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VCE Mathematical Methods ¾
Differentiation Exam Skills [2.3]
Homework Solutions

Homework Outline:

Section A: Compulsory Questions

Sub-Section [2.3.1]: Find General Derivatives With Functional Notation

Question 1

If f is a differentiable function, find $\frac{dy}{dx}$ for the following:

a. $y = \sin(f(x))$

$$f'(x)\cos(f(x))$$

b. $y = f(2x^3)$

$$6x^2 f'\left(2x^3\right)$$

Question 2

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = \log_e\big(f(x)\big) \cdot \cos(x)$

$$\frac{\cos(x)f'(x)}{f(x)} - \sin(x)\log(f(x))$$

b. $y = \frac{xf(x)}{g(x)}$

$$\frac{xg(x)f'(x) - xf(x)g'(x) + f(x)g(x)}{g(x)^2}$$

Question 3

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = e^{f(x)g(x)} \cdot \sin(x^2)$

$$g(x)\sin\left(x^2\right)e^{f(x)g(x)}f'(x) + f(x)\sin\left(x^2\right)e^{f(x)g(x)}g'(x) + 2x\cos\left(x^2\right)e^{f(x)g(x)}$$
$$= \sin\left(x^2\right)e^{f(x)g(x)}\left(g(x)f'(x) + f(x)g'(x)\right) + 2x\cos\left(x^2\right)e^{f(x)g(x)}$$

b. $y = \frac{\log_e\left(f(g(x))\right)}{[g(x)]^2}$

$$\frac{g'(x)f'(g(x))}{g(x)^2 f(g(x))} - \frac{2g'(x)\log(f(g(x)))}{g(x)^3} = \frac{g'(x)\left(g(x)f'(g(x)) - 2f(g(x))\log(f(g(x)))\right)}{g(x)^3 f(g(x))}$$

Sub-Section [2.3.2]: Apply Differentiability to Join Two Functions Smoothly

Question 4

A hybrid function is defined as:

$$f(x) = \begin{cases} ax + b, & x < 3 \\ x^2 - 3x + 4, & x \geq 3 \end{cases}$$

Find the values of a and b such that $f(x)$ is smooth and continuous at $x = 3$.

Continuous: $3a + b = 4$
Smooth: $a = 3$
So $a = 3$ and $b = -5$

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Question 5

A function $f(x)$ is given by:

$$f(x) = \begin{cases} ax^2 + bx + 1, & x < 2 \\ x^3 - 2x + 3, & x \geq 2 \end{cases}$$

Find the values of a and b such that $f(x)$ is both continuous and differentiable at $x = 2$.

Continuous: $1 + 4a + 2b = 7$
Smooth: $4a + b = 10$
Solving simultaneously yields $a = \frac{7}{2}$ and $b = -4$

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Question 6

Consider the hybrid function:

$$f(x) = \begin{cases} 2\sin(x-a)+b, & x<1 \\ x^2-2x+2, & x \geq 1 \end{cases}$$

Find all possible values of a and b so that f is a smooth and continuous function for all $x \in \mathbb{R}$.

Continuous: $b + 2\sin(1-a) = 1$

Smooth: $2\cos(1-a) = 0 \implies \cos(1-a) = \frac{\pi}{2} + 2n\pi$ or $\cos(1-a) = -\frac{\pi}{2} + 2n\pi$

In the first case $\sin(1-a) = 1 \implies b+2 = 1 \implies b = -1$. So one possible set of solutions is

$$b = -1 \text{ and } a = 1 - \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$

In the second $\sin(1-a) = -1 \implies b - 2 = 1 \implies b = 3$. So another possible set of solutions is

$$b = 3 \text{ and } a = 1 + \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$

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Sub-Section: Exam 1 Questions

Question 7

a. Let $f: (-2, \infty) \to \mathbb{R}, f(x) = \frac{x+3}{x+2}$.

Find $f'(x)$.

Note $f(x) = 1 + \frac{1}{x+2}$. Then $f'(x) = -\frac{1}{(x+2)^2}$

b. Let $g(x) = (3 - x^3)^3$. Evaluate $g'(1)$.

$g'(x) = -9x^2(3 - x^3)^2$.
$g'(1) = -9(2)^2 = -36$

c. Evaluate $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$.

$$\lim_{x \to 2} \frac{(x-2)(x+4)}{x-2} = \lim_{x \to 2}(x+4) = 6$$

Question 8

Let $f(x) = 2x^3 + 3x^2 - 12x + 12$.

a. Find the coordinates of all stationary points of f.

$f'(x) = 6x^2 + 6x - 12 = 0$. Solve

$$x^2 + x - 1 = 0$$
$$(x+2)(x-1) = 0$$
$$\implies x = -2, 1$$

$f(-2) = -16 + 12 + 24 + 12 = 32$ and $f(1) = 2 + 3 - 12 + 12 = 5$.
So stationary points at $(-2, 32)$ and $(1, 5)$

b. State the nature of any stationary points found in **part a.**

$(-2, 32)$ is a local maximum and $(1, 5)$ is a local minimum.

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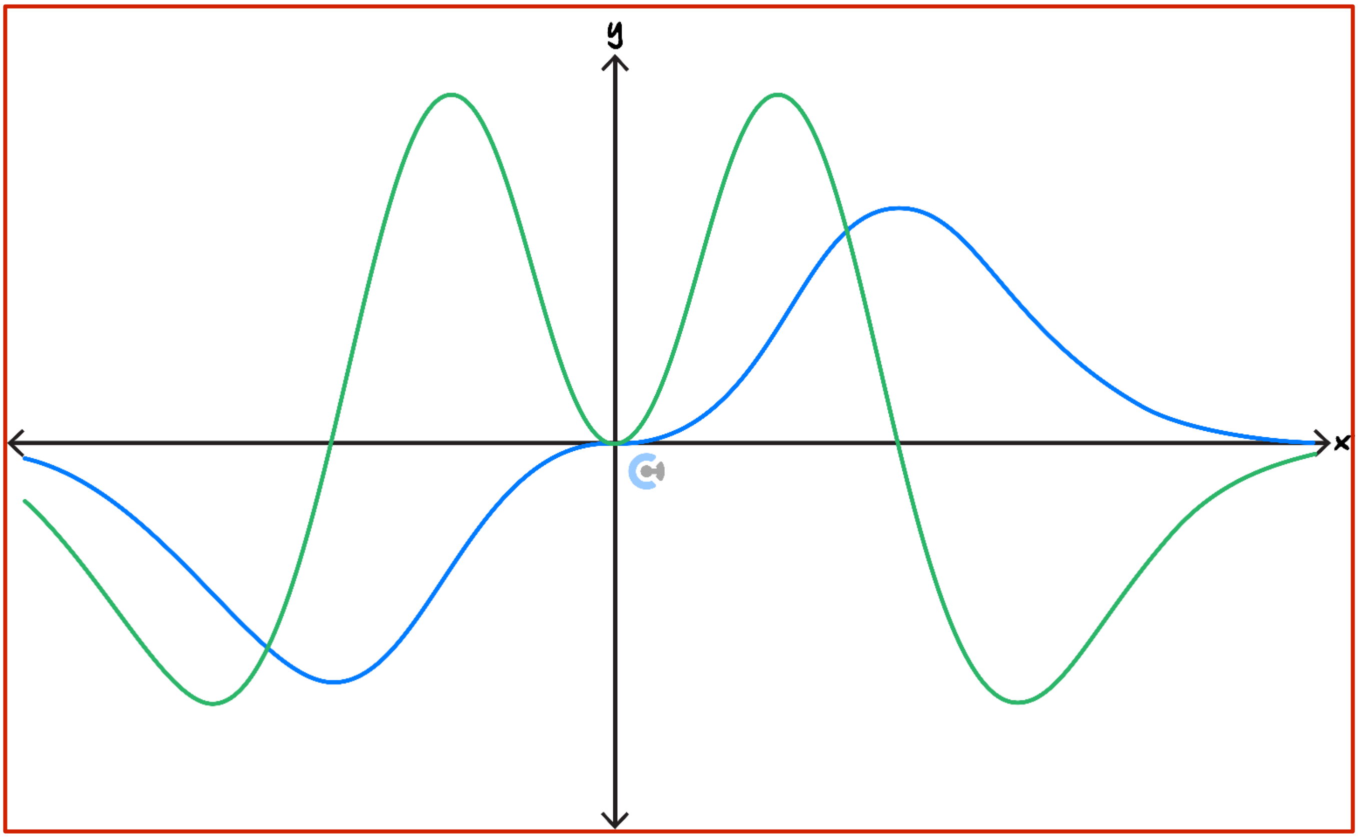
Question 9

The graph of f is shown on the axes below. Sketch the graph of f' on the same set of axes.



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Question 10

Consider the function $f(x) = x^3e^{-x^2}$.

a. Find $f'(x)$ in the form $ax^2e^{-x^2}(b - cx^2)$ for positive integers $a, b,$ and c.

$$f'(x) = 3x^2e^{-x^2} - 2x \cdot x^3e^{-x^2} = x^2e^{-x^2}(3 - 2x^2)$$

b. Hence, find the coordinates for any stationary points of f.

Stationary points when $f'(x) = 0$. Therefore $x = 0$ or

$$\begin{aligned} 3 - 2x^2 &= 0 \\ x^2 &= \frac{3}{2} \\ x &= \pm\sqrt{\frac{3}{2}} \\ &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

$f(0) = 0$ and $f\left(\frac{\sqrt{6}}{2}\right) = \frac{6\sqrt{6}}{8}e^{-3/2}$

The stationary points are $\left(-\frac{\sqrt{6}}{2}, -\frac{3\sqrt{6}}{4e^{3/2}}\right)$, $(0, 0)$ and $\left(\frac{\sqrt{6}}{2}, \frac{3\sqrt{6}}{4e^{3/2}}\right)$.

c. Determine the nature of any stationary points of f.

We can obtain $f''(x) = 2e^{-x^2}x\left(x^2-3\right)\left(2x^2-1\right)$.
From here we see that $f''(0) = 0$ so $(0,0)$ is a staionary point of inflection.
$f''\left(-\frac{\sqrt{6}}{4}\right) > 0$ so $\left(-\frac{\sqrt{6}}{2}, -\frac{3\sqrt{6}}{4e^{3/2}}\right)$ is a local minimum and $f''\left(\frac{\sqrt{6}}{4}\right) < 0$ so $\left(\frac{\sqrt{6}}{2}, \frac{3\sqrt{6}}{4e^{3/2}}\right)$ is a local maximum.

Probably simpler to just investigate points close to the stationary points.
For $f'(x)$ note that $x^2e^{-x^2} \geq 0$ so we just investigate the sign change of $g(x) = 3-2x^2$ close to stationary points.
Note that $1 < \frac{\sqrt{6}}{2} < 2$.
Now $g(-2) < 0$ and $g(-1) > 0$ so $\left(-\frac{\sqrt{6}}{2}, -\frac{3\sqrt{6}}{4e^{3/2}}\right)$ is a local minimum.
Now $g(-1) > 0$ and $g(1) > 0$ so $(0,0)$ is a staionary point of inflection.
Finally $g(1) > 0$ and $g(2) < 0$ so $\left(\frac{\sqrt{6}}{2}, \frac{3\sqrt{6}}{4e^{3/2}}\right)$ is a local maximum.

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Sub-Section: Exam 2 Questions

Question 11

If $y = 2x^2 + 4x + 3$, the rate of change of y with respect to x at $x = k$ is:

A. $4k + 4$

B. $2k + 4$

C. $k^2 + 4$

D. $k^3 + 4$

Question 12

Let $f(x) = (ax + b)^3$ and let g be the inverse function of f.

Given that $f(0) = 1$, what is the value of $g'(1)$?

A. $\frac{3}{a}$

B. 1

C. $\frac{1}{3a}$

D. 0

Question 13

If $f(x) = e^{g(x^3)}$, where g is a differentiable function, then $f'(x)$ is equal to:

A. $3x^2e^{g(x^3)}$

B. $3x^2g(x^3)e^{g(x^3)}$

C. $3x^2g'(x^3)e^{g(x^3)}$

D. $3x^2g'(3x^2)e^{g(x^3)}$

Question 14

For two differentiable functions f and g the derivative of $f(3x) \times g(x^2)$ is:

A. $6xf'(3x)g'(x^2)$

B. $x^2f(3x)g'(x^2) + 3xf'(3x)g(x^2)$

C. $3f(3x)g'(x^2) + 3xf'(3x)g(x^2)$

D. $2xf(3x)g'(x^2) + 3f'(3x)g(x^2)$

Question 15

Consider the function:

$$f(x) = \begin{cases} 2x^2 + ax + 1 & x \leq 2 \\ x^2 + 3x + b & x > 2 \end{cases}$$

If f is a smooth and continuous function for all $x \in \mathbb{R}$ then the values of a and b are:

A. $a = 1, b = 3$

B. $a = -1, b = -3$

C. $a = 1, b = -3$

D. $a = -1, b = 3$

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Question 16

Tammy Jones is exploring the jungle looking for a lost civilisation when she is struck by a blowgun dart, fired by the local tribesman.

The dart is poisoned and the concentration of poison, in mg/L, in Tammy's blood t minutes, after she is hit, is given by the continuous function:

$$C(t) = \begin{cases} \dfrac{350}{70-t} & 0 \le t \le k \\ m, & k < t \le 60 \end{cases}$$

a. What is the initial concentration of poison in Tammy's blood?

$C(0) = 5mg/L$

b. Find an expression for m in terms of k.

Function is continuous so $C(k) = m \implies m = \dfrac{350}{70-k}$

c. Find the minimum and maximum values of m.

Minimum when $k = 0 \implies m = 5$
Max when $k = 60 \implies m = \dfrac{350}{10} = 35$

d. Find the domain and rule for the function $C'(t)$.

Sharp points and endpoint are not differentiable.

$$C'(t) = \begin{cases} \dfrac{350}{(70-t)^2} & 0 < t < k \\ 0 & k < t < 60 \end{cases}$$

e. If the rate at which the concentration of poison in Tammy's blood was increasing was 2 mg/L per minute, find the value of t. Express your answer correct to two decimal places.

Solve $C'(t) = 2 \implies t = 56.7712$.
Thus $t = 56.77$ minutes.

Tammy will not survive if the concentration of poison in her blood exceeds 12 mg/L.

f. Given that Tammy is unable to receive any treatment for 60 minutes, find the possible values of k in order for her to survive.

We have $m = \dfrac{350}{70-k}$. For Tammy to survive we require that $m < 12 \implies \dfrac{350}{70-k} < 12$.
This yields $k < \dfrac{245}{6}$ or $k \triangleright 70$. We reject $k > 70$.
So to survive $k \in \left[0, \dfrac{245}{6}\right)$

Question 17

a. Let $f(x) = (x^2 + bx + c)\sqrt{3x - 4}$.

i. Use calculus to find $f'(x)$.

We use the product rule.

$$f'(x) = (2x + b)\sqrt{3x - 4} + \frac{3(x^2 + bx + c)}{2\sqrt{3x - 4}}$$

ii. Hence, express the derivative of $(x^2 + bx + c)\sqrt{3x - 4}$ in the form $\frac{q(x)}{2\sqrt{3x-4}}$ where $q(x)$ is a quadratic function.

$$\begin{aligned} f'(x) &= \frac{(2x + b)\sqrt{3x - 4} \cdot 2\sqrt{3x - 4}}{2\sqrt{3x - 4}} + \frac{3(x^2 + bx + c)}{2\sqrt{3x - 4}} \\ &= \frac{9bx - 8b + 3c + 15x^2 - 16x}{2\sqrt{3x - 4}} \\ &= \frac{15x^2 + (9b - 16)x - 8b + 3c}{2\sqrt{3x - 4}} \end{aligned}$$

b. Let $g(x) = (x^2 + bx - 1)\sqrt{3x - 4}$.

i. State the domain of $g(x)$.

$$\text{dom } g = \left[\frac{4}{3}, \infty\right)$$

ii. Find the values of b for which g has a stationary point.

$$g'(x) = \frac{15x^2 + (9b - 16) - 8b - 3}{2\sqrt{3x - 4}} = 0$$

So we require

$$15x^2 + (9b - 16) - (8b + 3) = 0 \implies x = \frac{1}{30}\left(16 - 9b \pm \sqrt{81b^2 + 192b + 436}\right)$$

We note that the inside of the square roots is always > 0.
No stationary points if both these roots are outside the domain of g.
Therefore solve $\frac{1}{30}\left(16 - 9b \pm \sqrt{81b^2 + 192b + 436}\right) < \frac{4}{3} \implies b > -\frac{7}{12}$.
Therefore g has stationary point for $b < -\frac{7}{12}$. Note that $b = -7/12$ is not included because the derivative at the starting point of g is not defined.

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Section B: Supplementary Questions

Sub-Section [2.3.1]: Find General Derivatives With Functional Notation

Question 18

If f is a differentiable function, find $\frac{dy}{dx}$ for the following:

a. $y = f(x)\tan(x)$

$$\frac{dy}{dx} = f'(x)\tan(x) + \frac{f(x)}{\cos^2(x)}$$

b. $y = \sqrt{f(x)}$

$$\frac{dy}{dx} = \frac{f'(x)}{2\sqrt{f(x)}}$$

Question 19

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = f(e^x) \cdot g(x)$

$$\frac{dy}{dx} = e^x f'(e^x)g(x) + g'(x)f(e^x)$$

b. $y = f(g(\cos(3x))$

$$\frac{dy}{dx} = -3\sin(3x)g'(\cos(3x))f'(g(\cos(3x)))$$

Question 20

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = \sqrt{f(3x^2) + g(2x + f(x))}$

$$\frac{dy}{dx} = \frac{6xf'(3x^2) + (2 + f'(x))g'(2x + f(x))}{2\sqrt{f(3x^2) + g(2x + f(x))}}$$

b. $y = \frac{e^{f(x^2)}}{g(f(x^2)) + f(x^2)}$

$$\frac{dy}{dx} = 2xf'(x^2)\frac{e^{f(x^2)}(g(f(x^2)) + f(x^2)) - (g'(f(x^2)) + 1)e^{f(x^2)}}{(g(f(x^2)) + f(x^2))^2}$$

Question 21

If f and g are differentiable increasing functions, with $g'(x)$ also being one-to-one, what is the maximum amount of stationary points that $y = f(x) + 3x + g(-f(x) - 3x)$ has?

We know that $\frac{dy}{dx} = f'(x) + 3 - (f'(x) + 3)g'(-f(x) - 3x) = (f'(x) + 3)(1 - g'(-f(x) - 3))$.
If we solve it to be 0, since $f'(x) \geq 0$, we must have $(1 - g'(-f(x) - 3)) = 0$.
Since $g'(x)$ is one to one, and so is $-f(x) - 3$, we will only have one stationary point.

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Sub-Section [2.3.2]: Apply Differentiability to Join Two Functions Smoothly

Question 22

A hybrid function is defined as:

$$f(x) = \begin{cases} e^{2x} - 2, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$$

Find the values of a and b such that $f(x)$ is smooth and continuous at $x = 0$.

Continuous: $b = -1$ Smooth: $a = 2$ So $a = 2$ and $b = -1$

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Question 23

A hybrid function is defined as:

$$f(x) = \begin{cases} \log_e(ax), & x < 1 \\ bx^2, & x \geq 1 \end{cases}$$

Where $a > 0$. Find the values of a and b such that $f(x)$ is both continuous and differentiable at $x = 1$.

Continuous: $\log_e(a) = b$
Smooth: $2b = 1$
Solving simultaneously yields $b = \frac{1}{2}$ and $a = \sqrt{e}$.

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Question 24

A hybrid function, $f : \mathbb{R} \to \mathbb{R}$, is defined as:

$$f(x) = \begin{cases} 2x + 4 & x < -2 \\ ax^3 + bx^2 + cx + d & -2 \leq x \leq 2 \\ x^2 - 6x + 10 & x > 2 \end{cases}$$

Find the values of a, b, c and d such that $f(x)$ is both continuous and smooth over its entire domain.

Let $g(x) = ax^3 + bx^2 + cx + d$.
For continuity we require $g(-2) = 0$ and $g(2) = 2$. This yields the equations,

$$-8a + 4b - 2c + d = 0 \quad \text{and} \quad 8a + 4b + 2c + d = 2$$

For smoothness we require $g'(-2) = 2$ and $g'[2] = -2$. This yields the equations,

$$12a - 4b + c = 2 \quad \text{and} \quad 12a + 4b + c = -2$$

Solving simultaneously yields $a = -\frac{1}{16}, b = -\frac{1}{2}, c = \frac{3}{4}$ and $d = 3$.

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Question 25 Tech-Active.

a. A hybrid function $f : \mathbb{R} \to \mathbb{R}$, is defined as:

$$f(x) = \begin{cases} \sin(x) + 3 & x < 0 \\ g_1(x) & 0 \leq x < 1 \\ g_2(x) & 1 \leq x < 2 \\ g_3(x) & 2 \leq x < 3 \\ \log_e\left(\frac{e^2x^3}{27}\right) & x \geq 3 \end{cases}$$

Where g_1, g_2 and g_3 are cubic polynomials. Find g_1, g_2, g_3 if both f and f' are smooth on $\mathbb{R}$.

We solve the following equations simultaneously,

$$g_1(0) = 3, g_1'(0) = 1, g_1''(0) = 0$$
$$g_2(1) = g_1(1), g_2'(1) = g_1'(1), g_2''(1) = g_1''(1)$$
$$g_3(2) = g_2(2), g_3'(2) = g_2'(2), g_3''(2) = g_2''(2)$$
$$2 = g_3(3), 1 = g_3'(3), \frac{1}{3} = g''(3)$$

Thus,

$$g_1(x) = -\frac{37}{54}x^3 + x + 3$$
$$g_2(x) = \frac{151}{108}x^3 - \frac{25}{4}x^2 + \frac{29}{4}x + \frac{11}{12}$$
$$g_3(x) = -\frac{83}{108}x^3 + \frac{27}{4}x^2 - \frac{75}{4}x + \frac{73}{4}$$

b. A different hybrid function, $h : \mathbb{R} \to \mathbb{R}$, is defined as:

$$h(x) = \begin{cases} \sin(x) + 3 & x < 0 \\ g_4(x) & 0 \leq x < 3 \\ \log_e\left(\frac{e^2x^3}{27}\right) & x \geq 3 \end{cases}$$

Where g_4 is a polynomial. If both h and h' are smooth on $\mathbb{R}$, what is the minimum degree of $g_4(x)$?

Degree 5, we will have 6 equations, and thus require 6 unknowns.

Sub-Section: Exam 1 Questions

Question 26

a. Let $f: (1, \infty) \to \mathbb{R}, f(x) = \frac{x^2 - 2x}{(x-1)^2}$. Differentiate f with respect to x.

We can rewrite $f(x)$ as, $f(x) = 1 - \frac{1}{(x-1)^2}$.

Hence $f'(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$

b. Let $g(x) = (x-3)^3(x+1)^2$. Solve $g'(x) = 0$ for x.

$$\begin{aligned} g'(x) &= 3(x-3)^2(x+1)^2 + 2(x+1)(x-3)^3 \\ &= (x+1)(x-3)^2(3x+3+2x-6) = (x+1)(x-3)^2(5x-3) \end{aligned}$$

Thus if $g'(x) = 0$ we see that $x = -1, \frac{3}{5}, 3$.

c. If $y = \frac{e^{x^3+2x}}{\sin(x^3+2x)}$, find $\frac{dy}{dx}$.

Observe that $y = (f \circ g)(x)$ where $f(x) = \frac{e^x}{\sin(x)}$ and $g(x) = x^3 + 2x$.

Using the fact that $f'(x) = \frac{e^x \sin(x) - e^x \cos(x)}{\sin^2(x)}$, we see that,

$$\begin{aligned} \frac{dy}{dx} &= g'(x) f'(g(x)) \\ &= (3x^2+2)\frac{e^{x^3+2x}\sin(x^3+2x) - e^{x^3+2x}\cos(x^3+2x)}{\sin^2(x^3+2x)} \end{aligned}$$

d. Let $h : [1, \infty) \to \mathbb{R}, h(x) = \sqrt{\log_e(x)}$. Evaluate $h'(e)$.

$h'(x) = \frac{1}{x} \times \frac{1}{2\sqrt{\log_e(x)}}$.

Hence $h'(e) = \frac{1}{2e\log_e(e)} = \frac{1}{2e}$

Question 27

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 e^{kx}$.

Find the value of k for which $f(x)$ and $f'(x)$ have exactly one point of intersection.

$f'(x) = 3x^2e^{kx} + kx^3e^{kx}$.

We solve $f'(x) = f(x)$, getting,

$$\begin{aligned} & 3x^2e^{kx} + kx^3e^{kx} = x^3e^{kx} \\ \implies & e^{kx}x^2(3 + kx - x) = 0 \\ \implies & x = 0 \quad \text{or} \quad 3 + kx - x = 0 \\ \implies & x = 0, \frac{3}{k-1} \end{aligned}$$

For us to have 1 solution we require $\frac{3}{k-1}$ to be invalid, hence $k = 1$.

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Question 28

Let $f : [-\pi, \pi] \to \mathbb{R}, f(x) = e^x \sin(x)$.

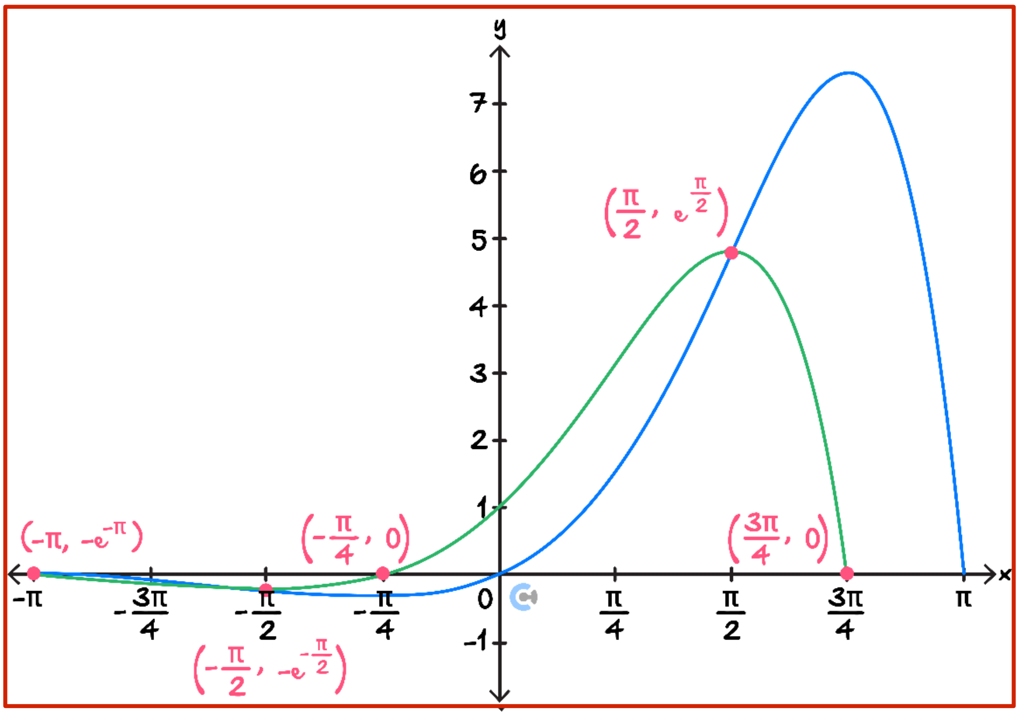
a. Find $f'(x)$.

$e^x \sin(x) + e^x \cos(x)$

b. Show that $f(x)$ has a stationary point when $x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

We solve $f'(x) = 0$, from which we see that $\sin(x) + \cos(x) = 0 \implies -\tan(x) = 0$.
As $x \in [-\pi, \pi]$ our solution to $\tan(x) = 0$ is $x = -\frac{\pi}{4}, \frac{3\pi}{4}$.
Hence $f(x)$ has a stationary point when $x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

c. On the set of axes below, sketch the graph of $y = f'(x)$ on the domain $\left[-\pi, \frac{3\pi}{4}\right]$, labelling the endpoints and points of intersection with the graph of $y = f(x)$ with their co-ordinates.



Question 29

Consider the piecewise function, $f : \mathbb{R} \to \mathbb{R}$, with the following rule;

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^n \log_e(x) & 0 < x \leq 1 \\ ax + b & x > 1 \end{cases}$$

a. Find the values of a and b such that the graph of $y = f(x)$ is smooth at $x = 1$.

If $g(x) = x^n \log_e(x)$, then $g(1) = 0$ and $g'(x) = x^{n-1}(1 + \log_e(x))$, hence $g'(1) = 1$.
Thus $a = 1$ and $b = -1$.

b. It is known that $\lim\limits_{x \to 0^+} x^k \log_e(x) = 0$ if and only if $k > 1$.

i. For what values of n is the graph of $y = f(x)$ continuous at $x = 0$?

$n > 1$ as per our hint.

ii. For what values of n is the graph of $y = f(x)$ smooth at $x = 0$?

We require $x^{n-1}(1 + \log_e(x)) \to 0$ as $x \to 0^+$.
This is only possible if $x^{n-1} \log_e(x) \to 0$ as $x \to 0^+$.
By the fact in our question this implies that $n > 2$.

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Sub-Section: Exam 2 Questions

Question 30

The gradient of the graph of $y = \log_e(2x)$ at the point where the graph crosses the horizontal axis is equal to:

A. 1

B. 2

C. e

D. $2e$

Question 31

Let f and g be differentiable functions.

If $f'(2) = 3, f'(5) = -1, g(1) = 5$ and $g'(1) = 2$, then what is the value of $(f \circ g)'(1)$?

A. -2

B. -5

C. 15

D. There is insufficient information to determine the correct answer.

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Question 32

Consider the function:

$$f(x) = \begin{cases} e^{x-1} & x \leq 1 \\ ax^2 + bx & x > 1 \end{cases}$$

If the derivative of f is a smooth function for all $x \in \mathbb{R}$, then the values of a and b are:

A. $a = 0$ and $b = 1$.

B. $a = 1$ and $b = 1$.

C. $a = 2$ and $b = 1$.

D. $a = \frac{1}{2}$ and $b = 0$.

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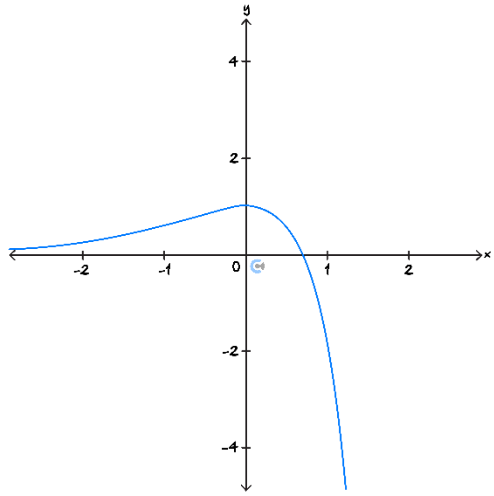
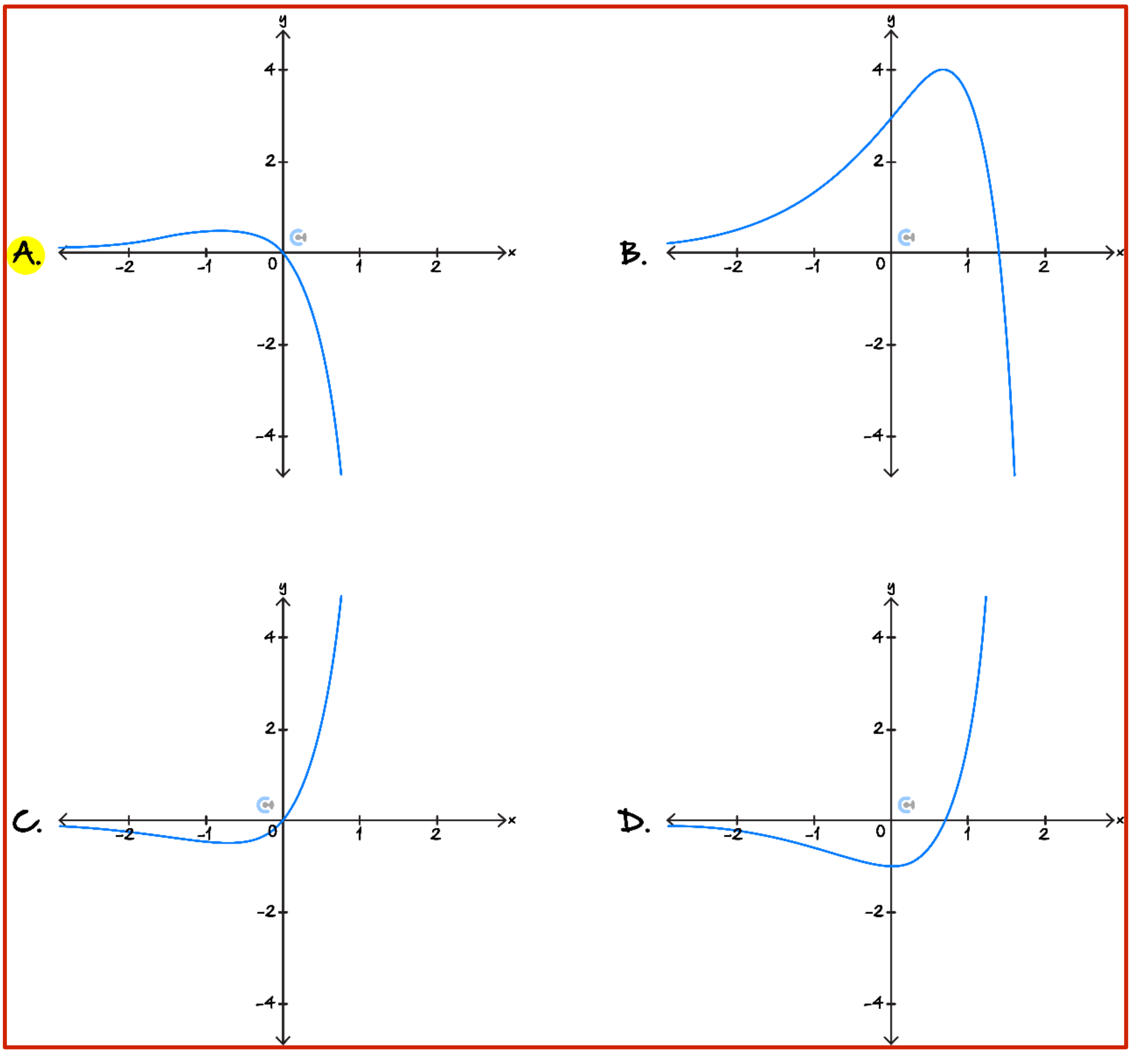
Question 33

The graph of $y = f(x)$ is shown below.



The graph of $y = f'(x)$, the first derivative of $f(x)$ with respect to x, could be:



Question 34

Suppose a function $f : [-2, 2] \to \mathbb{R}$ and its derivative $f' : [-2, 2] \to \mathbb{R}$ are defined and continuous in their domains.

If $f'(x) > f(x)$ for all x and $f'(0) = 0$, which one of these statements must be true?

A. f is increasing on $[0, 2]$.

B. f is increasing on $[-2, 0]$.

C. f is decreasing on $[0, 2]$.

D. f is decreasing on $[-2, 0]$.

Question 35

The population of cockatiels (in thousands) living in Australia, t years after 1800, is modelled by the smooth piecewise function:

$$f(t) = \begin{cases} 800 + 100\cos\left(\frac{\pi t + 10\pi}{30}\right) & 0 \le t < 100 \\ g(t) & 100 \le t < 145 \\ 1000e^{0.02(t-145)} + 120\sin\left(\frac{\pi t + 10\pi}{30}\right) & t \ge 145 \end{cases}$$

a. Write $f'(t)$ as a piecewise function in terms of $g'(t)$.

$$f'(x) = \begin{cases} -\frac{10\pi}{3}\sin\left(\frac{\pi t + 10\pi}{30}\right) & 0 \le t < 100 \\ g'(t) & 100 \le t < 145 \\ 20e^{0.02(t-145)} + 4\pi\cos\left(\frac{\pi t + 10\pi}{30}\right) & t \ge 145 \end{cases}$$

b. If $g(t) = a(t-100)^3 + b(t-100)^2 + c(t-100) + d$, construct simultaneous equations to show that, $a = \frac{48-\sqrt{3}\pi}{6075}$, $b = -\frac{42+4\sqrt{3}\pi}{135}$, $c = \frac{5\pi}{\sqrt{3}}$ and $d = 850$.

We require four conditions, specifically, $g(100) = f(100), g'(100) = f'(100), g(145) = f(145)$ and $g'(145) = f'(145)$.
These conditions can be expressed as,

$$\begin{aligned} g(100) = f(100) &\implies d = 850 \\ g'(100) = f'(100) &\implies c = \frac{5\pi}{\sqrt{3}} \\ g(145) = f(145) &\implies 91125a + 2025b + 45c + d = 940 \\ g'(145) = f'(145) &\implies 6075a + 90b + c = 20 - 2\sqrt{3}\pi \end{aligned}$$

We solve these equations simultaneously to get $a = \frac{48-\sqrt{3}\pi}{6075}, b = -\frac{42+4\sqrt{3}\pi}{135}, c = \frac{5\pi}{\sqrt{3}}$ and $d = 850$.

c. Let g be defined as in **part b.**

i. Solve $f'(t) = 0$ for $t \in [100, 145]$. Give your answer correct to 2 decimal places.

$t = 113.90, 131.05$

ii. Hence, state the values of $t \in [100, 145]$ for which $f(t)$ is strictly decreasing.

Give your answer correct to 2 decimal places.

$t \in [113.90, 131.05]$

d. Find the minimum value of $f'(t)$ for $t \in [100, 145]$ correct to 2 decimal places.

-1.55

e. We can approximate $g(t)$ using an alternative approximation, with:

$$g(t) = 850 + p\sin\left(\frac{\pi t}{30}\right) + q\cos\left(\frac{\pi t}{30}\right) + r\sin\left(\frac{\pi t}{15}\right) + s\cos\left(\frac{\pi t}{15}\right)$$

Find the values of p, q, r and s correct to 2 decimal places.

$p = -135.09, q = -29.69, r = -125.36$ and $s = 46.53$

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Question 36

Consider the composite function $g(x) = f(\tan(2x))$, where the function f is an unknown but differentiable function, whose derivative, $f'(x)$ is a decreasing function for all values of x.

Use the following table of values for f and f'.

x	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$f(x)$	-2	5	-7
$f'(x)$	2	0	-3

a. Express $g'(x)$ in terms of $f'(x)$.

$$g'(x) = \frac{2f'(\tan(2x))}{\cos^2(2x)}$$

b. Find two solutions to $g'(x) = \frac{16}{3}$ for all $x \in [0, \pi]$.

$g'(x) = \frac{16}{3}$ if $f'(\tan(2x)) = 2$ and $\cos^2(2x) = \frac{3}{4}$.

Hence $\tan(2x) = \frac{1}{\sqrt{3}}$ and $\cos(2x) = \pm\frac{\sqrt{3}}{2}$.

Thus $2x = \frac{\pi}{6}, \frac{7\pi}{6} \implies x = \frac{\pi}{12}, \frac{7\pi}{12}$.

c.

i. Solve $g'(x) = 0$ for all $x \in \mathbb{R}$.

$g'(x) = 0 \implies f'(\tan(2x)) = 0$ and $\cos(2x) \neq 0$.
Thus $\tan(2x) = 1 \implies 2x = \frac{\pi}{4} + k\pi$ where $k \in \mathbb{Z}$.
Hence $x = \frac{\pi}{8} + \frac{k\pi}{2}$ where $k \in \mathbb{Z}$.

ii. Explain why there cannot be any more solutions to the equation $g'(x) = 0$, than those you have provided in the previous part.

As $\frac{2}{\cos^2(2x)} \neq 0$ for all x, we see that $g'(x) = 0$ if and only if $f'(\tan(2x)) = 0$.
As f' is a decreasing function, we know that only $f'(1) = 0$.
Hence our only solutions are those that satisfy $\tan(2x) = 1$, which are exactly those in the previous part.

iii. Find the maximal subset of $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ for which $g(x)$ is decreasing.

Observe that $g'(x) \leq 0 \iff f'(\tan(2x)) \leq 0$.
As $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ is one period for $\tan(2x)$, the above inequality implies that $\tan(2x) \geq 1 \implies x \geq \frac{\pi}{8}$.
Hence our subset is $\left[\frac{\pi}{8}, \frac{\pi}{4}\right)$.

d. How many solutions are there to the equation $g(x) = 0$ over the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$? Explain your answer.

2

From the previous part we know that g is decreasing for $x \geq \frac{\pi}{8}$.

Similarly we can show that g is increasing for $x \leq \frac{\pi}{8}$.

As $g\left(\frac{\pi}{12}\right) < 0$ we know that there will be no solution for $x < \frac{\pi}{12}$.

As $g\left(\frac{\pi}{12}\right) < 0 < g\left(\frac{\pi}{8}\right)$, we know (using the fact that g is increasing) there will be one solution for $\frac{\pi}{12} < x < \frac{\pi}{8}$.

As $g\left(\frac{\pi}{6}\right) < 0 < g\left(\frac{\pi}{8}\right)$, we know (using the fact that g is decreasing) there will be one solution for $\frac{\pi}{8} < x < \frac{\pi}{6}$.

As $g\left(\frac{\pi}{6}\right) < 0$ we know that there will be no solution for $x > \frac{\pi}{6}$.

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