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VCE Mathematical Methods ¾
Differentiation II [2.2]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 14
Supplementary Questions	Pg 15 – Pg 25

Section A: Compulsory Questions

Sub-Section [2.2.1]: Evaluate Limits and Find Points Where the Function is not Continuous

Question 1

Evaluate the following limits:

a. $\lim_{x \to 2}(x^2 - 3)$

As $x^2 - 3$ is continuous at $x = 2$, we may substitute $x = 2$ to obtain $\lim_{x \to 2}(x^2 - 3) = 1$.

b. $\lim_{x \to 3}(3^x - 2x^2 + 3)$

As $3^x - 2x^2 + 3$ is continuous at $x = 3$, we may substitute $x = 3$ to obtain $\lim_{x \to 3}(3^x - 2x^2 + 3) = 12$

c. $\lim_{x \to 1}(f(x))$, where,

$$f(x) = \begin{cases} 2x + 1, & x < 1 \\ 5x - 2, & x \geq 1 \end{cases}$$

At $x = 1$ we see that both the left and right limits are equal and so $\lim_{x \to 1} f(x) = 3$

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Question 2

Find the points x for which the following functions are **discontinuous** and state a reason as to why they are discontinuous.

a. $f(x) = \begin{cases} 2x, & x < 2 \\ 2x+1, & x \geq 2 \end{cases}$

Note that $\lim_{x \to 2^-} f(x) = 4$ but $\lim_{x \to 2^+} f(x) = 5$. Since the left and right limits are not equal at $x = 2$, we conclude that $f(x)$ is discontinuous at $x = 2$.

b. $f(x) = \frac{6}{x^2-x-2}$

We solve $x^2 - x - 2 = 0$ to find that $x = -1$ or $x = 2$. Thus, $f(x)$ is discontinuous at $x = -1$ and $x = 2$ as the function is not defined at these points.

c. $f(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$

The function is not defined at the point $x = 0$ and is therefore not continuous at $x = 0$.

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Question 3

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} 2x^2 - 4x + 3, & x < 2 \\ ax + 4, & x \geq 2 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Note that $f(x)$ is already continuous for all $x \neq 2$, and all that remains is to make $f(x)$ continuous at $x = 2$. Thus, we need $3 = 2a + 4 \implies a = -\frac{1}{2}$.

Question 4 Tech-Active.

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^2 - 3x + 2, & x < 2 \\ a^2 - ax - 3, & 2 \leq x < 4 \\ 2x - 14, & x \geq 4 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

We have $f(2^-) = 0$ then $f(2^+) = a^2 - 2a - 3 = 0 \implies a = -1, 3$.
But also $f(4^+) = -6$ so $f(4^-) = a^2 - 4a - 3 = -6 \implies a = 1, 3$.
Only $a = 3$ satisfies both of these equations.

Sub-Section [2.2.2]: Apply Differentiability to Find Points Where Functions are not Differentiable, Domain of the Derivative and Unknowns of a Function

Question 5

Find the values of x such that the following functions are not differentiable.

a. $f(x) = \begin{cases} -x+2, & x < 1 \\ x, & x \geq 1 \end{cases}$

A rough sketch shows that there is a sharp point $x = 1$. Hence, the function is not differentiable at $x = 1$.

b. $f(x) = \frac{1}{x-3}$

The function is not defined whenever the denominator vanishes, i.e. at $x = 3$. Thus, the function is not differentiable at $x = 3$.

c. $f(x) = \begin{cases} 2x-4, & x < -1 \\ 2x-6, & x \geq -1 \end{cases}$

By inspection, we can see a discontinuity at $x = -1$. If a function is not continuous at a given point, it is also not differentiable there either.

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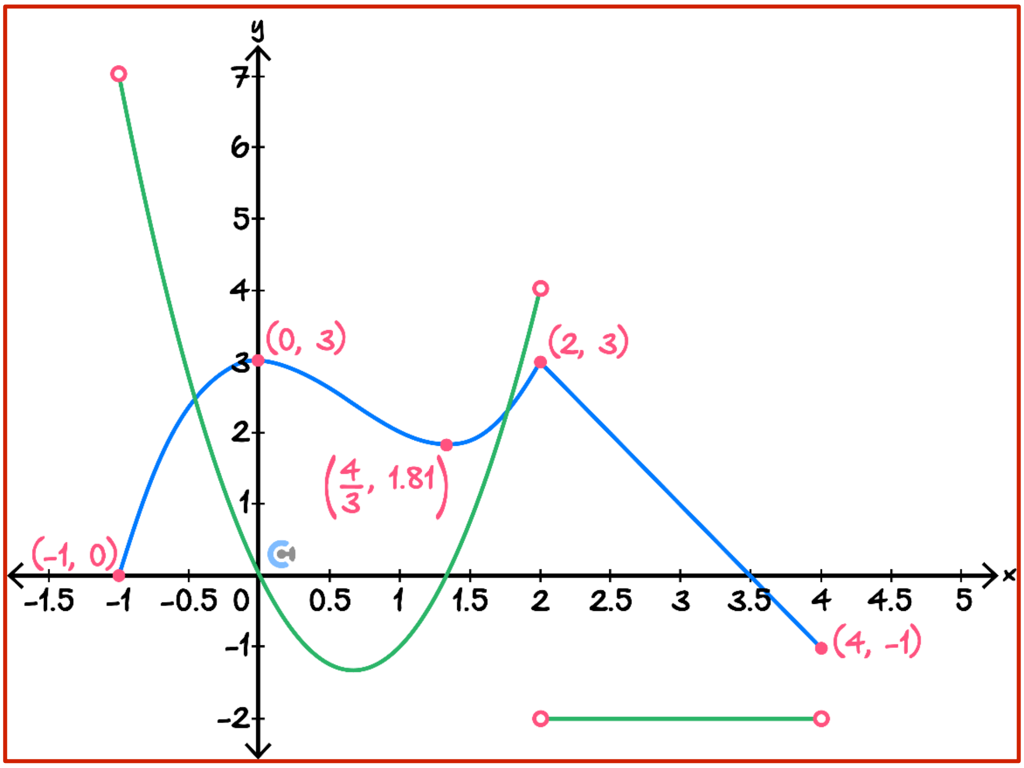
Question 6

Consider the following function. Sketch the corresponding derivative function on the same set of axes.



a. Sketch the corresponding derivative function on the same set of axes above.

b. Furthermore, state the domain of the derivative function.

The function will be differentiable everywhere except the endpoints and the discontinuity, i.e. dom $f' = (-1, 4) \setminus \{2\}$.

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Question 7

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^2 - 4x + 3, & x < 1 \\ ax + b, & x \geq 1 \end{cases}$$

Find the value of a and b such that $f(x)$ is differentiable at $x = 1$.

Since the two rules must join at $x = 1$, imposing $f(1^+) = f(1^-)$ gives $a + b = 0$. Also, $f'(1^+) = f'(1^-)$ gives $a = -2$. By solving the resulting system of equations, we conclude that $a = -2$ and $b = 2$.

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Question 8 Tech-Active.

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^3 - x^2 - 2x + 3, & x < 2 \\ -x^2 + bx + c, & x \geq 2 \end{cases}$$

Find the value of a and b such that $f(x)$ is differentiable at $x = 2$.

Let $f(x) = x^3 - x^2 - 2x + 3$ and let $g(x) = -x^2 + bx + c$.
We must have $f(2) = g(2)$ and $f'(2) = g(2)$. Solve using CAS to obtain

$$b = 10 \text{ and } c = -13$$

```
In[36]:= f[x_] := x^3 - x^2 - 2 x + 3
In[37]:= g[x_] := -x^2 + b x + c
In[38]:= Solve[f[2] == g[2] && f'[2] == g'[2]]
Out[38]= {{b → 10, c → -13}}
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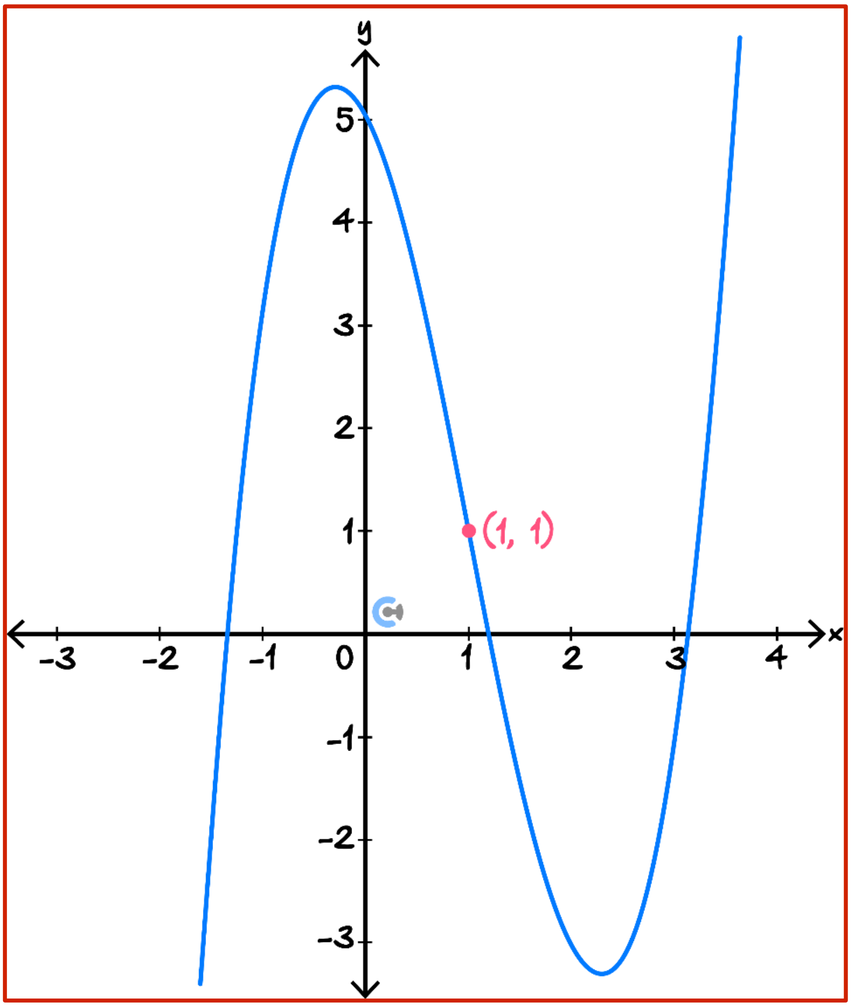
Sub-Section [2.2.3]: Identify Concavity and Find Inflection Points

Question 9

Consider the following graph for $f(x)$.



a. Circle the point of inflection on the above graph.

b. State the values of x such that the function is concave up.

Notice how the gradient of the function increases after $x = 1$ (i.e. it is becoming less negative, and after the turning point becomes more positive). So, the answer is $x > 1$.

c. State the values of x such that the function is concave down.

Similar to above, $x < 1$.

Question 10

Consider a function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 + 2x^3 - 12x^2 + 6x + 4$.

a. Calculate the second derivative of the function $f(x)$.

$f''(x) = 12x^2 + 12x - 24 = 12(x + 2)(x - 1)$.

b. Find the points of inflection of the function $f(x)$.

The points of inflection occur where $f''(x) = 0$. This is easily solved from our factored form $f(x) = 12(x + 2)(x - 1) = 0 \implies x = -2, 1$.
Therefore points of inflection at $x = -2, 1$

c. Find the values of x where the function is concave up.

We solve for the values of x such that $f''(x) > 0$. Thus, the function is concave up whenever $x < -2$ or $x > 1$

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Question 11

Suppose that a function $f(x)$ is double differentiable for all $x \in (0,2)$, and satisfies the following properties:

- $f''(1) = 0$
- $f'(0) = 2$
- $f'(0.5) = 0$
- $f'(0.75) = -0.5$
- $f'(1) = -2$
- $f'(1.25) = -0.5$
- $f'(1.5) = 0$
- $f'(1.75) = 0.5$

Find the values of x such that the function is concave down.

Point of inflection occur at $x = 1$. Note that the gradient is decreasing for $x < 1$ and increasing for $x > 1$ Thus, the values of x such that the function is concave down is $x \in (0, 1)$.

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Question 12 Tech-Active.

Find a rule of a polynomial $f(x)$ so that $f(0) = 3, f(1) = 2, f'(2) = -8$, and so that there is a point of inflection when $x = 2$.

Since there are four equations, we should use a polynomial that involves four parameters – i.e. a cubic. Hence, assume $f(x) = ax^3 + bx^2 + cx + d$.
We solve these equations using CAS.
Solving these equations gives us the following solution $a = 1, b = -6, c = 4$ and $d = 3$. Therefore, a possible rule for the cubic could be $x^3 - 6x^2 + 4x + 3$.

```
In[96]:= f[x_] := a x^3 + b x^2 + c x + d

In[97]:= Solve[f[0] == 3 && f[1] == 2 && f'[2] == -8 && f''[2] == 0]

Out[97]= {{a → 1, b → -6, c → 4, d → 3}}
```

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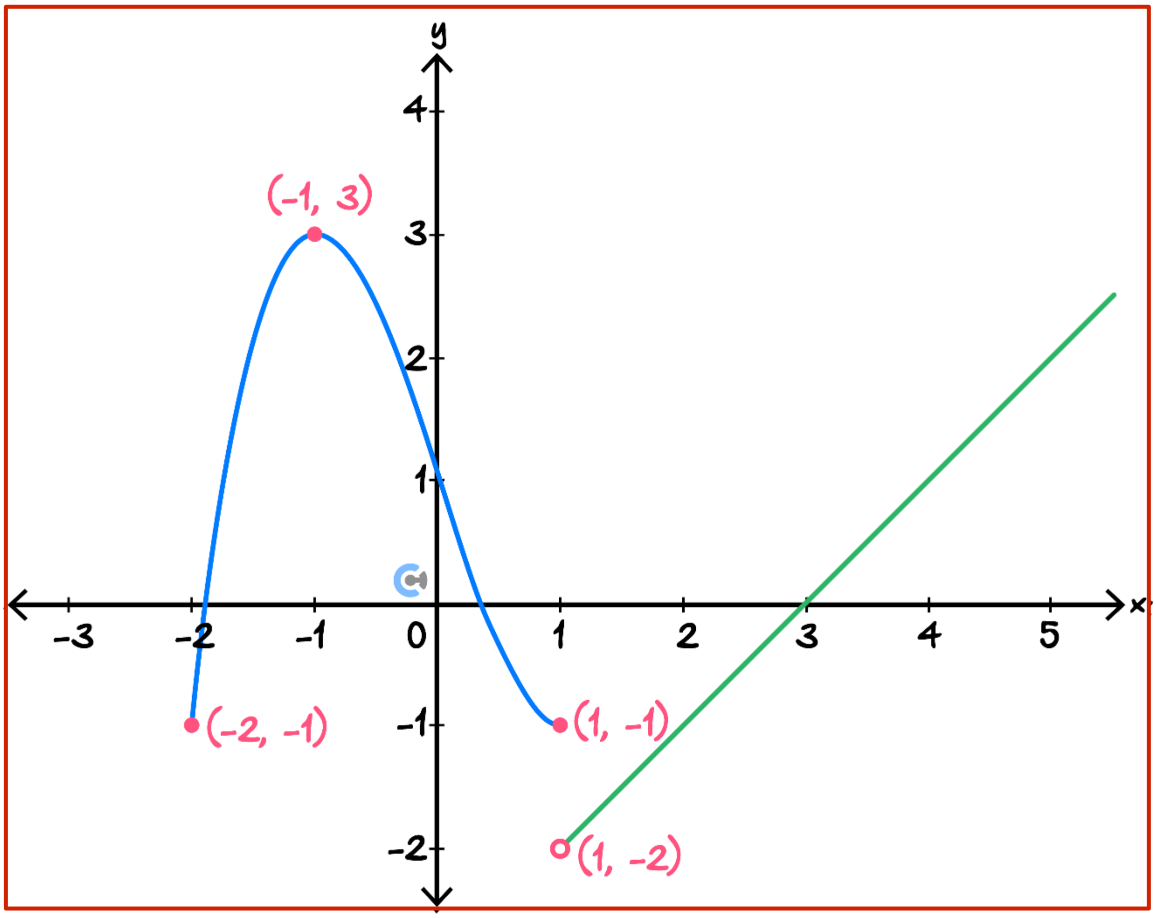
Sub-Section: The 'Final Boss'

Question 13

Consider the hybrid function:

$$f(x) = \begin{cases} x^3 - 3x + 1, & -2 \leq x \leq 1 \\ x - 3, & x > 1 \end{cases}$$

a. Sketch the graph of $y = f(x)$ on the axes below.



b. Define the derivative function $f'(x)$, specifying its domain.

$$f'(x) = \begin{cases} 3x^2 - 3 & -2 < x < 1 \\ 1 & x > 1 \end{cases}$$

c. State the point of inflection for the function $f_1(x) = x^3 - 3x + 1$.

$f_1''(x) = 0 \implies 6x = 0 \implies x = 0$

d. The function:

$$g(x) = \begin{cases} x^3 - x^2 + a - 2, & x \le 1 \\ x^2 + bx + 3, & x > 1 \end{cases}$$

is continuous and differentiable for all $x \in \mathbb{R}$. Find the values of a and b.

Let $g_1(x) = x^2 - x^2 + a - 2$ and $g_2(x) = x^2 + bx + 3$. We have
$g_1(1) = a - 2$ and $g_1'(1) = 1$.
$g_2(1) = 4 + b$ and $g_2'(1) = 2 + b$
We need $g_1'(1) = g_2'(1) = \implies 1 = 2 + b \implies b = -1$
Then $a - 2 = 4 + b \implies a = 5$

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Section B: Supplementary Questions

Sub-Section [2.2.1]: Evaluate Limits and Find Points Where the Function is not Continuous

Question 14

Evaluate the following limits:

a. $\lim_{x \to 3}(x^3 - 2x^2 + 5)$

As $x^3 - 2x^2 + 5$ is continuous at $x = 3$, we may substitute $x = 3$ to obtain $\lim_{x \to 3}\left(x^3 - 2x^2 + 5\right) = 14$.

b. $\lim_{x \to 4}(2^{\sqrt{x}} + \log_3(x^3 + 2x))$

As $2^{\sqrt{x}} + \log_3\left(x^3 + 2x\right)$ is continuous at $x = 4$, we may substitute $x = 4$ to obtain $\lim_{x \to 4}\left(2^{\sqrt{x}} + \log_3\left(x^3 + 2x\right)\right) = 4 + \log_3(72)$

c. $\lim_{x \to 3}(f(x))$, where,

$$f(x) = \begin{cases} 2x + 1, & x < 3 \\ 3x - 2, & x \geq 3 \end{cases}$$

At $x = 3$, we can check that $\lim_{x \to 3^-} f(x) = 2(3) + 1 = 7$ and $\lim_{x \to 3^+} f(x) = 3(3) - 2 = 7$. As the left and right limits are equal, we may conclude that $\lim_{x \to 3} f(x) = 7$.

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Question 15

Find the points x for which the following functions are **discontinuous** and state a reason as to why they are discontinuous.

a. $f(x) = \begin{cases} x, & x < 1 \\ x+1, & x \geq 1 \end{cases}$

Note that $\lim_{x \to 0^-} f(x) = 1$ but $\lim_{x \to 0^+} f(x) = 2$. Since the left and right limits are not equal at $x = 1$, we conclude that $f(x)$ is discontinuous at $x = 1$.

b. $f(x) = \frac{50}{x^2 - 7x + 6}$

We solve $x^2 - 7x + 6 = 0$ to find that $x = 1$ or $x = 6$. Thus, $f(x)$ is discontinuous at $x = 1$ and $x = 6$ as the function is not defined at these points.

c. $f(x) = \frac{x^2 - 4x + 3}{x - 3}$

We solve $x - 3 = 0$ to find that $x = 3$. Thus, $f(x)$ is not continuous at $x = 3$ as it is not defined at this point.

Note that $\lim_{x \to 3} f(x)$ *exists, but the function is not defined at* $x = 3$.

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Question 16

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} 3^{x-2} + 5x, & x < 2 \\ ax + 6, & x \geq 2 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Note that $f(x)$ is already continuous for all $x \neq 2$, and all that remains is to make $f(x)$ continuous at $x = 2$. Thus, we need $3^{2-2} + 5 \cdot 2 = 2a + 6$, so that $a = 5/2$.

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Question 17

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^2 - 4x - 12, & x < 7 \\ a^2 - ax + 1, & 7 \leq x < 10 \\ -x - 5, & x \geq 10 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Since $f(x)$ is continuous, we require that the right and left limits for $x = 7$ should be equal – i.e. $f(7^+) = f(7^-)$. Hence, we obtain the equation $9 = a^2 - 7a + 1$, i.e. $a^2 - 7a - 8 = 0$. Thus, $a = -1$ or $a = 8$.

Furthermore, we require that the right and left limits for $x = 10$ should be equal – i.e. $f(10^+) = f(10^-)$. Hence, we obtain a new equation $-15 = a^2 - 10a + 1$, i.e. $a^2 - 10a + 16 = 0$. Therefore, $a = 2$ or $a = 8$.

Since we require that $f(x)$ is continuous for all $x \in \mathbb{R}$, we need the value of a which satisfies the equations at both $x = 7$ and $x = 10$, so the final answer is $a = 8$.

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Sub-Section [2.2.2]: Apply Differentiability to Find Points Where Functions are not Differentiable, Domain of the Derivative and Unknowns of a Function

Question 18

Find the values of x such that the following function are not differentiable.

a. $f(x) = \begin{cases} -x+5, & x < 2 \\ x+1, & x \geq 2 \end{cases}$

By inspection of the sketch, there is a sharp point $x = 2$. Hence, the function is not differentiable at $x = 2$.

b. $f(x) = \frac{1}{x^2-4x+3}$

By inspection of the sketch, the function is not defined whenever the denominator vanishes, i.e. at $x = 1$ and $x = 3$. Thus, the function is not differentiable at $x = 1$ and $x = 3$.

c. $f(x) = \begin{cases} 2x, & x < 0 \\ 2x+1, & x \geq 0 \end{cases}$

By inspection, we can see a discontinuity at $x = 0$. If a function is not continuous at a given point, it is also not differentiable there either.

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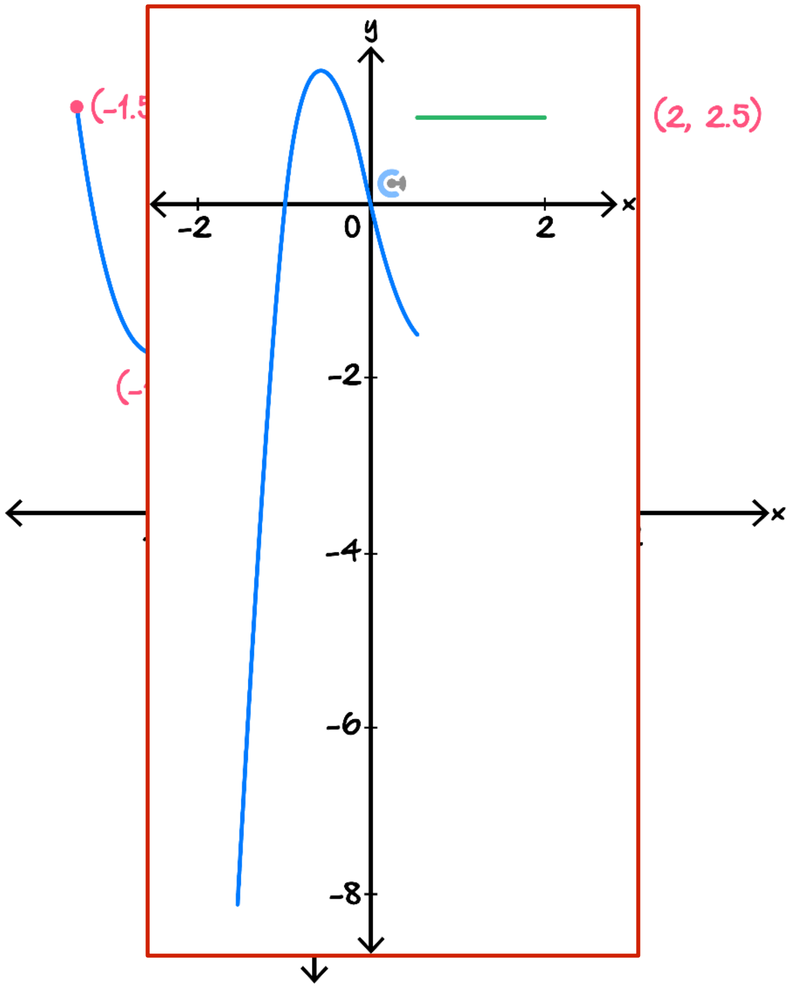
Question 19

Consider the following function.



a. Sketch the corresponding derivative function on the same set of axes above.

b. Furthermore, state the domain of the derivative function.

The function will be differentiable everywhere except the endpoints and the discontinuity, i.e. dom $f' = (-1.5, 2) \backslash \{0.5\}$.

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Question 20

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} 2x^2 - 6x + 5, & x < 2 \\ ax + b, & x \geq 2 \end{cases}$$

Find the value of a and b such that $f(x)$ is differentiable at $x = 2$.

Since the two rules must join at $x = 2$, imposing $f(2^+) = f(2^-)$ gives $2a + b = 1$. Also, $f'(2^+) = f'(2^-)$ gives $a = 2$. By solving the resulting system of equations, we conclude that $a = 2$ and $b = -3$.

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Question 21

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^3 - 3x + 5, & x < -1 \\ g(x), & -1 \le x < 1 \\ x^2 - 5x + 2, & x \ge 1 \end{cases}$$

The goal for this question is to find a suitable rule $g(x)$ making $f(x)$ differentiable for all $x \in \mathbb{R}$.

a. State the four equations that $g(x)$ and $g'(x)$ must satisfy at $x = 1$ and $x = -1$.

$g(1) = -2, g'(1) = -3, g(-1) = 7$ and $g'(-1) = 0$.

b. A natural choice would be to let $g(x)$ be a polynomial. As there are four equations that need to be satisfied, explain why it is suitable to set $g(x)$ to be a cubic polynomial.

A cubic polynomial $y = ax^3 + bx^2 + cx + d$ has four parameters, which should possibly give a unique solution for a, b, c and d based on the system of four equations obtained in the previous part.

c. Hence, find a suitable rule for $g(x) = ax^3 + bx^2 + cx + d$ assuming $g(x)$ is a polynomial. It may be necessary to use a CAS to solve the system of equations obtained in the working.

By using the first question, we see that a, b, c and d must satisfy

$$\begin{aligned} a + b + c + d &= -2 \\ 3a + 2b + c &= -3 \\ -a + b - c + d &= 7 \\ 3a - 2b + c &= 0 \end{aligned}$$

Hence, $a = \frac{3}{2}, b = -\frac{3}{4}, c = -6$ and $d = \frac{13}{4}$. Furthermore, $g(x) = \frac{3}{2}x^3 - \frac{3}{4}x^2 - 6x + \frac{13}{4}$ is a possible rule for the polynomial.

Note: This is a common technique called cubic splines used for interpolation.

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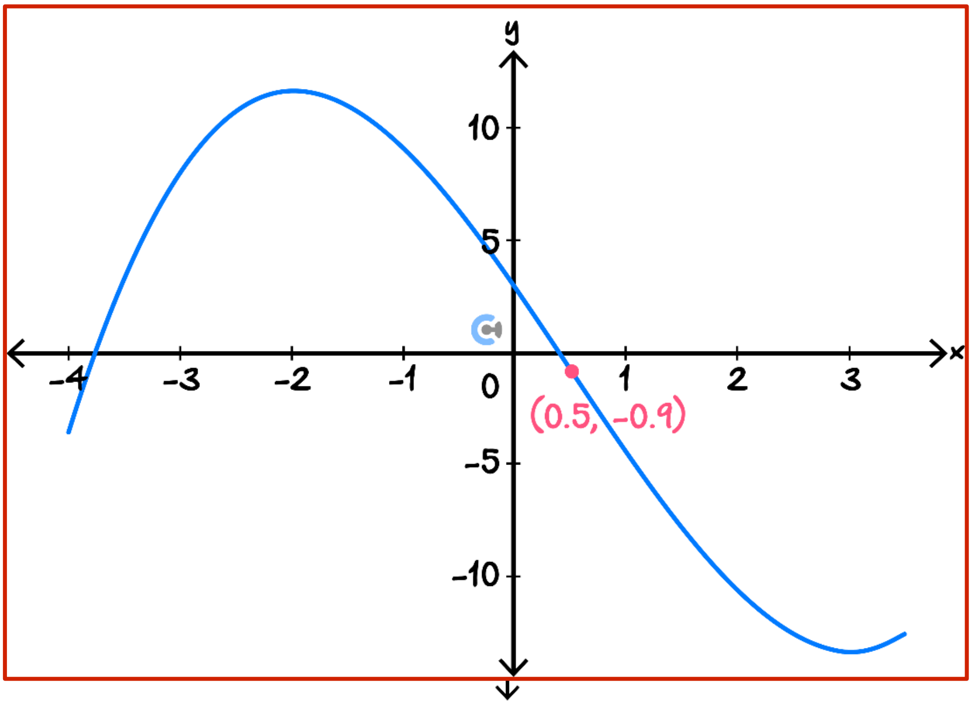
Sub-Section [2.2.3]: Identify Concavity and Find Inflection Points

Question 22

Consider the following graph for $f(x)$.



a. Circle the point of inflection on the above graph.

b. State the values of x such that the function is concave up.

Notice how the gradient of the function increases after $x = 1/2$ (i.e. it is becoming less negative, and after the turning point becomes more positive). So, the answer is $x > 1/2$.

c. State the values of x such that the function is concave down.

Similar to above, $x < 1/2$.

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Question 23

Consider a function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 - 2x^3 - 36x^2 + 5x + 1$.

a. Calculate the second derivative of the function $f(x)$.

$f''(x) = 12x^2 - 12x - 72 = 12(x^2 - x - 6) = 12(x-3)(x+2)$.
Note: You did not need to factorise the final answer, but it is helpful for the following question where you need to solve for the points of inflection.

b. Find the points of inflection of the function $f(x)$.

The points of inflection occur where $f''(x) = 0$, i.e. when $12(x-3)(x+2) = 0$. Therefore, the points of inflection occur when $x = -2$ or $x = 3$. Note also that $f''(2.9) < 0$, $f''(3.1) > 0$, $f''(-2.1) > 0$ and $f''(-1.9) < 0$, so $f''(x)$ does indeed switch signs around $x = -2$ and $x = 3$.

c. Find the values of x where the function is concave up.

We solve for the values of x such that $f''(x) > 0$. Thus, the function is concave up whenever $x < -2$ or $x > 3$
Note: Remember to exclude the endpoints!

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Question 24

Suppose that a function $f(x)$ is double differentiable for all $x \in (0,2)$, and satisfies the following properties:

- $f''(1) = 0$
- $f'(0) = 1$
- $f'(0.5) = 0$
- $f'(0.75) = -0.71$
- $f'(1) = -1$
- $f'(1.25) = -0.71$
- $f'(1.5) = 0$

Find the values of x such that the function is concave up.

Notice that the point of inflection occurs at $x = 1$ and that the gradient starts increasing after $x > 1$ (e.g. it becomes less negative). Thus, the values of x such that the function is concave up is $x \in (1, 2)$.

Question 25

Find a rule of a polynomial $g(x)$ so that $g(0) = 12$, $g(1) = 9$, $g(2) = 0$, and so that there is a point of inflection when $x = 2$.

Since there are four equations, we should use a polynomial that involves four parameters – i.e. a cubic. Hence, assume $g(x) = ax^3 + bx^2 + cx + d$. Based on the conditions, we obtain four equations

$$\begin{aligned} d &= 1 \\ a + b + c + d &= 9 \\ 8a + 4b + 2c + d &= 0 \\ 12a + 2b &= 0 \end{aligned}$$

The last equation comes from substituting $x = 2$ into $g''(x) = 6a + 2b$. Solving these equations gives us the following solution $a = 1, b = -6, c = 2$ and $d = 12$. Therefore, a possible rule for the cubic could be $x^3 - 6x^2 + 2x + 12$.

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