CONTOUREDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Differentiation II [2.2]
Workbook

Outline:

Learning Objectives:

- ☐ **MM34 [2.2.1]** - Evaluate Limits and Find Points Where the Function is Not Continuous
- ☐ **MM34 [2.2.2]** - Apply Differentiability to Find Points Where Functions are Not Differentiable, Domain of the Derivative and Unknowns of a Function
- ☐ **MM34 [2.2.3]** - Identify Concavity and Find Inflection Points

Section A: Limits

Context: Limits and its Link to Calculus

Sub-Section: Understanding Limits

What is a limit?

Limits

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches __L__ as x approaches __a__."

- Limit is the value that a function (y-value) __approach__ as the x-value approaches a value.

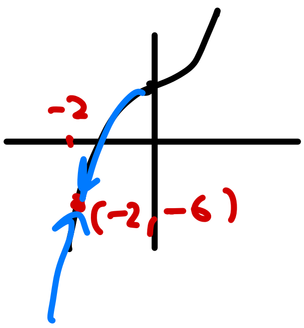
Question 1 Walkthrough.

Evaluate the following limit:

$(-2)^3+2$

$$\lim_{x \to -2} (x^3 + 2)$$

$= -6$



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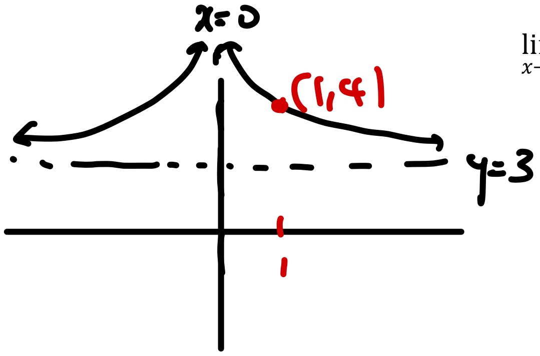
Question 2

Evaluate the following limit:

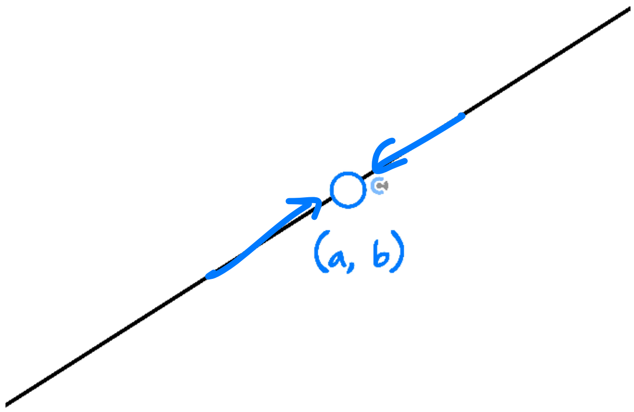


$$\lim_{x\to 1}\left(3+\frac{1}{x^2}\right) = 4 = f(1)$$

What is the difference between a limit and simply finding the y-value?

<u>Exploration</u>: **Purpose of Limits**

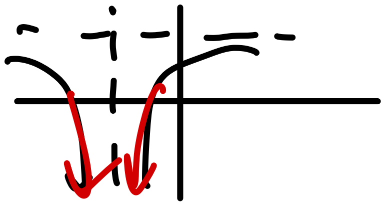
- Consider the following graph:



- What is $f(a)$ equal to? Undef
- What about $\lim\limits_{x\to a}(f(x))$? $= b$.
- Does the function need to be defined for the limit to be defined? [yes/no]

Question 3

Evaluate $\lim_{x \to -2} \left(\frac{-1}{(x+2)^2} + 4 \right) = -\infty$



TIP: Sketch the function and see the y-value that the function approaches.

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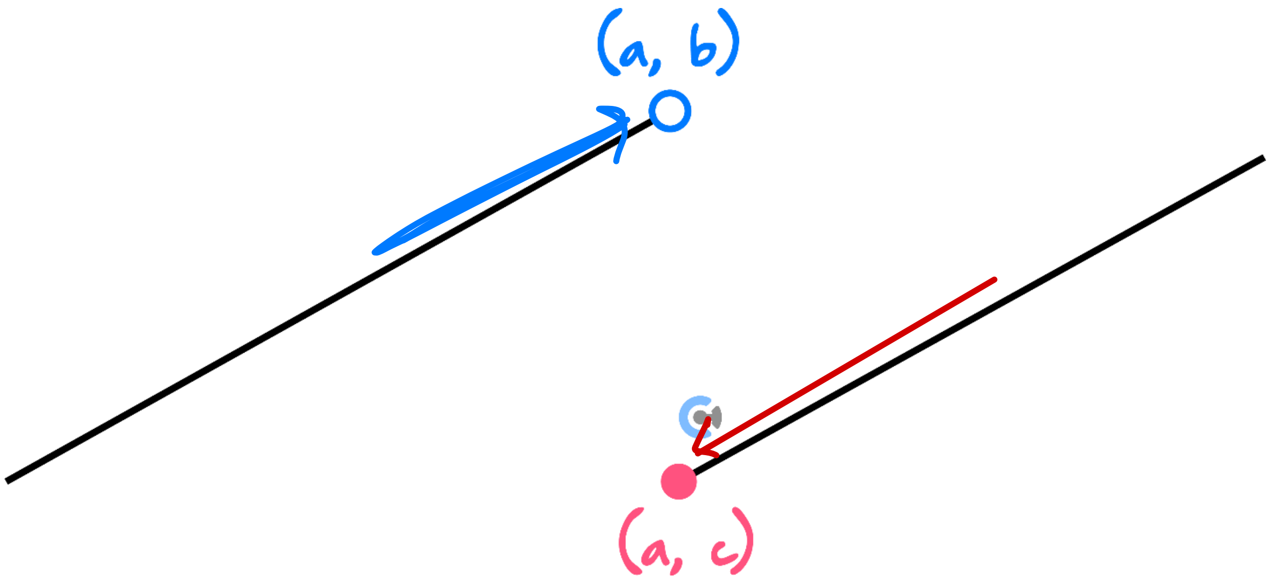
Sub-Section: Validity of Limits

Limit $\lim_{x \to a} f(x)$ seems invincible compared to the actual $f(a)$!

Do limits themselves also break?

Exploration: Validity of Limits



- What does the above function approach from the left-hand side of $x = a$? **(Left Limit)**

$$\lim_{x \to a^-} f(x) = \underline{\ b\ }$$

- What does the above function approach from the right-hand side of $x = a$? **(Right Limit)**

$$\lim_{x \to a^+} f(x) = \underline{\ c\ }$$

- Hence, what is the **overall limit?**

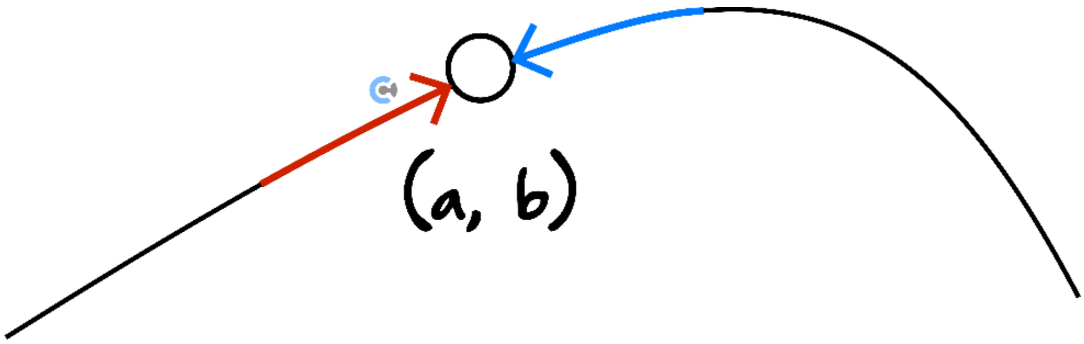
$$\lim_{x \to a} f(x) = \underline{\text{ Undefined. }}$$

- Hence, **for an overall limit to exist:**

$$\text{Right Limit} = \text{Left Limit}$$

Definition

Validity of Limits



$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

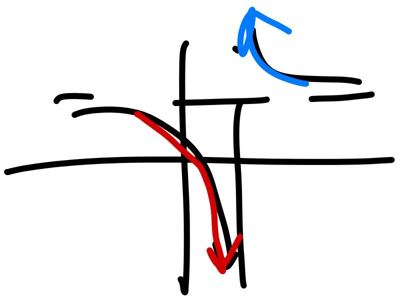
- Limit is defined when the left limit equals to the right limit.

Question 4 Walkthrough.

Consider $f(x) = \frac{1}{x-2} + 4$.

Evaluate the left and right limits of $f(x)$ for $x = 2$, and hence, state whether the limit is defined.

$\lim_{x \to 2^-} f(x) = -\infty$



$\lim_{x \to 2^+} f(x) = \infty$.

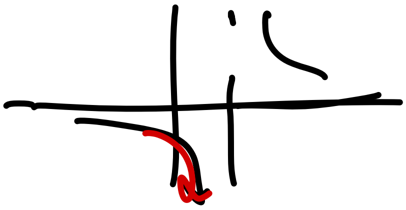
$\lim_{x \to 2} f(x)$ doesnt exist

Question 5

Evaluate the left and right limits for each of the following, and hence, state whether the limit is defined.

a. $f(x) = \frac{3}{x-1}$ for $x = 1$.

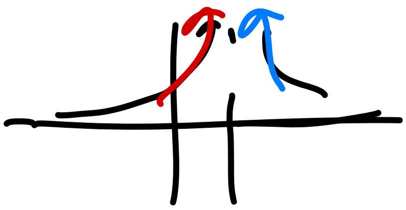
$\lim_{x \to 1^-} f(x) = -\infty$. $\lim_{x \to 1^+} f(x) = \infty$



$\therefore \lim_{x \to 1} f(x)$ is undef

b. $g(x) = \frac{1}{(x-3)^2}$ for $x = 3$.

$\lim_{x \to 3^-} f(x) = \infty$ $\lim_{x \to 3^+} f(x) = \infty$



$\therefore \lim_{x \to 3} f(x) = \infty$.

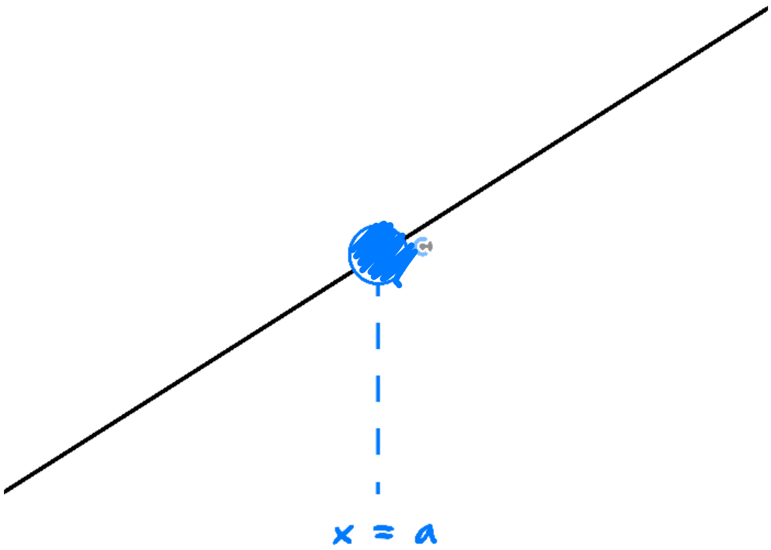
Section B: Continuity and Differentiability

Sub-Section: Continuity

What condition does the function need to satisfy for it to be continuous at a point?

Exploration: Case 1 of Continuity of a Function at $x = a$

- Consider the function below.



- Is the function continuous at $x = a$? [yes/no]
- What is missing at $x = a$? f(a) is undef.
- How could we have prevented this?

$f(a)$ needs to be defined.

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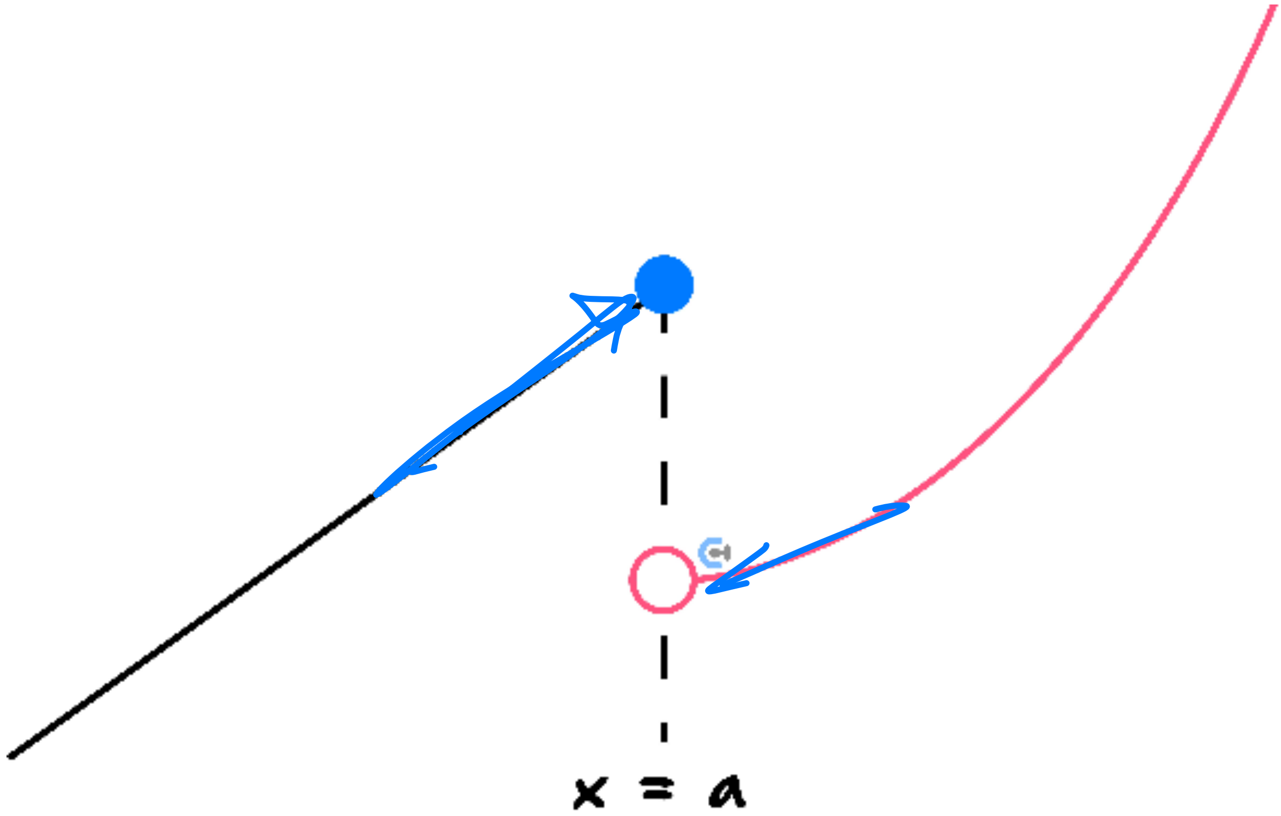
<u>Exploration</u>: Case 2 of Continuity of a Function at $x = a$

- Consider the function below.



- Is the function continuous at $x = a$? [yes/no]
- What is missing at $x = a$? $\lim_{x \to a} f(x)$ is undef
- How could we have prevented this?

$\lim_{x \to a} f(x)$ needs to be def.

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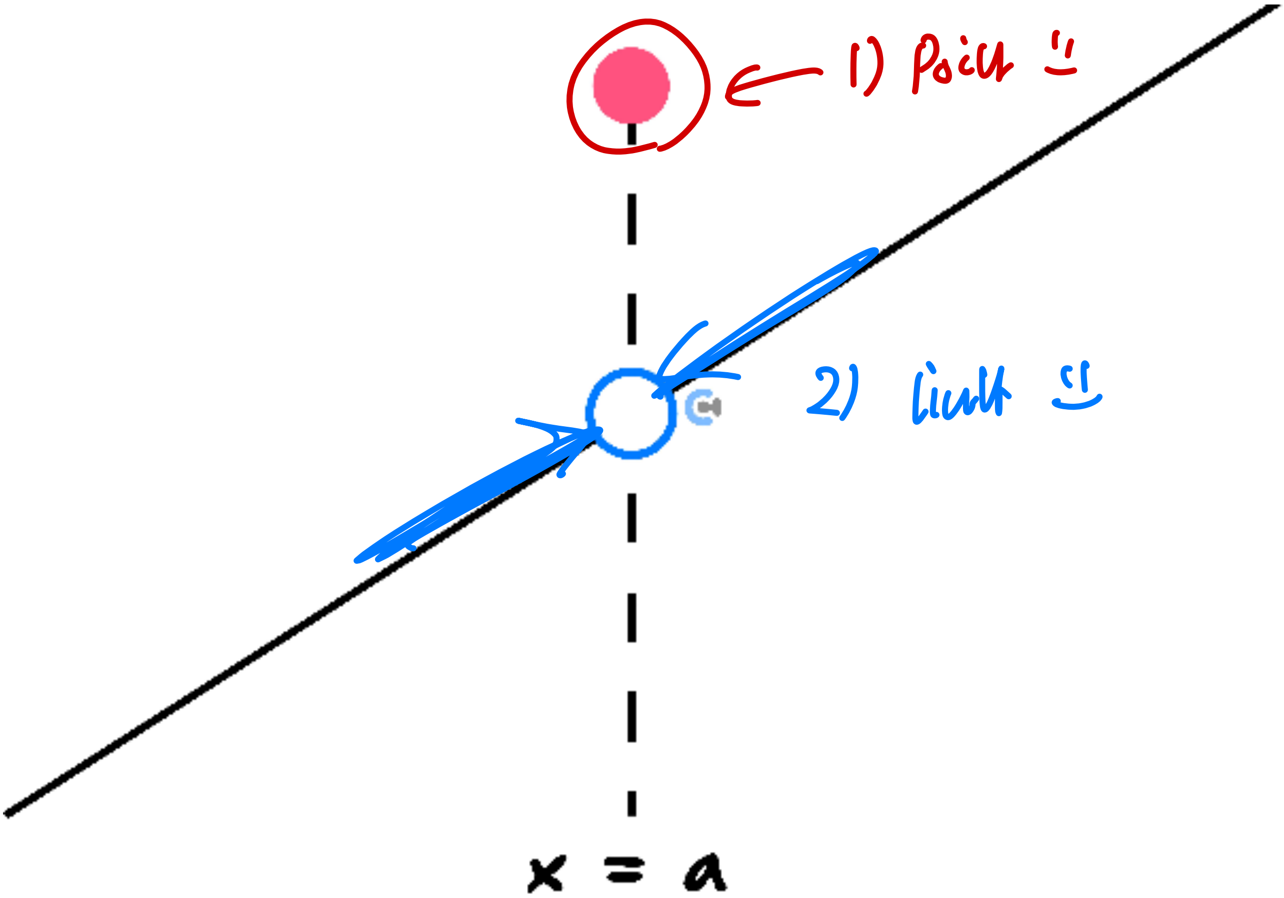
<u>Exploration</u>: Case 3 of Continuity of a Function at $x = a$

- Consider the function below.



- Is the function continuous at $x = a$? [yes/no]
- What is missing at $x = a$?
- How could we have prevented this?

$f(a) = \lim_{x \to a} f(x)$

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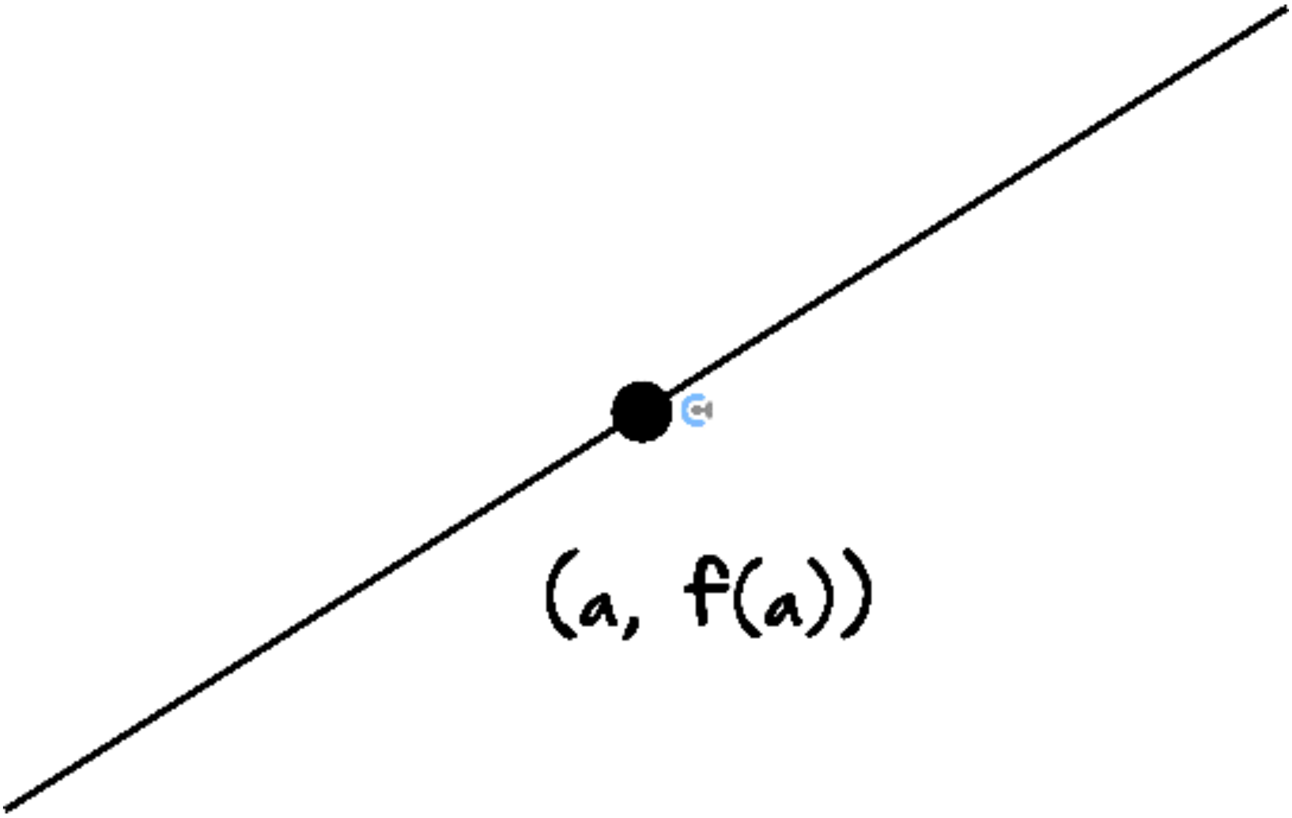
Let's summarise!

Continuity



- A function f is said to be continuous at a point $x = a$ if:

  1. $f(x)$ is defined at $x = a$.

  2. $\lim_{x \to a} f(x)$ exists.

  3. $\lim_{x \to a} f(x) = f(a)$.

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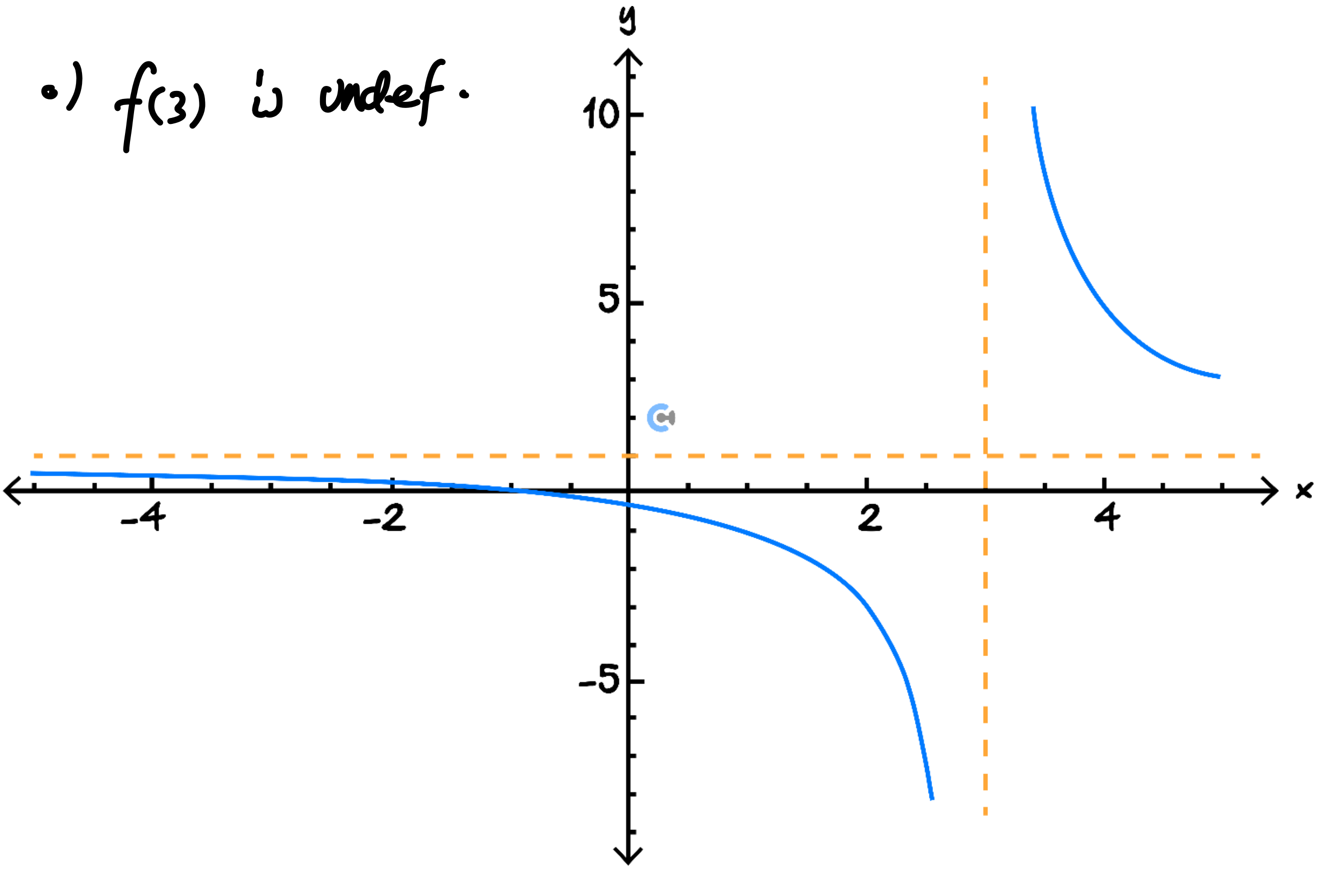
Question 6 Walkthrough.

Find the values of x for which the following function has a discontinuity, and state the reason.

$$f(x) = \frac{x+1}{x-3}$$

At $x = 3$

•) $f(3)$ is undef.



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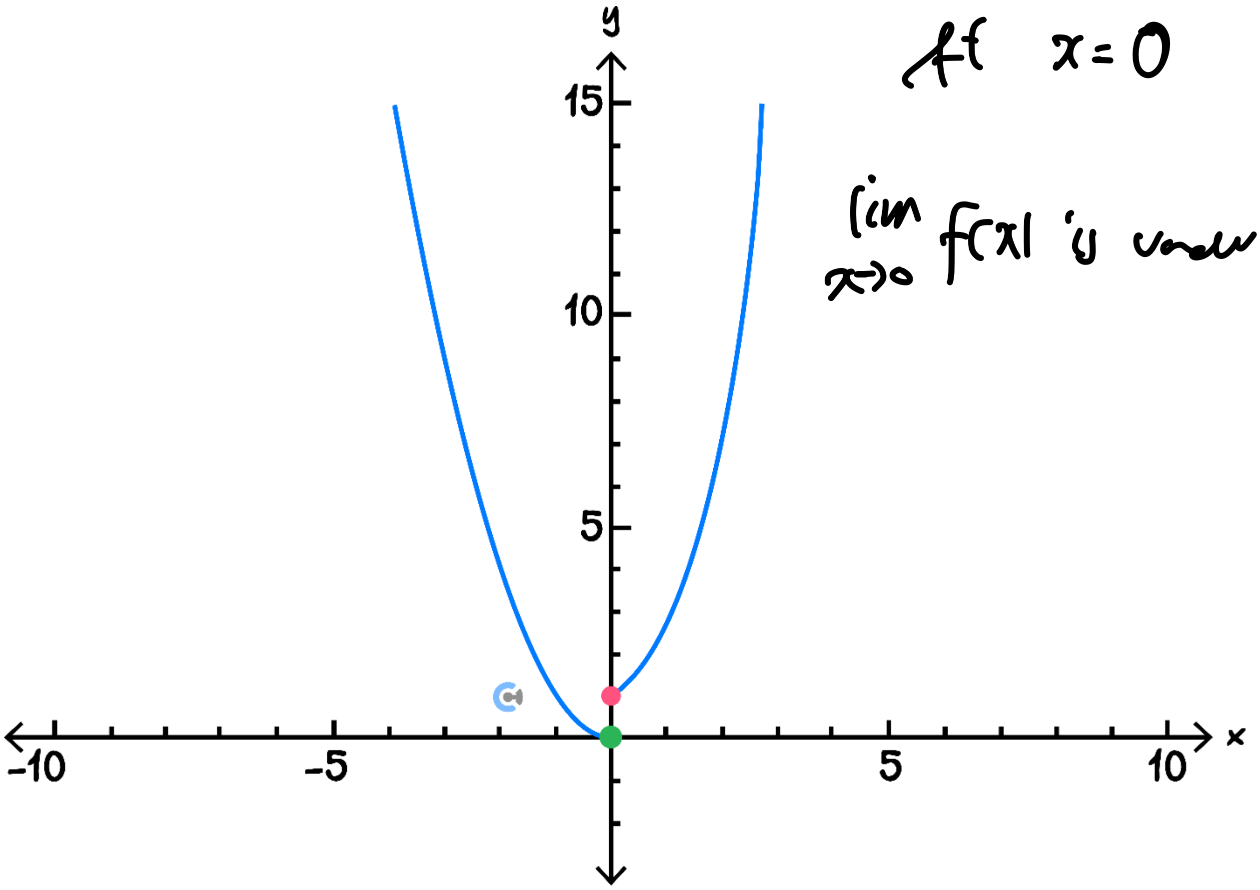
Question 7

Find the values of x for which the following functions have a discontinuity, and state the reason.

$$f(x) = \begin{cases} x^2, & x \in (-\infty, 0) \\ e^x, & x \in [0, \infty) \end{cases}$$



At $x = 0$

$\lim_{x \to 0} f(x)$ is undef

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Question 8 Extension.

Find the values of x for which the following functions have a discontinuity, and state the reason.

$$f(x) = \begin{cases} \sin(2x), x \in \left(-\infty, \frac{\pi}{2}\right) \\ \frac{3x}{\pi} - \frac{5}{2}, x \in \left[\frac{\pi}{2}, \pi\right] \\ \sin(2x) + \frac{1}{2}, x \in (\pi, \infty) \end{cases}$$

$x = \frac{\pi}{2}$ only. Left limit $= 0$ and right limit $= -1$

When $x = \pi$ both left and right limits are $\frac{1}{2}$ so, it is continuous.

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Sub-Section: Differentiability

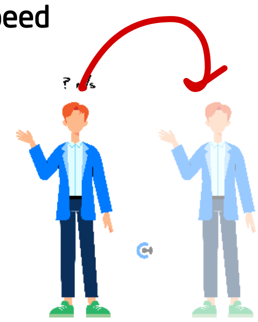
Discussion: What do you think the word differentiability means?

Differentiation = Finding gradient of a point

Differentiability = Whether point has a valid gradient

What conditions do the points need to satisfy to have a valid gradient?

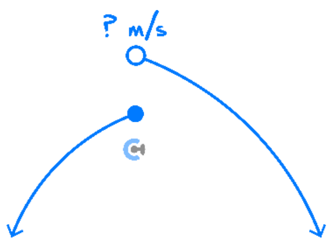
Analogy: Differentiability and Richard's Speed



- Richard has a special power.

  He can teleport from place to place!

- What is his speed when he teleports? Undef.



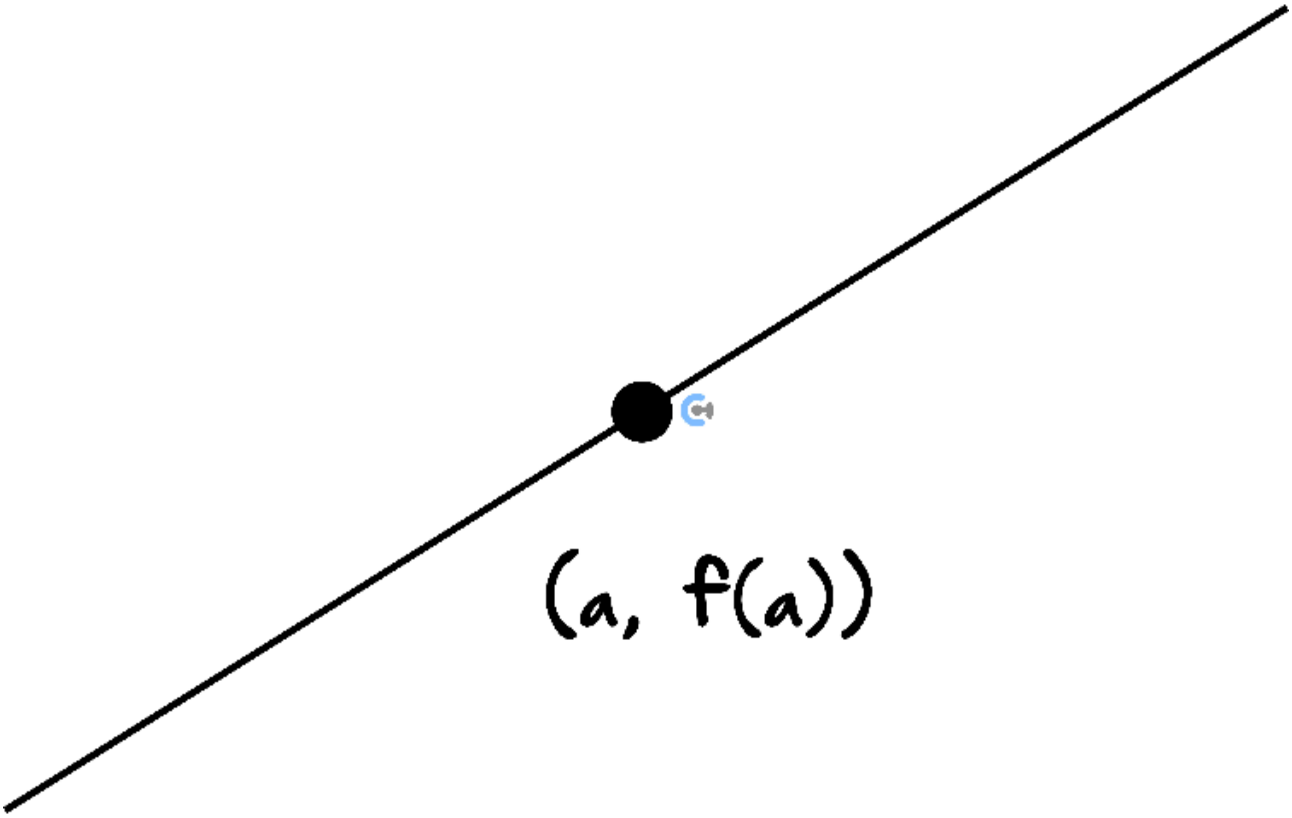
- Similarly, what is the gradient of a function when it teleports to another point (Not Continuous)?

  Undef

- Therefore, for a point to be differentiable, what must it be with the rest of the function?

  Continuous.

Differentiability



- A function f is said to be differentiable at a point $x = a$ if:

  1. $f(x)$ is continuous at $x = a$.
  2. $\lim_{x \to a} f'(x)$ exists.

     - Limit exists when the left and right limits are the same.
     - Gradient on the LHS & RHS must be the same.

- We **cannot** differentiate:

  1. Discontinuous Points
  2. Sharp Points
  3. Endpoints

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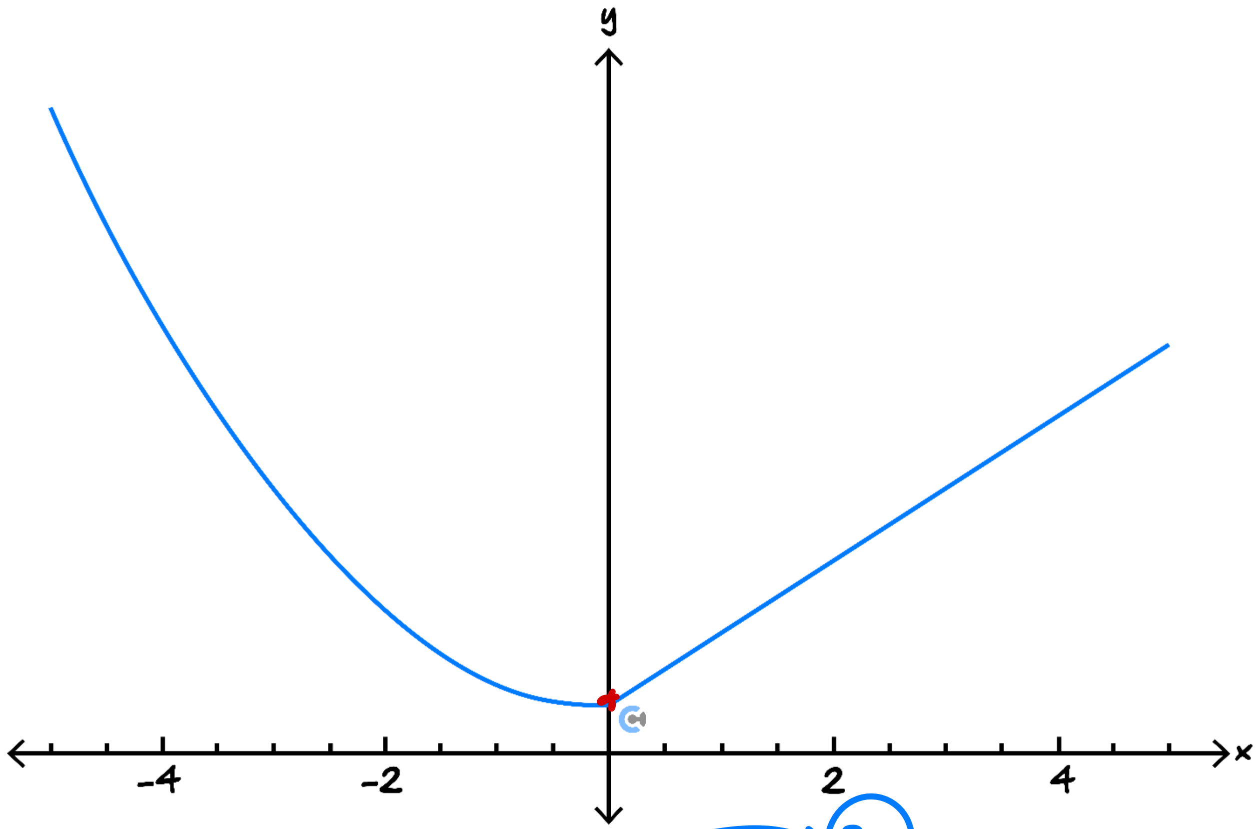
Question 9 Walkthrough.

Consider the function below:



$$f(x) = \begin{cases} x^2 + 2, & x \leq 0 \\ 3x + 2, & x > 0 \end{cases}$$

$2x$

3

State the points that are not differentiable and state the reason.

At $x = 0$.

Not differentiable

1) Continuum (Join) $\lim_{x \to 0^-} f(x) = 2 = \lim_{x \to 0^+} f(x)$

2) $\lim_{x \to a} f'(x)$ (Smooth) $\lim_{x \to 0^-} f'(x) = 0.$ $\neq$ $\lim_{x \to 0^+} f'(x) = 3$

NOTE: Left and right limit of the gradient is simply the gradient from the LHS and RHS.

ALSO NOTE: We call this a sharp point!

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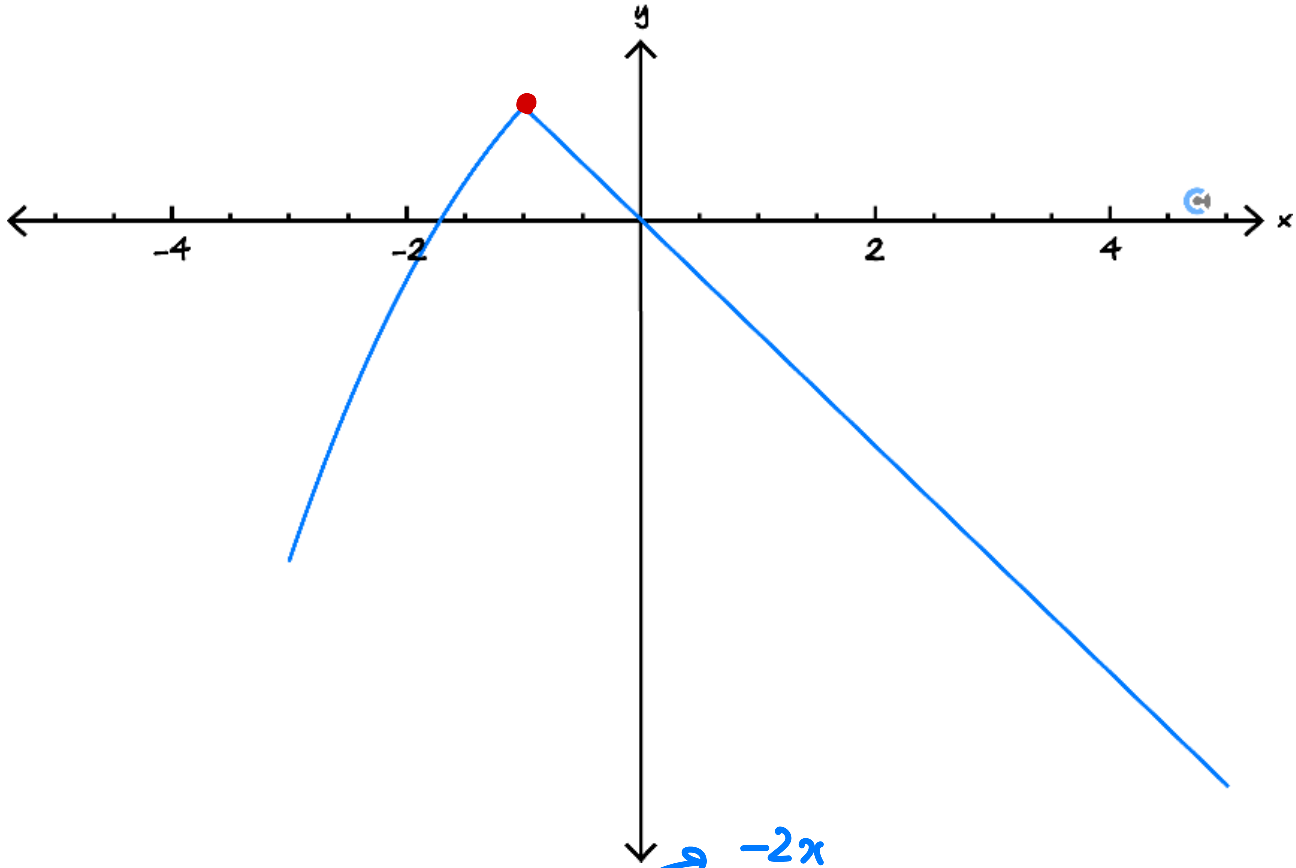
Question 10

Consider the function below:



$-2x$

$$f(x) = \begin{cases} -x^2 + 3, & -3 \leq x \leq -1 \\ -2x, & x > -1 \end{cases}$$

-2

State the points that are not differentiable and state the reason.

$x=-1$ Join: $(-1)^2+3=2 \; . = \; -2(-1)=2$

Smooth: $-2(-1)=2 \neq -2$

$x=-3$: Endpt

NOTE: We cannot differentiate endpoints as they only have a left or right limit.

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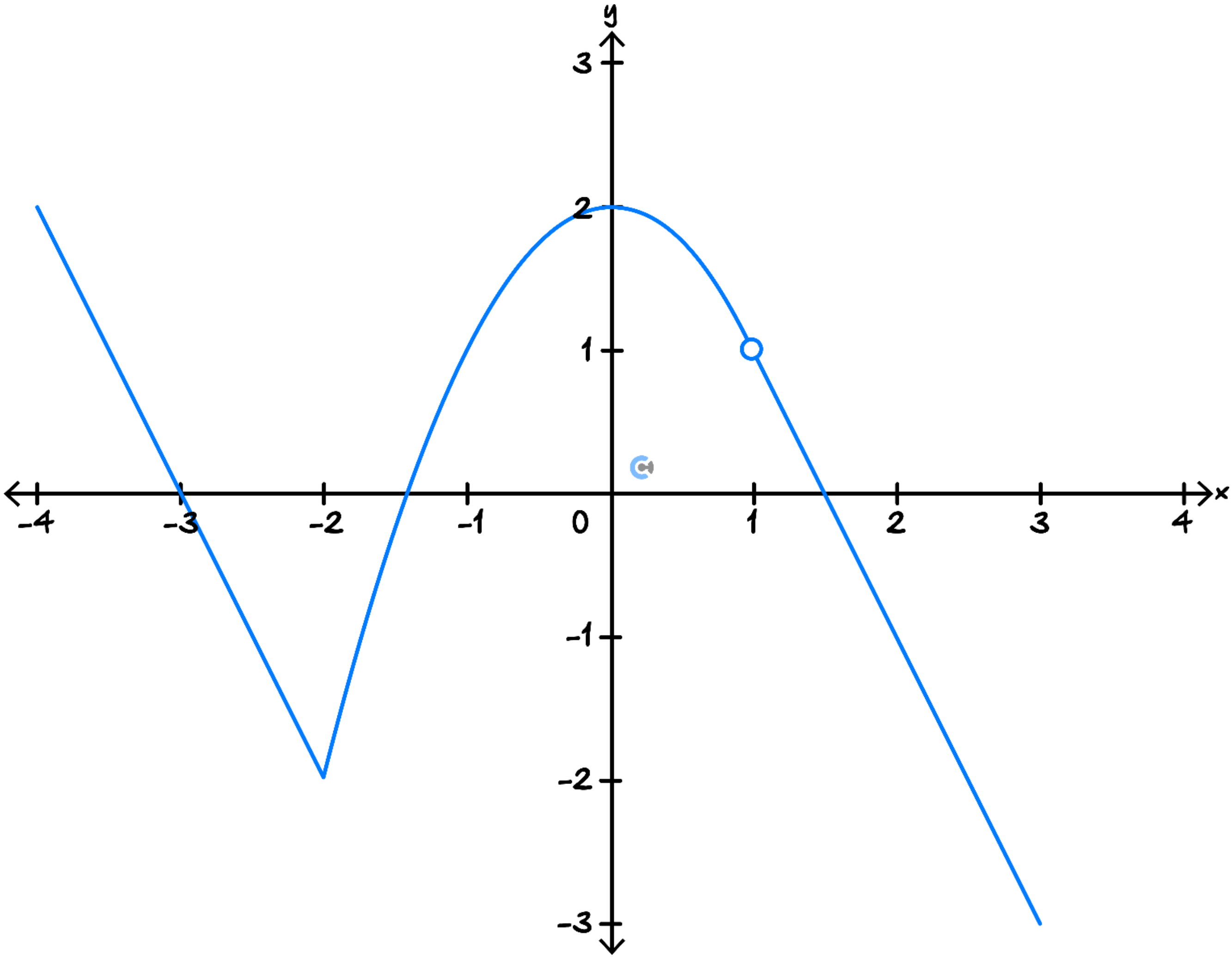
Question 11 Extension.

Consider the function below:



$$f(x) = \begin{cases} -2x - 6, & -4 \leq x \leq -2 \\ -2x, & -2 < x < 1 \\ -2x + 3, & x > 1 \end{cases}$$

State the points that are not differentiable and state the reason.

$x = -4$ (endpoint)

$x = -2$ (sharp point)

$x = 1$ (discontinuous)

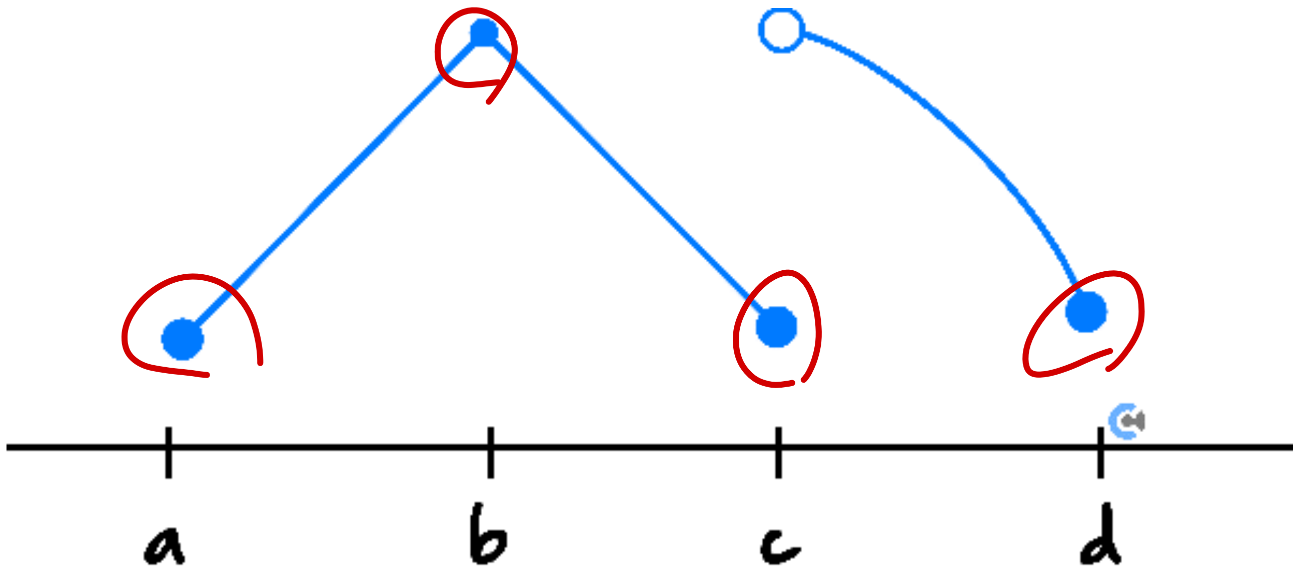
Sub-Section: Domain of the Derivative Function

Discussion: If a point is not differentiable, can its x value be part of the derivative's domain? Does the derivative function exist at that x-value?

No.

Question 12 Walkthrough.

Find the domain of the derivative function for the function shown below.



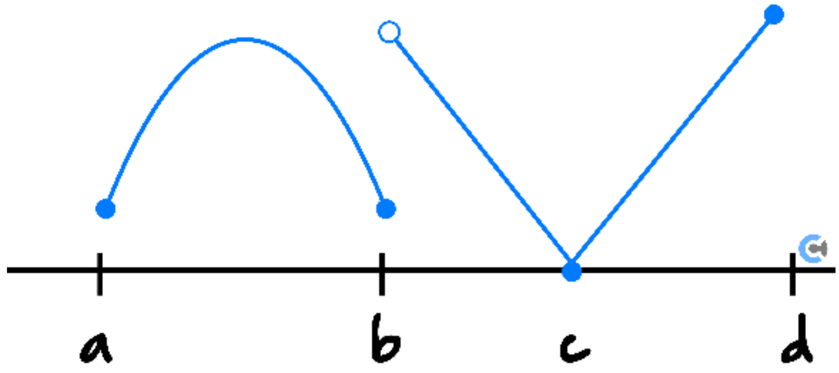
Dom: $[a, d]$

Dom: $\left(\frac{dy}{dx}\right)$: $(a, d) \setminus \{b, c\}$

NOTE: Endpoints, sharp points, and points of discontinuity need to be taken out.

Question 13

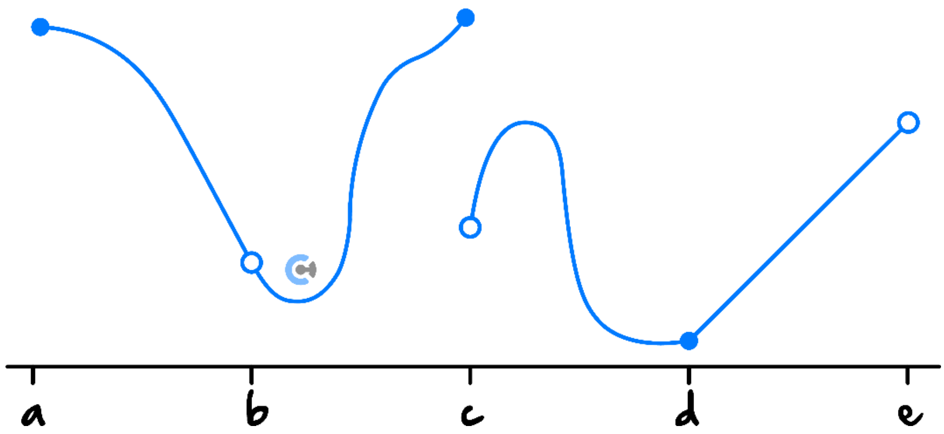
Find the domain of the derivative function for the function shown below.



$(a,d)\setminus\{b,c\}$.

Question 14 Extension.

Find the domain of the derivative function for the function shown below.

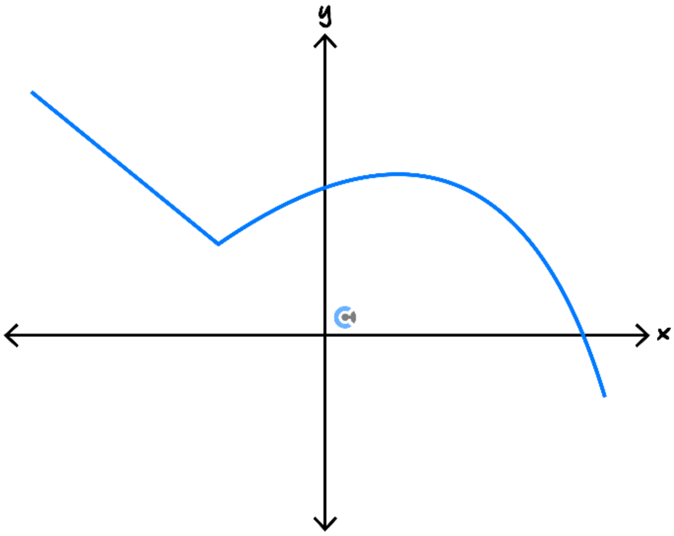


$\boxed{(a,e)\setminus\{b,c,d\}}$

Sub-Section: Defining Derivative Functions

Finding the Derivative of Hybrid Functions

Definition



1. Simply derive each function.

2. Reject the values for x that are not differentiable from the domain.

Question 15 Walkthrough.

For the following function, define the derivative function.

$$f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 3x - 4, & x > 0 \end{cases}$$

$x = 0$

$$f'(x) = \begin{cases} 2x, & x < 0 \\ 3, & x > 0 \end{cases}$$

Join: $\lim_{x \to 0^-} f(x) = -4$

$=$

$\lim_{x \to 0^+} f(x) = -4$

Smooth: $\lim_{x \to 0^-} f'(x) = 0$

$\neq$

$\lim_{x \to 0^+} f'(x) = 3$

$\therefore$ Not diff

Question 16

For the following function, define the derivative function.

$$f(x) = \begin{cases} 3\cos\left(\frac{x}{2}\right) + 1, & x \le 0 \\ 3x + 4, & x > 0 \end{cases}$$

$f'(x) = \begin{cases} -\frac{3}{2}\sin\left(\frac{x}{2}\right), & x < 0 \\ 3, & x > 0 \end{cases}$

Joins: $\lim_{x \to 0^-} f(x) = 3\cos(0) + 1 = 4$

$\lim_{x \to 0^+} f(x) = 3(0) + 4 = 4$

Smooth: $\lim_{x \to 0^-} f'(x) = -\frac{3}{2}\sin(0) = 0$

$\neq$

$\lim_{x \to 0^+} f'(x) = 3$

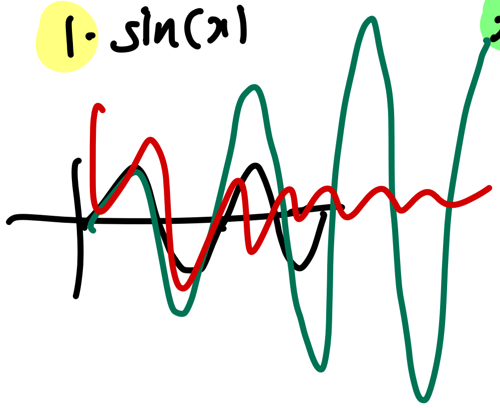
$\therefore$ Not differentiable

Question 17 Extension.

For the following function, define the derivative function.

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$1 \cdot \sin(x)$ $\quad$ $x^2 \sin(x)$ $\quad$ $\frac{1}{x}\sin(x)$



Section C: Concavity and Points of Inflection

Sub-Section: Concavity and Second Derivative

Discussion: What would the derivative's derivative represent?

Diff = trend of the func.

Diff's Diff = Trend of the gradient

Second Derivatives

$\frac{d^2x}{dt^2}$ $\frac{d^2y}{dt^2}$

- The diff's diff.
- To get the second derivative, we can **differentiate the original function twice**.

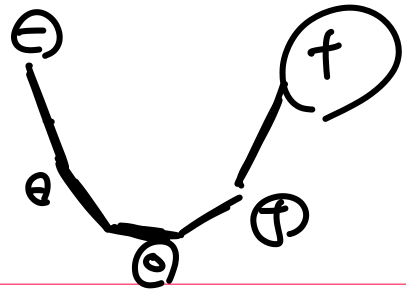
$\frac{d}{dx}\left(\frac{d}{dx}(y)\right)$

$$\frac{d^2y}{dx^2} = f''(x)$$

Discussion: What does it mean when the second derivative is negative? (Gradient's gradient is negative.)

Gradient is decreasing

<u>Discussion:</u> What does it mean when the second derivative is positive? (Gradient's gradient is positive.)



The trend of the gradient is given by concavity!

<u>Concavity</u>

- Concave up is when the gradient is increasing.

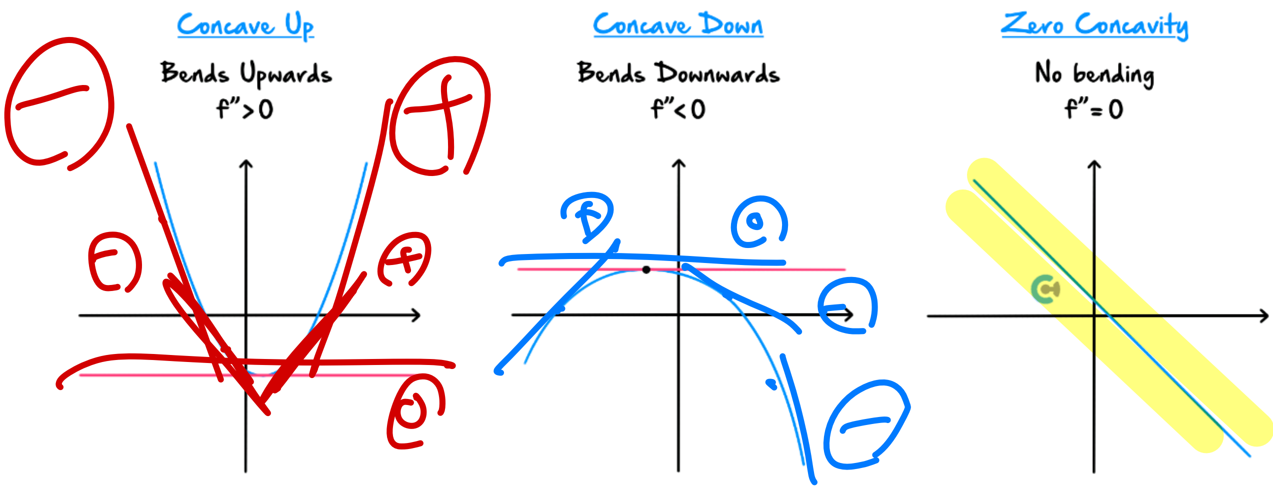
$$f''(x) > 0 \ \rightarrow \text{Concave Up}$$

- Concave down is when the gradient is decreasing.

$$f''(x) < 0 \ \rightarrow \text{Concave Down}$$

- Zero concavity is when the gradient is neither increasing nor decreasing.

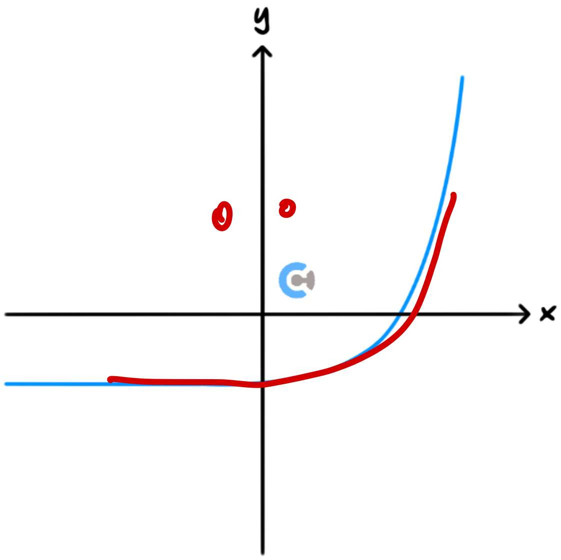
$$f''(x) = 0 \ \rightarrow \text{Zero Concavity}$$

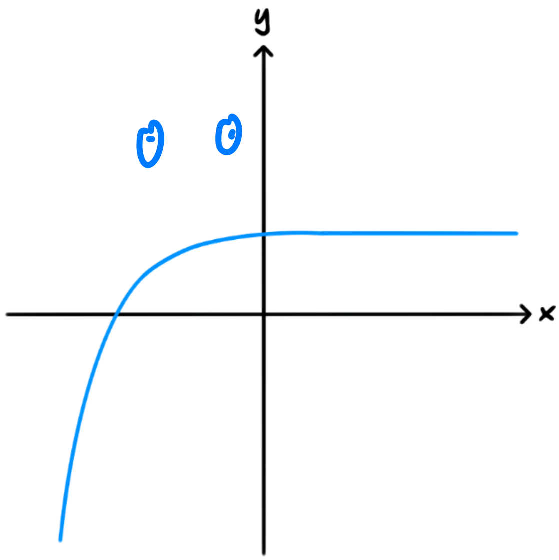


Concavity is also linked to how the curve is bent.

Question 18 Walkthrough.

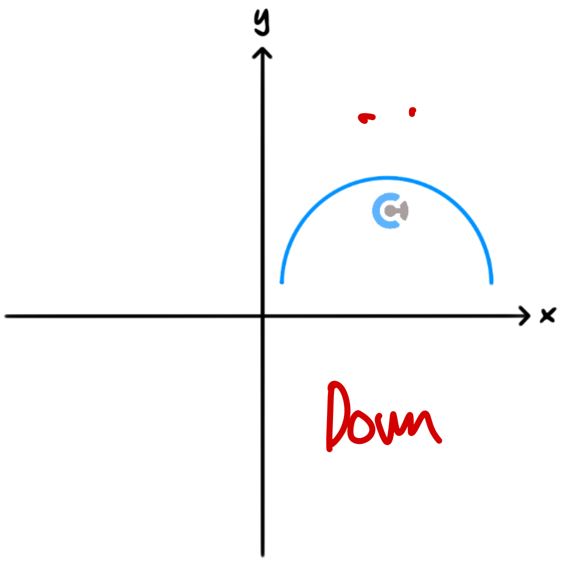
Classify the following curves as concave up or down.

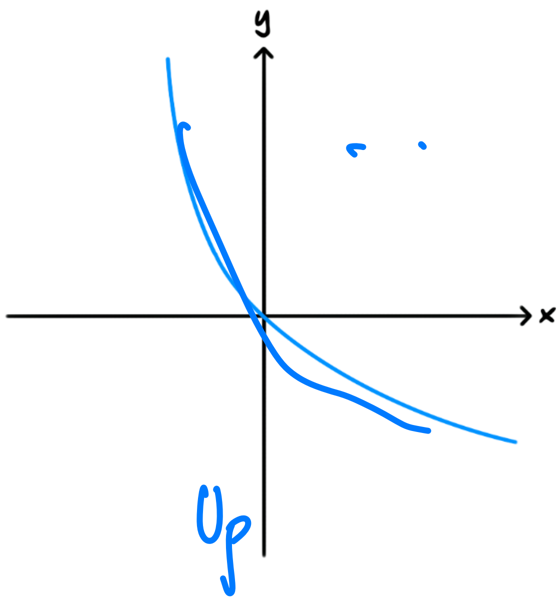




Question 19

Classify the following curves as concave up or down.





Remember that concavity is dictated by the sign of the double derivative!

Active Recall: Concavity and Double Derivative

$f''(x) > 0$ → Concave Up

$f''(x) < 0$ → Concave Down

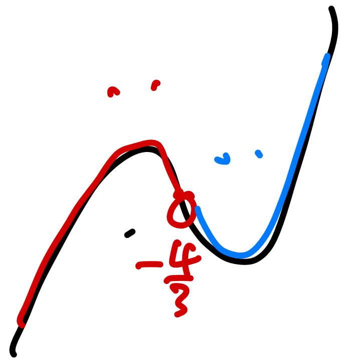
Question 20

Consider the function $f(x) = x^3 + 4x^2 - 5x + 6$. Find the second derivative and, hence, find the value(s) of x for which the function is concave down.

$f'(x) = 3x^2 + 8x - 5$

$f''(x) = 6x + 8 < 0$

$x < -\frac{4}{3}$



NOTE: Simply find the sign of the double derivative.

Discussion: For the previous question, what happens at $x = -\frac{4}{3}$?

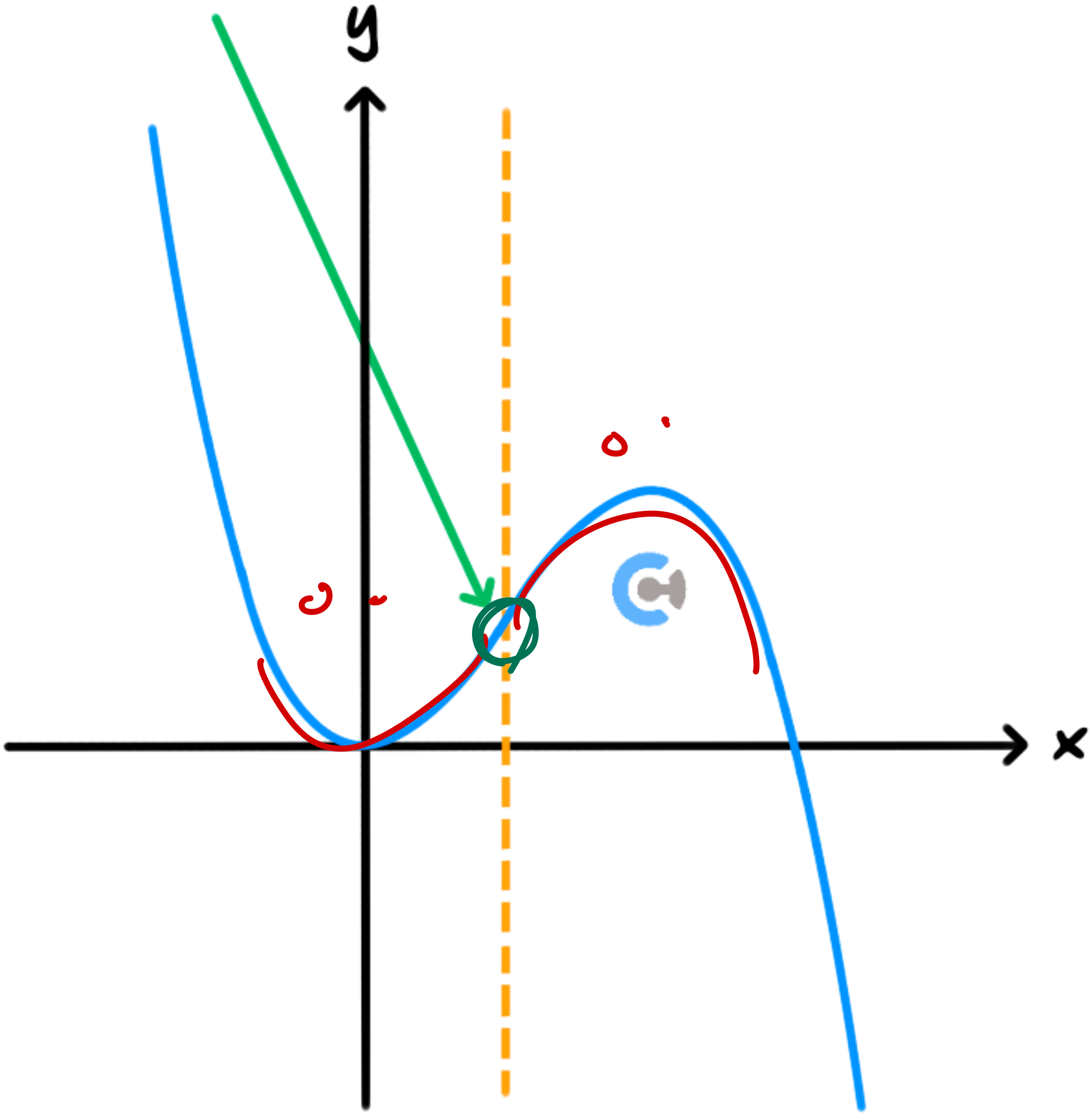
Concavity changed.

Sub-Section: Points of Inflection

The point where the concavity changes is called the point of inflection!

Points of Inflection

- A point at which a curve **changes concavity** is called a **point of inflection**.

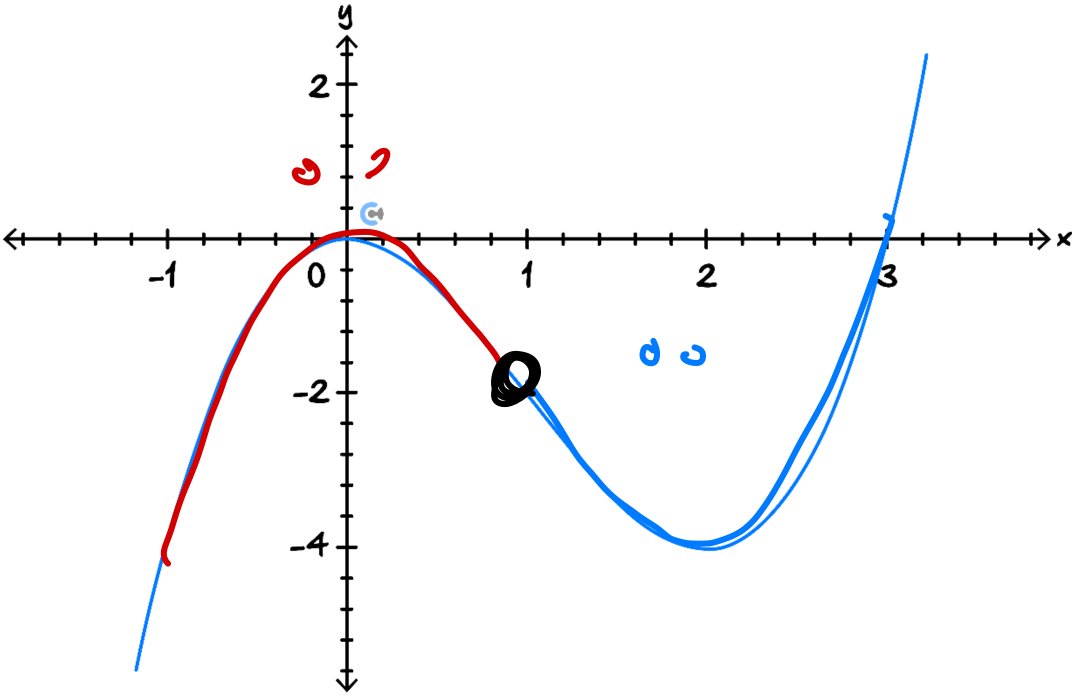


$$\boxed{f''(x) = 0}$$

Simply, it is when the ___concavity___ changes.

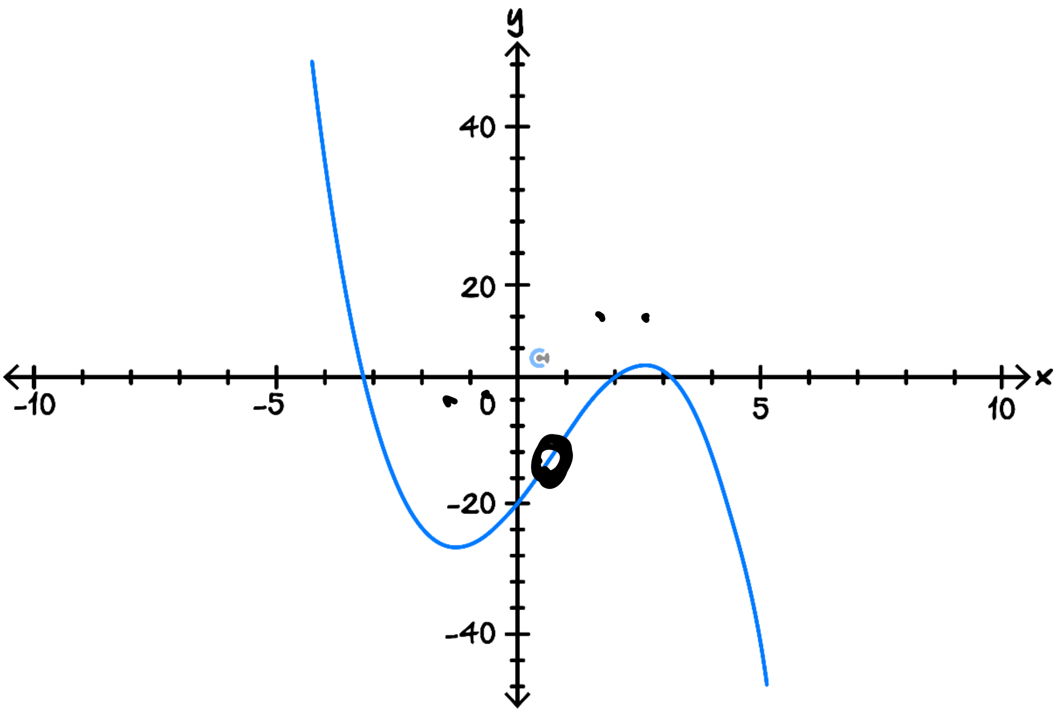
Question 21 Walkthrough.

Circle the point of inflection on the graph below.



Question 22

Circle the point of inflection on the graph below.



Question 23 Extension.

Let $f(x) = xe^{-x}$ where $x \in R$. The graph of f has exactly one point of inflection. Determine its coordinates.

$f'(x) = 1 \cdot e^{-x} + x \cdot e^{-x} \cdot -1$

$= (1-x)e^{-x}$

$f''(x) = -1 \cdot e^{-x} + (1-x) \cdot e^{-x} \cdot -1$

$= e^{-x} \cdot [-1 + x - 1]$

$= e^{-x} \cdot (x-2) = 0$

$x = 2$

$\boxed{(2, 2e^{-2})}$

Space for Personal Notes

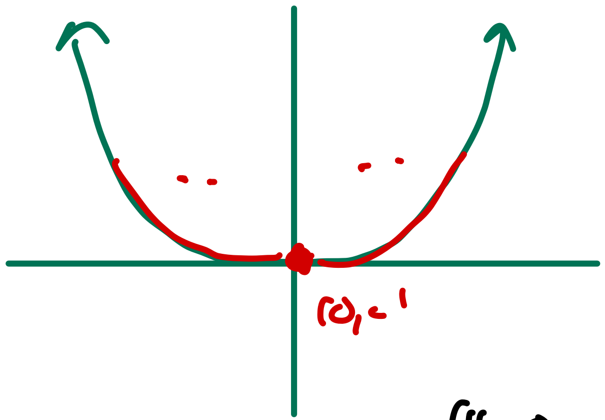
$f(x) = x^4$

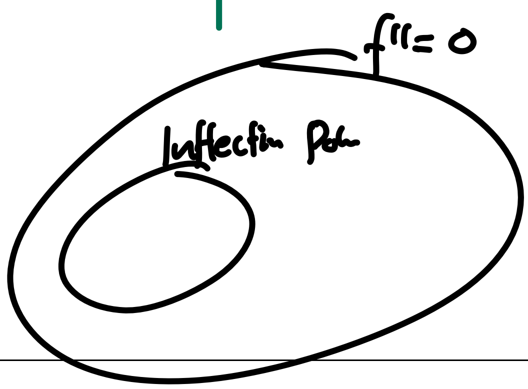
$f'(x) = 4x^3$

$f''(x) = 12x^2 = 0$

$x = 0$

$(0, 0)$

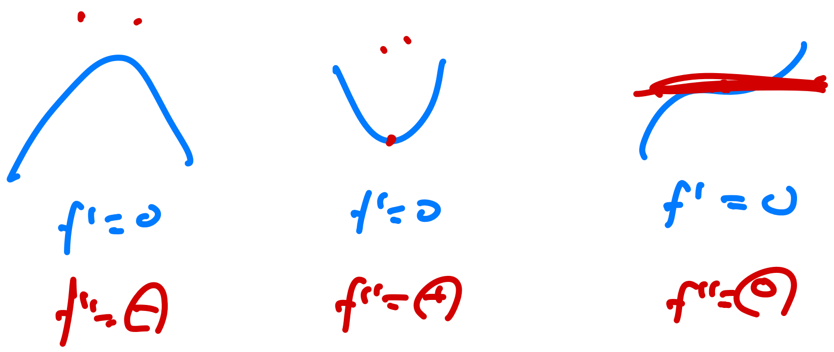




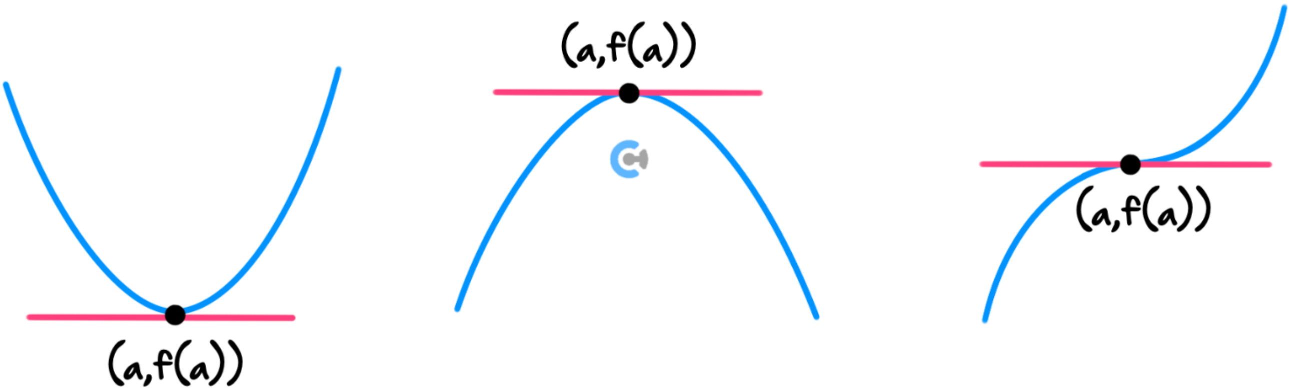
Sub-Section: Second Derivative Test

"Check if your teachers will let"

Discussion: How can we use the concavity to identify the nature of the stationary point?



The Second Derivative Test



- Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
  - Concave up gives us a local minimum.

$$f''(x) > 0 \ \rightarrow \text{Local Minimum}$$

  - Concave down gives us a local maximum.

$$f''(x) < 0 \ \rightarrow \text{Local Maximum}$$

  - Zero concavity gives us a stationary point of inflection.

$$f''(x) = 0 \ \rightarrow \text{Stationary Point of Inflection}$$

Question 24 Walkthrough.

Consider the function $f(x) = \log_e(x^2 + 4)$.

Find the stationary point and identify its nature by using the second derivative test.

$f'(x) = \frac{1}{x^2+4} \times 2x = 0$

$\frac{2x}{x^2+4}$

$x = 0$

$(0, \ln(4))$

$f''(x) = \frac{2\cdot(x^2+4) - 2x\cdot 2x}{(x^2+4)^2}$

$f''(0) = \frac{2(4) - 0}{(0+4)^2} = \frac{8}{16} = \frac{1}{2} = (+)$

Local Min

NOTE: This is much faster than using the table (comparing neighbouring gradients) from [2.1].

Question 25

Consider the function $f(x) = e^{x^2+4}$.

Find the stationary point and identify its nature by using the second derivative test.

```
Solve[f'[x] == 0 && y == f[x], {x, y}]
풀이 함수

Solve: Inconsistent or redundant transcendental equation.
Solve: Inverse functions are being used by Solve, so some
{{x → 0, y → e^4}}

f''[0]
2 e^4

(* Concave up hence its local minimum *)
```

Question 26 Extension.

Consider the function $f(x) = xe^{-x^2-x+3}$.

Find the stationary points and identify their nature by using the second derivative test.

```
In[254]:= Solve[f'[x] == 0 && y == f[x]] // Quiet
```

Out[254]= $\left\{\left\{x \to \frac{1}{2}, y \to \frac{e^{9/4}}{2}\right\}, \left\{x \to -1, y \to -e^3\right\}\right\}$

```
In[252]:= f''[-1]
```

Out[252]= $3\,e^3$

```
In[253]:= f''[1/2]
```

Out[253]= $-3\,e^{9/4}$

Local min when $x = -1$ and local max when $x = \frac{1}{2}$.

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Contour Checkoff

Learning Objective: [2.2.1] - Evaluate Limits and Find Points Where the Function is Not Continuous

Key Takeaways

- Limit Definition:

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a."

- Validity of Limit:

$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

  - Limit is defined when the left limit equals to right right limit.

- Continuity:

  - A function f is said to be continuous at a point $x = a$ if:

    1. $f(x)$ is defined at $x = a$.
    2. $\lim_{x \to a} f(x)$ is defined.
    3. $\lim_{x \to a} f(x) = f(a)$.

Learning Objective: [2.2.2] – Apply Differentiability to Find Points Where Functions are Not Differentiable, Domain of the Derivative and Unknowns of a Function

Key Takeaways

- **Differentiability:**
  - **A function f is said to be differentiable at a point $x = a$ if:**
    1. **$f(x)$ is continuous at $x = a$.**
    2. **$\lim_{x \to a} f'(x)$ exists.**
       - Limit exists when the left and right limits are the same.
       - Gradient on the LHS & RHS must be the same.
- We **cannot** differentiate:
  1. Point of discontinuity
  2. Sharp points
  3. End points
- **Finding the Derivative of Hybrid Functions**
  1. Simply diff each function.
  2. Reject the values for x that are not differentiable from the domain.

Learning Objective: [2.2.3] - Identify Concavity and Find Inflection Points

Key Takeaways

Second Derivatives

- The diff's diff.
- To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

Concavity

- Concave up is when the gradient is increasing.

$$f''(x) > 0 \ \rightarrow \text{Concave Up}$$

- Concave down is when the gradient is decreasing.

$$f''(x) < 0 \ \rightarrow \text{Concave Down}$$

- zero concavity is when the gradient is neither increasing nor decreasing.

$$f''(x) = 0 \ \rightarrow \text{Zero Concavity}$$

Points of Inflection

- A point at which a curve's concavity changes is called a **point of inflection**.

- The Second Derivative Test
  - Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
    - Concave up gives us ___Local Min___.

$$f''(x) > 0 \rightarrow \text{Local Minimum}$$

    - Concave down gives us ___Local Max___.

$$f''(x) < 0 \rightarrow \text{Local Maximum}$$

    - Zero concavity gives us ___Stationary Point of Inflection___.

$$f''(x) = 0 \rightarrow \text{Stationary Point of Inflection}$$

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