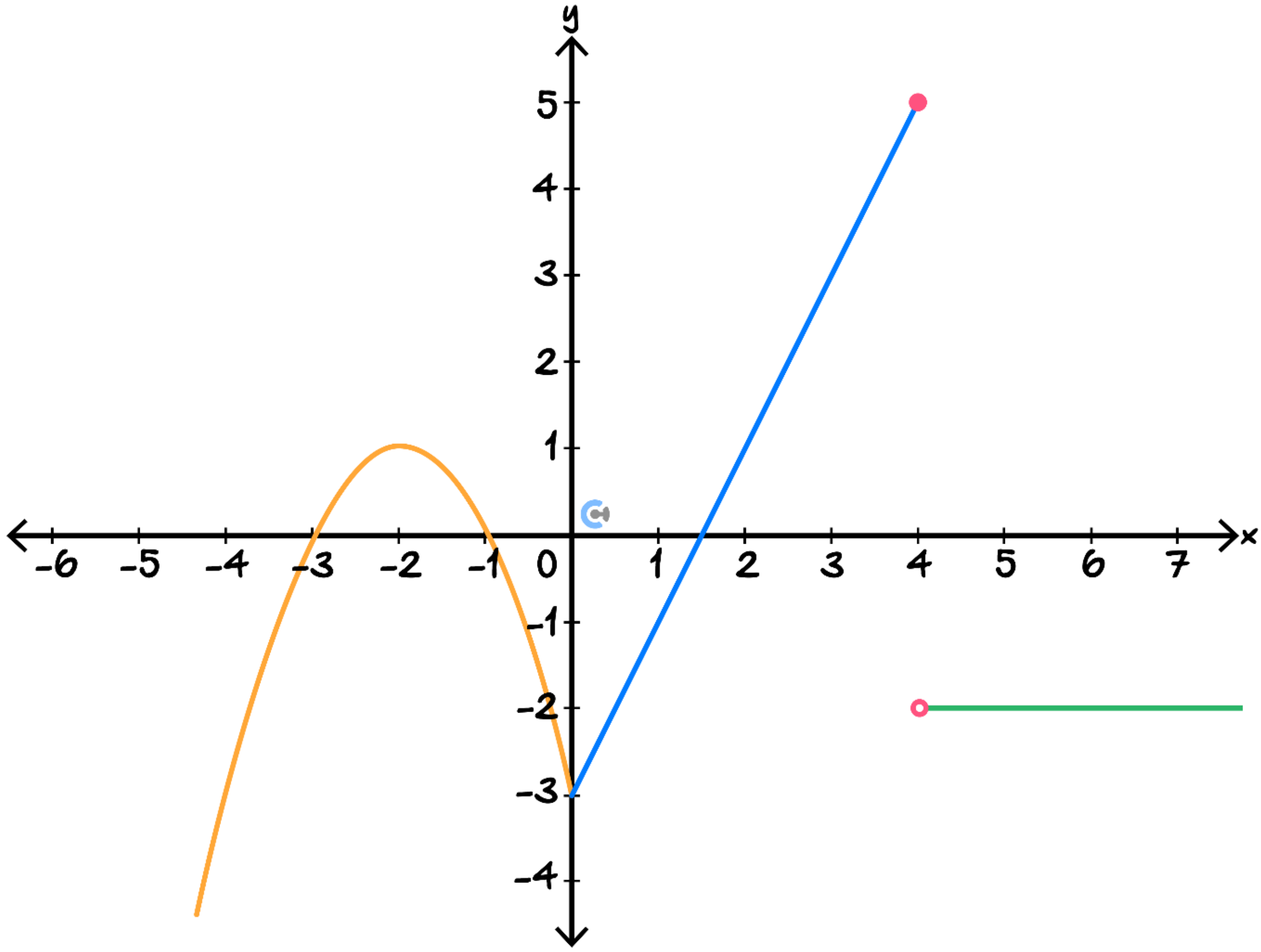
Question 24 (4 marks)

The diagram below shows the graph of a function with domain R.



a. For the graph shown above, sketch on the same set of axes, the graph of the derivative function. (3 marks)

b. Hence, state the domain of the derivative function. (1 mark)

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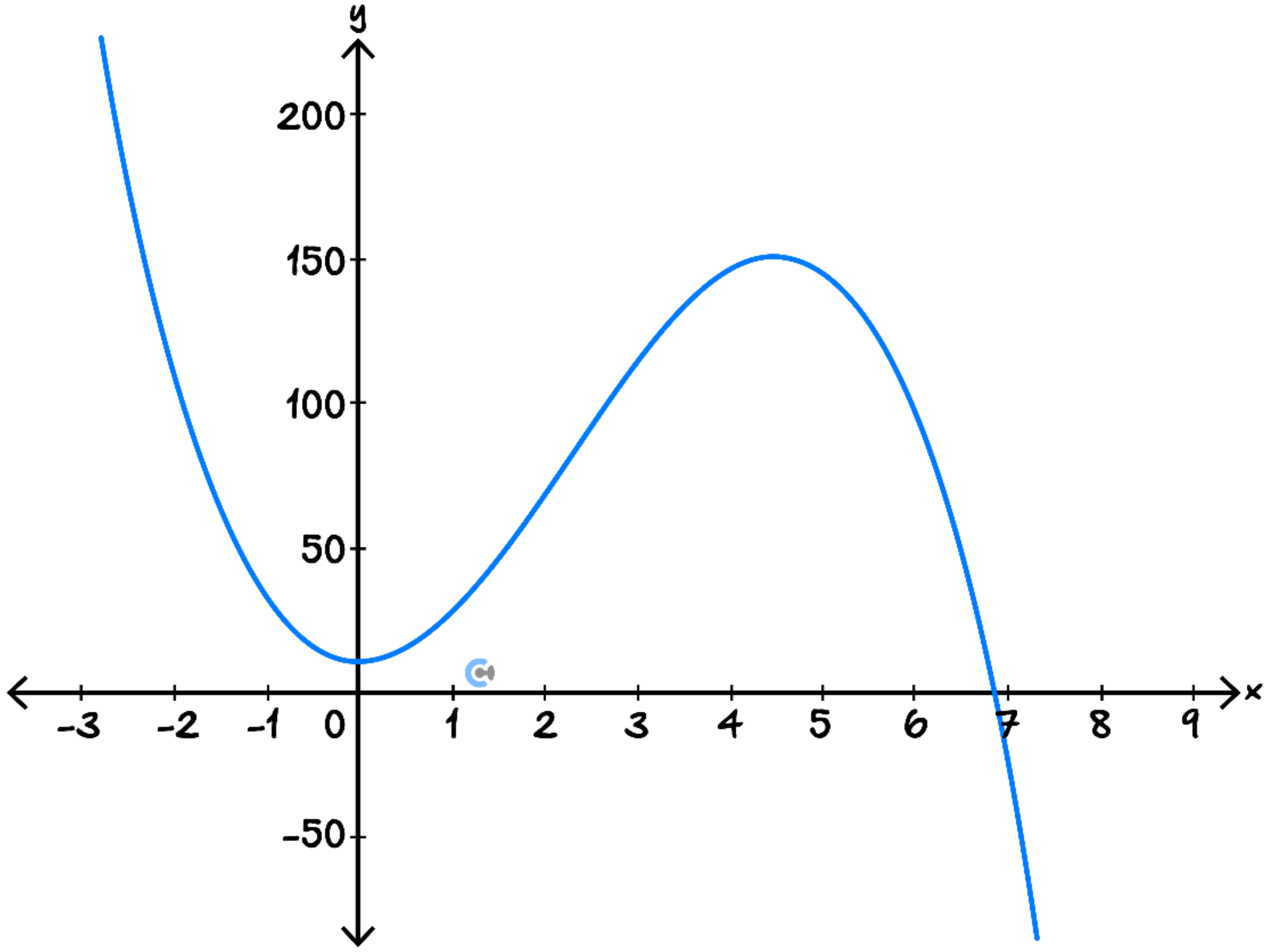
Question 25 (3 marks)

a. Circle the point of inflection on the graph below. (1 mark)



b. State whether at $x = 1$ the gradient is increasing or decreasing. Hence, state the concavity at $x = 1$. (2 marks)

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Section C: Differentiation Exam Skills

Sub-Section: Functional Notational Derivatives with Nested Chain and Product Rule

What happens when $\log_e(f(x))$ becomes $\log_e(f(g(x)))$?

Question 26 Walkthrough.

Find the derivative of $\sin(f(2x))$ with respect to x.

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Question 27

Find the derivative of $\log_e(x)f(x^2+1)$ with respect to x.

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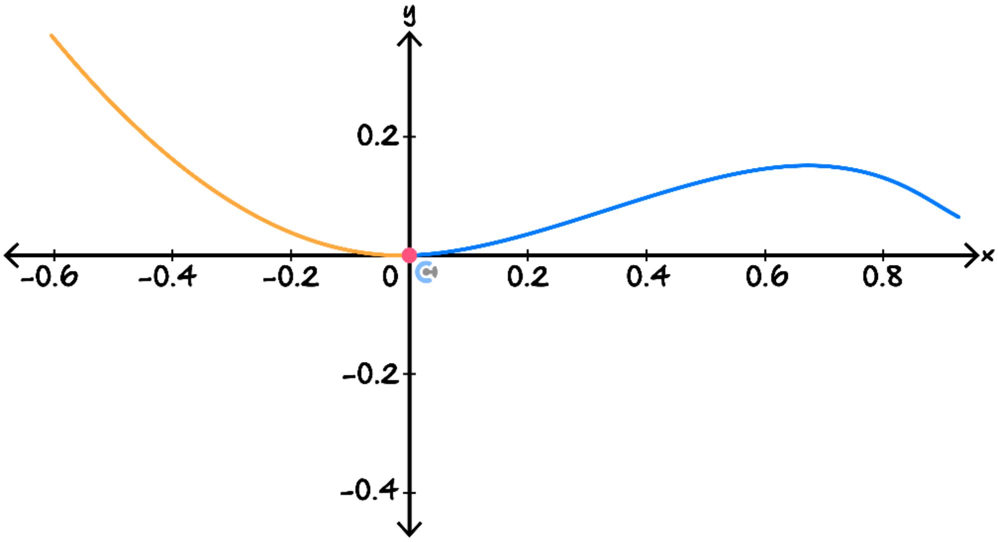
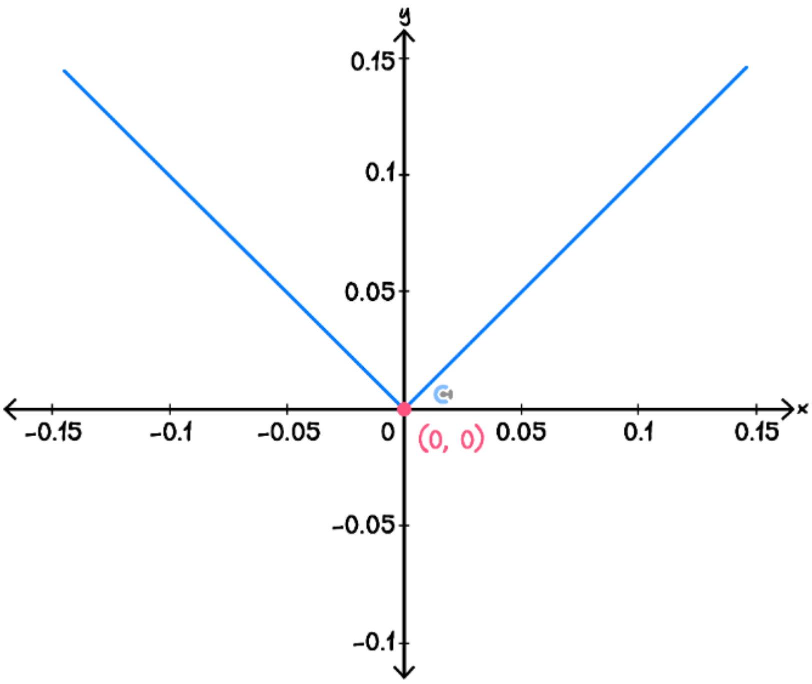
Sub-Section: Apply Differentiability to Join Two Functions Smoothly

Exploration: Joining Smoothly





- For these two hybrid functions, which one of the functions looks like it joins smoothly at $(0,0)$?

- Both graphs have different branches with the same value at $(0,0)$, but for the first graph, the two branches also have the same ______________________.

Definition

Joining Smoothly

- Let two different curves be defined as $f(x)$ and $g(x)$. For these two curves to join smoothly at $x = a$, they have to satisfy:
  - $f(a) = g(a)$
  - $f'(a) = g'(a)$
- In other words, the function must be **continuous** and **differentiable** at that point!

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Question 28 Walkthrough.

Consider the function:

$$f(x) = \begin{cases} x^2 + 4x + a, & x < 0 \\ -x^2 + bx + 3, & x \geq 0 \end{cases}$$

Find the values of a and b such that the graph of f is joins smoothly at $x = 0$.

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Question 29

Consider the function:

$$f(x) = \begin{cases} x - a, & x < 2 \\ x^2 - bx + 2, & x \geq 2 \end{cases}$$

Find the real integer values of a and b such that the graph of f is joining smoothly at $x = 2$.

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Section D: Exam 1 (22 Marks)

INSTRUCTION: 22 Marks. 26 Minutes Writing.

Question 30 (4 marks)

a. Let $y = x^3 \sin(2x)$. Find $\frac{dy}{dx}$. (2 marks)

b. Evaluate $f'(2)$ where $f(x) = e^{-3x^2+4x+3}$. (2 marks)

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Question 31 (4 marks)

Let $f(x) = x^3 - 3x^2 - 4x + 2$.

a. Find the average rate of change of f over the interval $x \in [-1, 2]$. (2 marks)

b. The gradient of f when $x = a$ is the same as the average rate of change of f over the interval $x \in [-1, 2]$. Find the possible values of a. (2 marks)

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Question 32 (4 marks)

Let $f(x) = x^3 - 6x^2 + 9x + 3$.

a. Find the coordinates of all stationary point(s) of f. (3 marks)

b. State the nature of any stationary point found in **part. a.** (1 mark)

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Question 33 (4 marks)

Let $f: \left(-\infty, \frac{1}{2}\right] \to R$, where $f(x) = \sqrt{1-2x}$.

a. Define $f'(x)$. (2 marks)

b. Find the angle θ from the positive direction of the x-axis to the tangent to the graph of f at $x = -1$, measured in the anti-clockwise direction. (2 marks)

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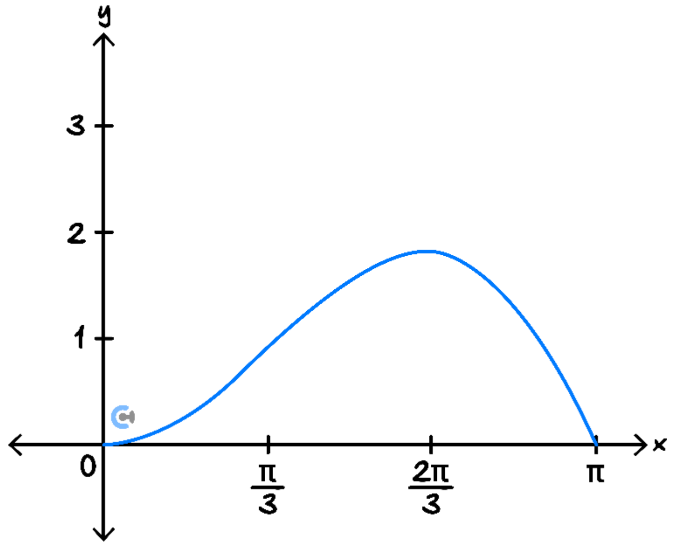
Question 34 (6 marks)

Part of the graph of $f: [-\pi, \pi] \to R, f(x) = x\sin(x)$ is shown below.



a.

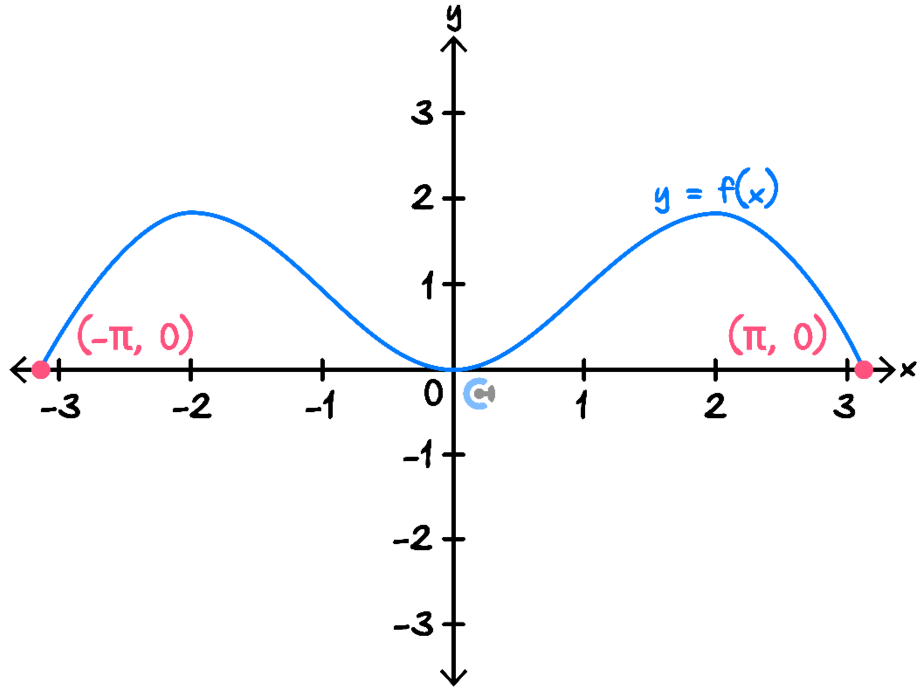
i. Find $f'(x)$. (1 mark)

ii. Determine the range of $f'(x)$ over the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. (1 mark)

iii. Hence, verify that $f(x)$ has a stationary point for $x \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. (1 mark)

b. On the set of axes below, sketch the graph of $y = f'(x)$ on the domain $[-\pi, \pi]$, labelling the endpoints with their coordinates.

You may use the fact that the graph of $y = f'(x)$ has a local minimum at approximately $(-1.1, -1.4)$ and a local maximum at approximately $(1.1, 1.4)$. (3 marks)



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Section E: Tech-Active Exam Skills

Sub-Section: Finding a Derivative Function

Calculator Commands: Finding Derivatives

Mathematica	TI	Casio
`f'[x]`	Shift Minus $\frac{d}{dx}(f(x))$	Math 2 $\frac{d}{dx}(f(x))$

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Sub-Section: Solve for Strictly Increasing and Decreasing Using Technology

Definition

Steps for Finding Strictly Increasing/Decreasing Regions

1. Plot the graph on CAS.
2. Find stationary points.
3. Use a graph to determine which regions are increasing/decreasing.

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Question 35

Find the values of x for which the function,

$$f: [0, 8] \to R, f(x) = -2x^3 + 3x^2 + 36x - 13$$

is strictly increasing.

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Sub-Section: Find Derivatives with Functional Notation

Simply derive on your technology!

Question 36 Walkthrough. Tech-Active.

If $y = e^{(f(x))^2}$ then, $\frac{dy}{dx}$ is equal to:

A. $f(x)e^{f(x)}$

B. $2f(x)f'(x)e^{(f(x))^2}$

C. $f(x)f'(x)e^{(f(x))^2}$

D. $f'(x)e^{(f(x))^2}$

Question 37

If $y = f(x)\left(g(x)\right)^2$ then, $\frac{dy}{dx}$ is equal to:

A. $f(x)g(x)$

B. $2f'(x)g(x)$

C. $2f(x)g'(x)g(x) + f'(x)\left(g(x)\right)^2$

D. $2g'(x)g(x) + f'(x)\left(g(x)\right)^2$

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Sub-Section: Joining Smoothly

Calculator Commands: Joining Smoothly

Mathematica

```
f[x_] := One Function
g[x_] := Another Function
Solve[f[x value] == g[x value] && f'[x value] == g'[x value]]
```

TI and Casio

- Define each branch as $f(x)$ and $g(x)$.
- TI: Define its derivative as $df(x)$ and $dg(x)$

  Casio: Define them as different names
- Solve $f(a) = g(a)$ and $df(a) = dg(a)$ simultaneously.

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Question 38 Tech-Active.

Consider the function:

$$f(x) = \begin{cases} \log_e(x-a), & x < 2 \\ x^2 - bx + 2, & x \geq 2 \end{cases}$$

Find the real integer values of a and b such that the graph of f is joining smoothly at $x = 2$.

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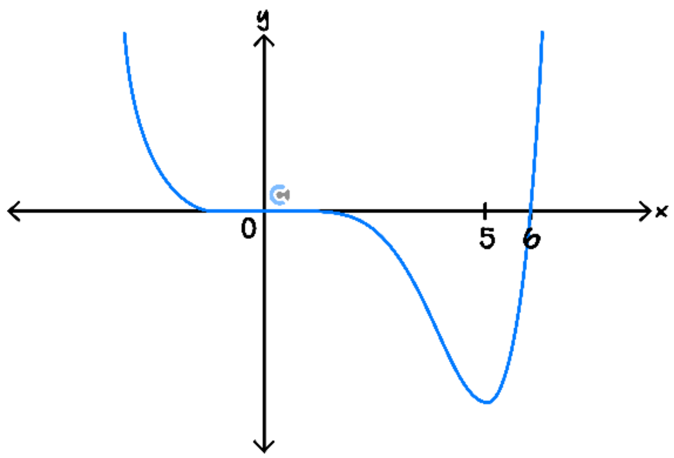
Section B: Exam 2 (21 Marks)

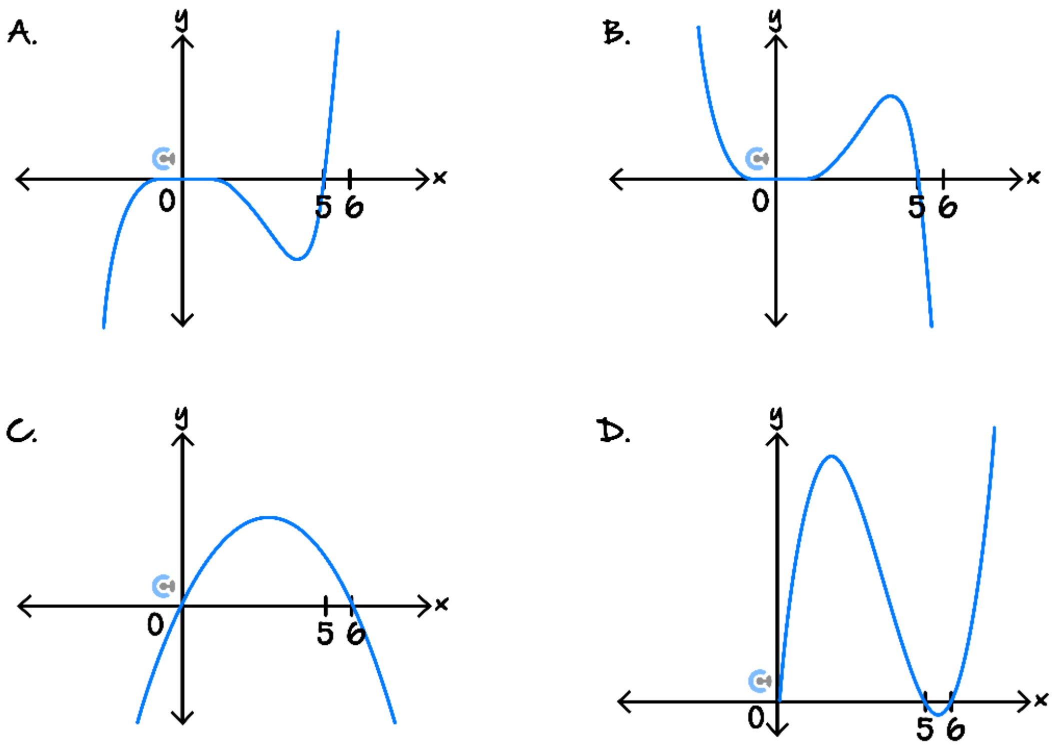
INSTRUCTION: 21 Marks. 25 Minutes Writing.

Question 39 (1 mark)

Part of the graph of $y = f(x)$ is shown below.



The corresponding part of the graph of $y = f'(x)$ is best represented by:



Question 40 (1 mark)

If $y = f(x)\log_e(g(2x))$, then $\frac{dy}{dx}$ is equal to:

A. $f'(x)\log_e(g(2x))$

B. $f'(x)\log_e(g(2x)) + \frac{2f(x)g'(2x)}{g(2x)}$

C. $f'(x)\log_e(g(2x)) + \frac{f(x)}{g(2x)}$

D. $f(x)\log_e(g'(2x)) + f'(x)g(2x)$

Question 41 (1 mark)

Let f and g be two functions defined such that $f(1) = 2$, $f'(1) = -8$, $g(1) = -1$, and $g'(1) = 3$.

If $k(x) = f(x)g(x)$, then the value of $k'(1)$ is equal to:

A. 12

B. 14

C. 6

D. 4

Question 42 (1 mark)

The average rate of change of the function with the rule $f(x) = x^2 - 2x$ over the interval $[1, a]$, where $a > 1$, is 8.

The value of a is:

A. 9

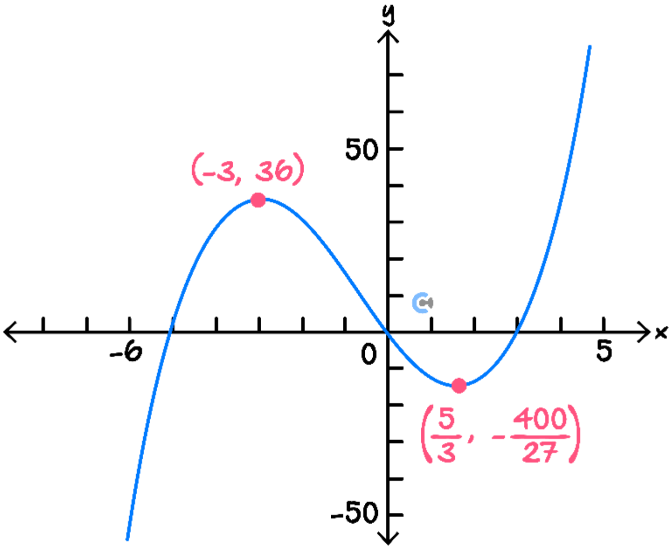
B. 8

C. 7

D. 4

Question 43 (1 mark)

Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.



$f'(x) < 0$ for the interval:

A. $(0, 3)$

B. $(-\infty, -5) \cup (0, 3)$

C. $(-\infty, -3) \cup \left(\frac{5}{3}, \infty\right)$

D. $\left(-3, \frac{5}{3}\right)$

Question 44 (1 mark)

The function $f(x) = x^3 - 5x^2 + 6x + 4$ is strictly decreasing for:

A. $\frac{5-\sqrt{7}}{3} \leq x \leq \frac{5+\sqrt{7}}{3}$

B. $\frac{5-\sqrt{7}}{3} < x < \frac{5+\sqrt{7}}{3}$

C. $\frac{5-\sqrt{3}}{3} \leq x \leq \frac{5+\sqrt{3}}{3}$

D. $\frac{5-\sqrt{3}}{3} < x < \frac{5+\sqrt{3}}{3}$

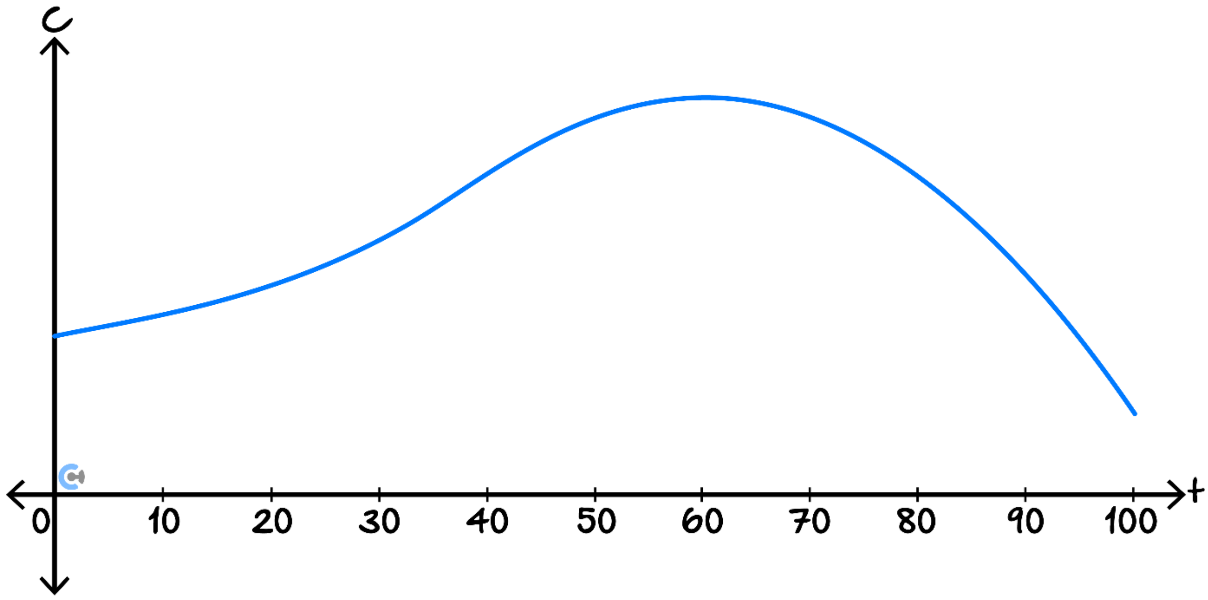
Question 45 (7 marks)

Special agent Jane Blonde has been trapped in a chamber where poisonous gas is leaking.
The concentration C, in mg/m^3, of the gas, t minutes after Jane becomes trapped is given by the smooth and continuous function:

$$C(t) = \begin{cases} \dfrac{320}{80-t}, & 0 \le t \le 40 \\ a(t-60)^2 + b, & 40 < t \le 100 \end{cases}$$

where a and b are constants.

A graph of the function is shown below.



a. What is the initial concentration of the gas in mg/m^3? (1 mark)

b. Show that $a = -\frac{1}{200}$ and $b = 10$. (3 marks)

c. Hence, find the maximum concentration of the gas in mg/m^3. (1 mark)

d. The gas is lethal if exposed to a concentration greater than 9 mg/m^3 for a duration greater than 30 minutes.

Will Jane survive the gas? (2 marks)

Question 46 (8 marks)

The population of rats on an island t days after a virus is introduced is modelled by $P = 5000e^{-\frac{t}{10}}$.

a. Find the initial number of rats on the island. (1 mark)

b. Find the time it takes for the population of rats to halve. Give your answer correct to the nearest day. (1 mark)

c. Write down an expression to calculate the rate of decrease of the population at a time t. (1 mark)

d. Find the rate of decrease of the population after 18 days. Give your answer to the nearest whole number of rats/day. (1 mark)

After 20 days, the remaining rats begin to build an immunity to the virus and start to breed. The population is now given by $P_1 = 5000\,e^{-\frac{t}{10}} + 2(t - 20)\log_e(t - 15)$, where t is the number of days since the virus was first introduced.

e. State the domain of P_1. (1 mark)

f. Find, to the nearest day, the time it takes for the rat population to reach the same number as when the virus was first introduced. (1 mark)

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g. Determine the lowest population of rats that the island ever has. Give your answer correct to the nearest rat. (2 marks)

____

____

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Contour Checklist

☐ Learning Objective: [2.3.1] – Find Derivatives with Functional Notation

Key Takeaways

☐ To derive composite functions like $\sin(f(x))$, apply the __________ rule.

☐ Learning Objective: [2.3.2] – Apply Differentiability to Join Two Functions Smoothly

Key Takeaways

☐ When two functions join smoothly at a point, the ____________ and ____________________ of each function are both equal at that point.

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