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VCE Mathematical Methods ¾
Differentiation I [2.1]
Workbook

Outline:

Learning Objectives:

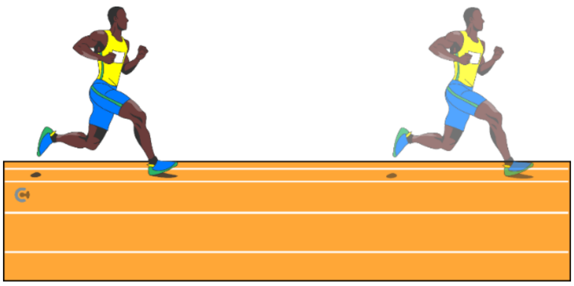
- ☐ **MM34 [2.1.1]** - Find the instantaneous rate of change and average rate of change
- ☐ **MM34 [2.1.2]** - Identify the nature of stationary points and trends (strictly increasing and decreasing)
- ☐ **MM34 [2.1.3]** - Graph Derivative Functions

Section A: Introduction to Differentiation

Sub-Section: Average Rate of Change

Context: Average Rate of Change

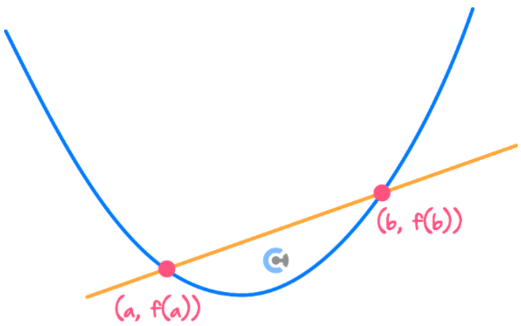
- Usain Bolt ran the world record $100\ m$ race in 9.58 seconds.
- Let's say we take a photo of Usain Bolt at the start and end of the race.



- How would the average speed between two instances be calculated?

$$\textit{Average Speed} = \frac{\textit{Change in } \underline{\text{position}} \textit{ from start to end}}{\textit{Change in } \underline{\text{time}}}$$

Average Rate of Change



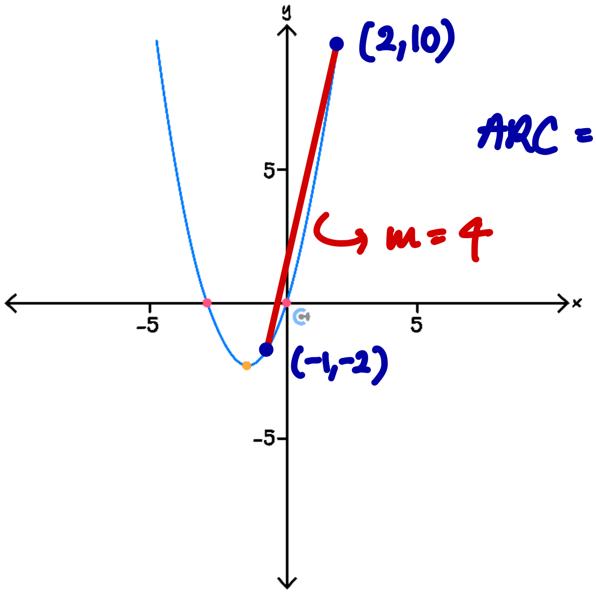
- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- It is the _gradient_ of the line joining the two points.

Question 1

Find the average rate of change of $y = x^2 + 3x$ over the interval $x \in [-1,2]$.



$$ARC = \frac{f(2) - f(-1)}{2-(-1)}$$

$$= \frac{10-(-2)}{2+1}$$

$$= \frac{12}{3} = 4$$

Discussion: Do we differentiate for average rate of change?

No! We diff for IRC.

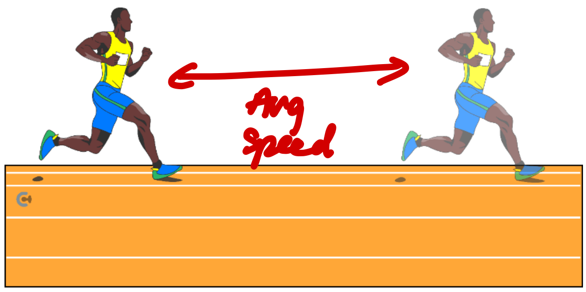
Instantaneous Rate of Change

Sub-Section: Instantaneous Rate of Change

How can we now find Usain Bolt's speed at a single moment?

Context: Instantaneous rate of change

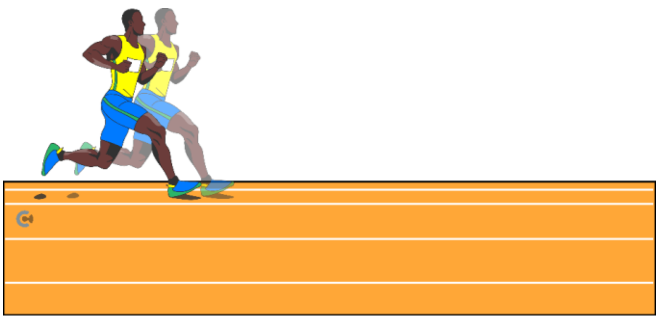
- Usain Bolt ran the world record $100\ m$ race in 9.58 seconds.



- We took two photos of Usain Bolt at the start and end of the race.
- We calculated the average speed over the entire race using these two photos.

$$\text{Average Speed} = \frac{\text{Distance}}{\text{Time Taken}}$$

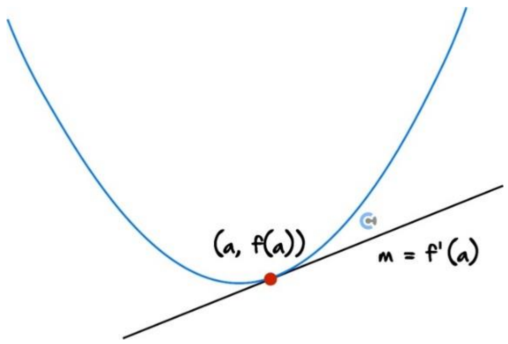
- How can we find his speed at the **start of the race?**
- Where should the two photos be taken? [At the start/At the end/Start and End]



- How closely should the two photos be taken to find the speed at a single moment? [Super Close/Super Far]
- This is the instantaneous rate of change!

Definition

Instantaneous Rate of Change



- **Instantaneous Rate of Change** is a gradient of a graph at a single point.

$$\textit{Instantaneous Rate of Change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.

Question 2

Consider the function $f(x)$ and its derivative $f'(x)$. It is known that $f(2) = 4$, $f(3) = 9$, $f'(2) = 3$ and $f'(3) = -4$.

Find the gradient of the function $f(x)$ at $x = 3$.

$$\therefore f'(3) = -4$$

Definition

Alternative Notation for Derivative

$f(x)$ $\qquad$ y

$$f'(x) = \frac{dy}{dx}$$

Discussion: How does the notation $\frac{dy}{dx}$ make sense?

Δ = delta "change" $\Rightarrow \dfrac{\Delta y}{\Delta x}$ ARC $\xRightarrow[\Delta x \to 0]{\Delta y \to 0} \dfrac{dy}{dx} \Rightarrow$ IRC

$dy \Rightarrow$ instantaneous change in y

$dx \Rightarrow$ instantaneous change in x

Sub-Section: Differentiation

How do we find derivative functions?

$f(x) = 2 \cdot x^0$

$f'(x) = 0 \cdot 2 \cdot x^{-1} = 0$

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \cdot x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = \sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$
Constant	0

1. Bring power down (x)
2. -1 from power

$\sec(x) = \frac{1}{\cos(x)}$

Question 3

Consider the function $f(x) = x^3 - 4x$.

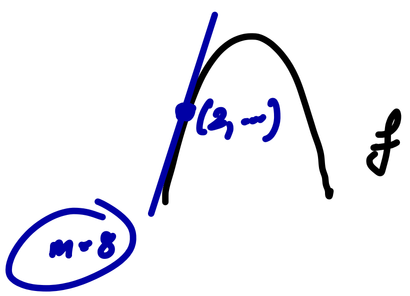
Find the gradient of the function at $x = 2$.

$f'(x) = 3x^2 - 4$

$f'(2) = 3(2)^2 - 4$

$= 8$

gradient of f at $x = 2$



Question 4

Consider the function $f(x) = 2e^x - 4$.

Find the gradient of the function at $x = 3$.

$f'(x) = 2e^x$

$f'(3) = 2e^3$ //

Question 5

Consider the function $f(x) = 2\log_e(x)$.

Find the gradient of the function at $x = 2e$.

$f'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$

$f'(2e) = \frac{2}{2e} = \frac{1}{e}$ //

Question 6

Consider the function $f(x) = \cos(x) + \sin(x)$.

Find the gradient of the function at $x = \frac{\pi}{4}$.

$f'(x) = -\sin(x) + \cos(x)$

$f'\left(\frac{\pi}{4}\right) = -\sin\left(\overset{\to 45^\circ}{\frac{\pi}{4}}\right) + \cos\left(\frac{\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$ //

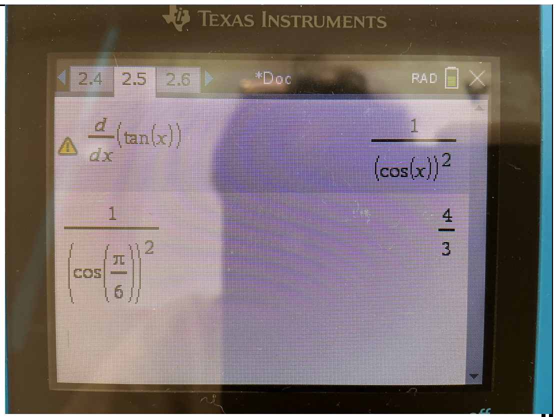
Calculator Commands: Finding Derivatives

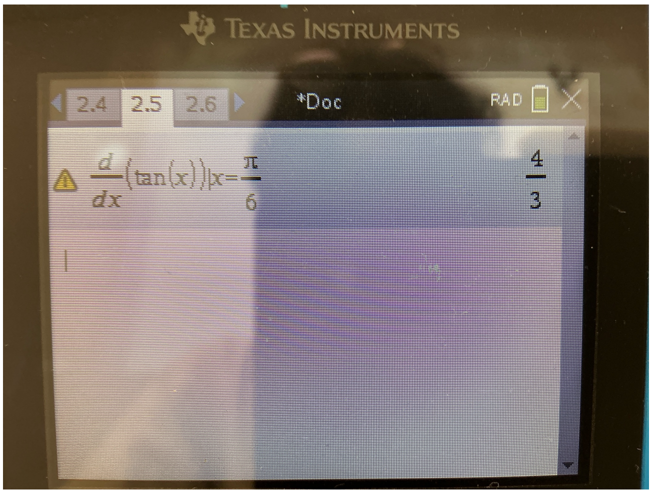
Mathematica	TI	Casio
f'[x]	Shift Minus	Math 2
	$\frac{d}{dx}(f(x))$	$\frac{d}{dx}(f(x))$

Question 7 Tech-Active.

Consider the function $f(x) = \tan(x)$.

Find the gradient of the function at $x = \frac{\pi}{6}$.

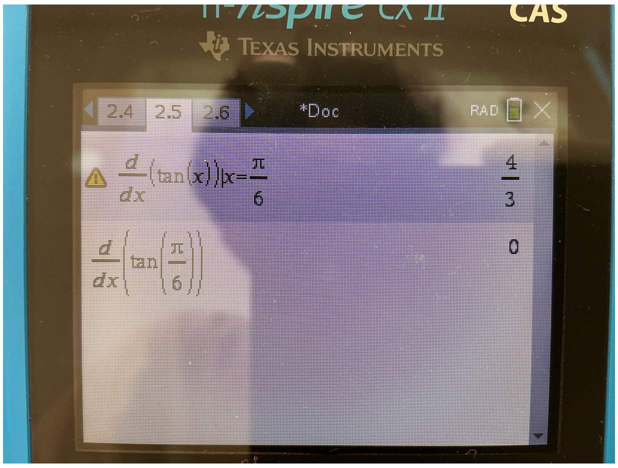




Diff operator : $\frac{d}{d\square}$: Menu → 4 → 1 / Shift → −

NOTE: You must substitute the x-value **after** finding the derivative function first.

Discussion: What would happen if you derived $f\left(\frac{\pi}{6}\right)$ instead of CAS?



$f\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right)$
$= \frac{\sqrt{3}}{3}$

Sub x-value first

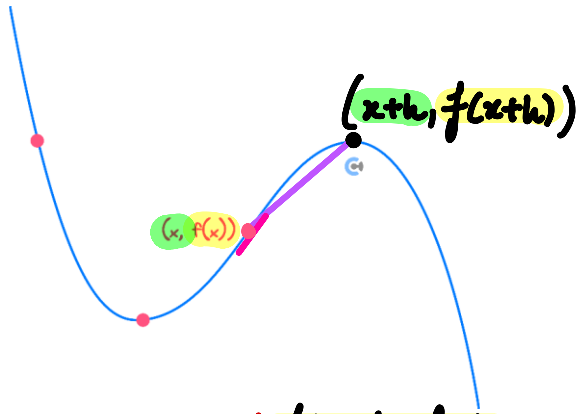
$\left(f\left(\frac{\pi}{6}\right)\right)' = \left(\frac{\sqrt{3}}{3}\right)'$
$= 0$

Diff constant

Sub-Section: First Principle

Where do all the derivative rules come from?

First Principle



$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{\cancel{x}+h-\cancel{x}}\right) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

- The fundamental method of differentiation

Exploration: Visualisation of First Principle

- Desmos link

Pascals:

1

1 1

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

Question 8

$f(x+h) = (x+h)^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$

Consider the function $f(x) = x^3$.

Find the derivative using the first principle.

1. Apply First Principle's Formula
2. Find $f(x+h)$
3. Simplify (& Remove h from denominator)
4. Set $h=0$ ("Activate" limit)

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{3x^2h + 3xh^2 + h^3}{h}\right) = \lim_{h \to 0}\left(3x^2 + \cancel{3xh} + \cancel{h^2}\right)$$

$\Downarrow h=0$

$= 3x^2$

NOTE: It's the same as the table above!

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Section B: Advanced Differentiation

Sub-Section: Product Rule

How do we find the derivative when two functions are multiplied?
For example: $x^2\sin(x)$.

$$(f(x)\cdot g(x))' = \cancel{f'(x)\cdot g'(x)}$$

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

NOTE: We never differentiate two functions at once!

Question 9 Walkthrough.

Find the derivative of $f(x) = x^3\tan(x)$.

$$f'(x) = (x^3)'(\tan(x)) + (x^3)(\tan(x))'$$

$$= 3x^2\tan(x) + x^3\sec^2(x)$$

NOTE: We **never** differentiate **both** functions at the same time!

Your turn!

Question 10

Find the derivatives of:

a. $f(x) = x^2e^x$

$$f'(x) = (x^2)'(e^x) + (x^2)(e^x)'$$
$$= 2xe^x + x^2e^x$$

b. $y = 3\sin(x)\cos(x)$

$3[\sin(x)\cos(x)]'$

$[(3\sin(x))\cdot(\cos(x))]'$

$$\frac{dy}{dx} = (3\sin(x))'(\cos(x)) + (\cos(x))'(3\sin(x))$$
$$= 3\cos(x)\cdot\cos(x) + (-\sin(x))\cdot 3\sin(x)$$
$$= 3\cos^2(x) - 3\sin^2(x)$$

c. $g(x) = \log_e(x) \cdot x$

$$g'(x) = (x)'(\log_e(x)) + (x)(\log_e(x))'$$
$$= 1\cdot\log_e(x) + x\cdot\frac{1}{x}$$
$$= \log_e(x) + 1$$

~~**Question 11 Extension.**~~

Find the derivative of $f(x) = x^3 \log_e(x)\sin(x)$.

$$f'(x) = (x^3)' \cdot \log_e(x) \cdot \sin(x) + x^3 \cdot (\log_e(x))' \cdot \sin(x) + x^3 \cdot \log_e(x) \cdot (\sin(x))'$$

$$= 3x^2 \log_e(x)\sin(x) + x^2 \sin(x) + x^3 \log_e(x)\cos(x)$$

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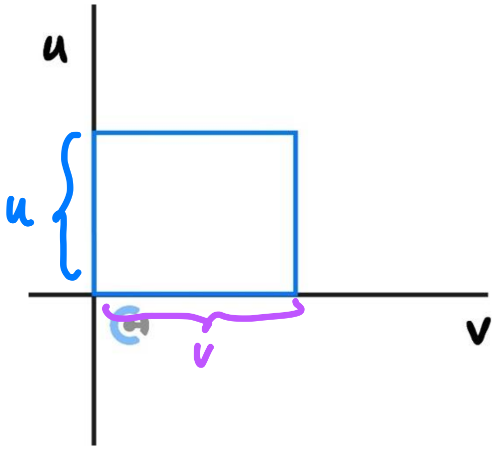
How does this work?

Extension: Understanding Product Rule

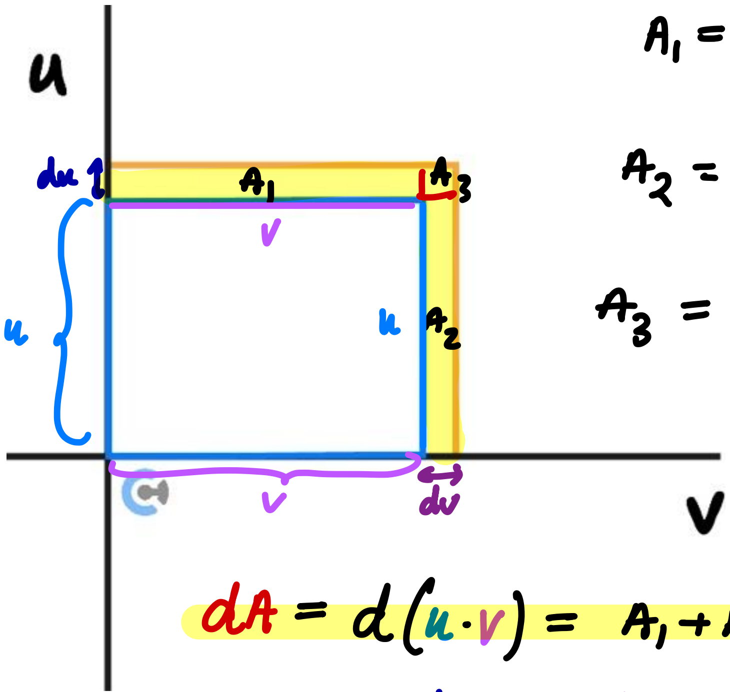
- Consider the rectangle in the diagram below.



- What would the area of the rectangle be?

$$\textit{Area of the rectangle} = \underline{\quad u \cdot v \quad}$$

- Now let's say the rectangle grew in size!
- Let's take a very close photo of this rectangle while it is growing (similar to Usain Bolt)



$$A_1 = v \cdot du$$

$$A_2 = u \cdot dv$$

$$A_3 = du \cdot dv \approx 0$$

$$dA = d(u \cdot v) = A_1 + A_2 + A_3$$

$$= v \cdot du + u \cdot dv + \underbrace{du \cdot dv}_{=0}$$

$$= v \cdot du + u \cdot dv$$

- How can we find the instantaneous **change of** $u \cdot v$?

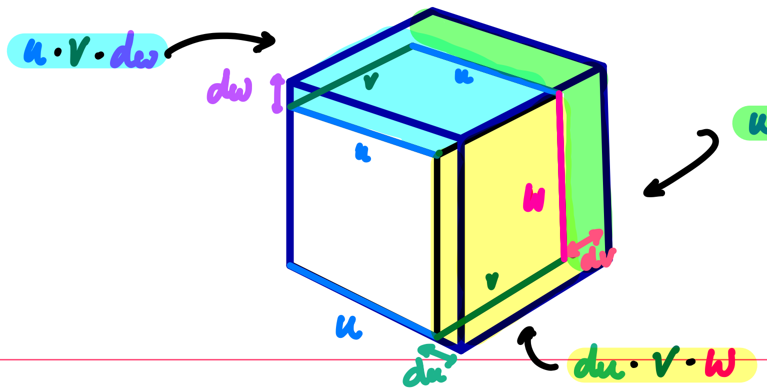
$$d(u \cdot v) = \underline{v \cdot du + u \cdot dv}$$

- Divide all sides by dx. What do you notice?

$$\frac{d(u \cdot v)}{dx} = \underline{v \cdot \frac{du}{dx} + u \cdot \frac{dv}{dx}}$$

This is a product rule!

~~**Discussion:** What shape do we use for proving product rule for three functions?~~



$\hookrightarrow \therefore = u'vw + uv'w + uvw'$

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Sub-Section: Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \cancel{\frac{f'(x)}{g'(x)}}$$

Definition

The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Always differentiate the top function first.

Question 12 Walkthrough.

Find the derivative of $y = \frac{x^2}{\sin(x)}$.

$$\frac{dy}{dx} = \frac{(x^2)'(\sin(x)) - (\sin(x))' \cdot (x^2)}{[\sin(x)]^2}$$

$$= \frac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)} //$$

NOTE: The order **matters** for the quotient rule! We differentiate the **top function first**.

Question 13

Find the derivatives of:

a. $\frac{e^x}{4x^3}$

$$f'(x) = \frac{4x^3e^x - 12x^2e^x}{(4x^3)^2}$$

$$= \frac{\cancel{4x^2}e^x(x-3)}{4\,\cancel{16}x^{\cancel{6}4}} = \frac{e^x(x-3)}{4x^4} //$$

b. $\frac{\log_e(x)}{x}$

$$f'(x) = \frac{\frac{1}{x}\cdot x - \log_e(x)\cdot 1}{x^2}$$

$$= \frac{1-\log_e(x)}{x^2} //$$

c. $g(x) = \frac{\sin(x)}{\cos(x)}$ $= \tan(x)$ $\quad g'(x) = \sec^2(x)$

$$g'(x) = \frac{(\sin(x))'\cdot\cos(x) - (\cos(x))'\cdot\sin(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - (-\sin(x))\cdot\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$\cos^2(x)+\sin^2(x) = 1$

$$= \frac{1}{\cos^2(x)} = \sec^2(x) //$$

NOTE: The last question is a derivative of tan.

~~**Question 14 Extension.**~~

Find the derivative of $y = \frac{x^2 e^x}{\log_e(x)}$.

$$(x^2e^x)' = 2xe^x + x^2e^x$$

$$\frac{dy}{dx} = \frac{(x^2e^x)'\log_e(x) - \frac{1}{x}(x^2e^x)}{(\log_e(x))^2}$$

$$= \frac{(2xe^x + x^2e^x)\log_e(x) - xe^x}{(\log_e(x))^2} //$$

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How does the quotient rule work? (Extended)

Exploration: Proving Quotient Rule

- Consider the rule:

$$y = \frac{u}{v}$$

- Instead of using the quotient rule, we can cross multiply and use product!

$$yv = u$$

$$y'v + v'y = u'$$

- What happens now when we make y' the subject?

$$y'v = u' - v'y$$

$$y' = \frac{u' - v'y}{v}$$

- Now substitute $y = \frac{u}{v}$ back again!

$$y' = \frac{u' - v'\left(\frac{u}{v}\right)}{v} = \frac{\left(\frac{u'v - v'u}{v}\right)}{v}$$

$$y' = \frac{u'v - v'u}{v^2} \;//$$

- What does this look like?

$$\begin{tikzcd} \text{Train} \arrow[r, bend left, "2x"] \arrow[rr, bend right, "8x"'] & \text{Cheetah} \arrow[r, bend left, "4x"] & \text{Lee} \end{tikzcd}$$

$$\begin{tikzcd} f(g(x)) \arrow[r, bend left, "f'(g(x))"] \arrow[rr, bend right, "f'(g(x)) \cdot g'(x)"'] & g(x) \arrow[r, bend left, "g'(x)"] & x \end{tikzcd}$$

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$$\begin{tikzcd} f(x) \arrow[r, bend left, "f'(x)"] & x \end{tikzcd}$$

$$\frac{d}{dx}\left(f(x)\right) = f'(x)$$

Sub-Section: Chain Rule

Definition

The Chain Rule

$$y = f\big(g(x)\big)$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$f \to g \times g \to x$$

- The process for finding derivatives of **composite functions.**

How does the chain rule work?

Exploration: Understanding chain rule.

- Consider the function we want to differentiate with respect to x is:

$$y = f\big(g(x)\big)$$

- We can remove the composition by letting the inside function equal to u.

$$\textit{Let } u = g(x)$$

$$\textit{Then } y = f(u)$$

- We can now derive y w/ respect to u.

$$\frac{dy}{du} = f'(u)$$

$$y \xrightarrow{f'(u)} u \longrightarrow x$$

$$\frac{dy}{du} \qquad \frac{du}{dx}$$

Note that we have $\frac{dy}{du}$ instead of $\frac{dy}{dx}$ as we derived in terms of u.

$$\frac{dy}{du} \cdot \boxed{\frac{du}{dx}} = \frac{dy}{dx}$$

- To find $\frac{dy}{dx}$, we simply multiply by $\frac{dy}{du}$ with $\frac{du}{dx}$.

$u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x)$$

$$\frac{du}{dx} = \frac{d}{dx}(u) = \frac{d}{dx}(g(x)) = g'(x)$$

- Finally, we can substitute $u = g(x)$.

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Question 15 Walkthrough.

Find the derivative of $f(x) = \sin(x^2)$.

Let $u = x^2$.

$y = \sin(u)$

$\frac{dy}{du} = \cos(u)$

$\frac{du}{dx} = 2x$

$\therefore \frac{dy}{dx} = 2x\cos(u)$

$= 2x\cos(x^2)$

$$\begin{tikzcd} y \arrow[r, "\cos(u)"'] \arrow[rr, bend left, "2x\cos(u)"] & u \arrow[r, "2x"'] & x \end{tikzcd}$$

$\cos(u) \times 2x$

1. Let $u = g(x)$ (inside function)
2. Find y in terms of u
3. Find $\frac{dy}{du}$
4. Find $\frac{du}{dx}$
5. Multiply $\frac{dy}{du}$ & $\frac{du}{dx}$ to get $\frac{dy}{dx}$.

5.5 Resub $u = g(x)$

NOTE: Always let the inside function equal to u.

Your turn!

Question 16

Find the derivatives of:

a. $e^{x^2+\frac{1}{2}x}$

Let $u = x^2 + \frac{1}{2}x$

$\therefore y = e^u$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = 2x + \frac{1}{2}$

$\therefore \frac{dy}{dx} = \left(2x+\frac{1}{2}\right) \cdot e^u$

$= \left(2x+\frac{1}{2}\right) \cdot e^{x^2+\frac{1}{2}x}$ //

b. $\left(4x + \frac{1}{x}\right)^3$

Let $u = 4x + \frac{1}{x}$:

$(x^{-1})'$

$= -1 \cdot x^{-2}$

$= \frac{-1}{x^2}$

$\therefore y = u^3$

$\therefore \frac{dy}{du} = 3u^2$

$\therefore \frac{du}{dx} = 4 - \frac{1}{x^2}$

$\therefore \frac{dy}{dx} = 3u^2\left(4-\frac{1}{x^2}\right)$

$= 3\left(4x+\frac{1}{x}\right)^2\left(4-\frac{1}{x^2}\right)$ //

c. $\log_e(x^2 - 4)$

Let $u = x^2 - 4$:

$\therefore y = \log_e(u)$

$\frac{dy}{du} = \frac{1}{u}$

$\frac{du}{dx} = 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{u} = \frac{2x}{x^2-4}$ //

~~**Question 17 Extension.**~~

Find the derivative of $f(x) = x^3 \log_e(x^2)\sin^2(x)$.

$f'(x) = (x^3)' \cdot \log_e(x^2) \cdot \sin(x^2) + x^3 \cdot (\log_e(x^2))' \sin^2(x) + x^3 \cdot \log_e(x^2) \cdot (\sin^2(x))'$

$= 3x^2 \log_e(x^2) \cdot \sin(x^2) + x^3 \cdot \frac{2x}{x^2} \cdot \sin(x^2) + x^3 \cdot \log_e(x) \cdot 2\sin(x) \cdot \cos(x)$

$= 3x^2 \log_e(x^2)\sin(x^2) + 2x^2 \sin(x^2) + 2x^3 \log_e(x)\sin(x)\cos(x)$ //

Is there a quicker way to do a chain rule?

Definition

Shortcut for Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

- Derive the outside function only.
- Multiply the function by the derivative of the inside.

1. Diff outside
2. Keep inside same
3. × by diff of inside

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Question 18 Walkthrough.

Using the quick method of the chain rule, find the derivative of $f(x) = \cos(x^3)$.

$$f'(x) = -\sin(x^3) \cdot 3x^2 = -3x^2\sin(x^3)$$

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Your turn!

Question 19

Using the quick method of chain rule, find the derivative of:

a. e^{3x^2-x}

$f'(x) = e^{3x^2-x} \cdot (6x-1)$ //

b. $\log_e(x^2 + 9x + 6)$

$f'(x) = \frac{1}{x^2+9x+6} \cdot (2x+9) = \frac{2x+9}{x^2+9x+6}$ //

c. $g(x) = \tan(x^2)$

$g'(x) = \sec^2(x^2) \cdot 2x$ //

Question 20 Extension.

Find the derivative of $g(t) = \log_e(\cos(\sqrt{t+1}))$.

$$g'(t) = \frac{1}{\cos(\sqrt{t+1})} \cdot \underline{(\cos(\sqrt{t+1}))'}$$

$$= \frac{1}{\cos(\sqrt{t+1})} \cdot \underline{-\sin(\sqrt{t+1}) \cdot (\sqrt{t+1})'}$$

$$= \frac{1}{\cos(\sqrt{t+1})} \cdot -\sin(\sqrt{t+1}) \cdot \underline{\frac{1}{2}(t+1)^{-\frac{1}{2}}} \quad //$$

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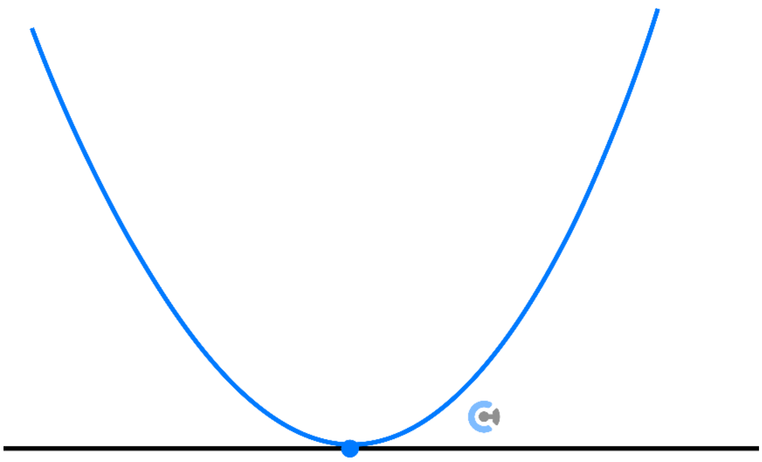
Section C: Stationary Points and Strictly Increasing

Sub-Section: Stationary Points

What would be the gradient of a point that is neither increasing nor decreasing?

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

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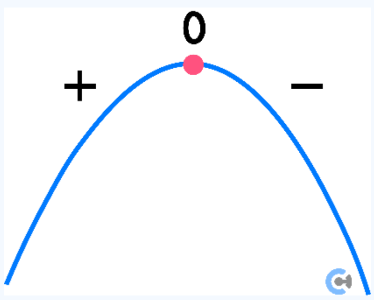
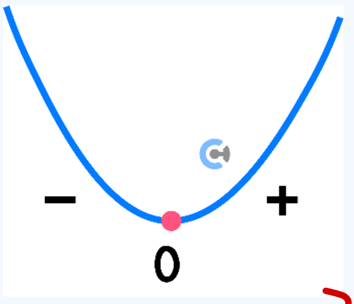
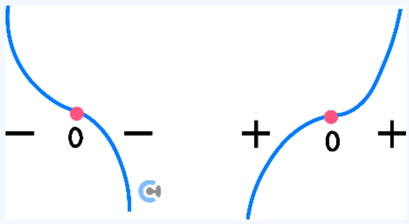
What are the types of stationary points?

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

TPs

Sign Test

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighbouring points.

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Question 21 Walkthrough.

Find and identify the nature of the stationary points of $y = -e^{x^2+4}$.

$\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \underbrace{-e^{x^2+4}}_{\neq 0} \cdot \underbrace{(2x)}_{=0} = 0$

$\therefore x = 0$

x	-1	0	1
$f'(x)$	$2e^5$	0	$-2e^5$
Shape	/	—	\

local max at $x=0$

$\left.\frac{dy}{dx}\right|_{x=-1} = (-2)(-e^5)$

$= 2e^5$

$\left.\frac{dy}{dx}\right|_{x=1} = (2)(-e^5)$

$= -2e^5$

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Question 22

Find and identify the nature of the stationary points of $y = \log_e(x^2 + 4)$.

$\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \underbrace{\frac{1}{x^2+4}}_{\neq 0} \cdot \underbrace{(2x)}_{=0} = 0$

$\Downarrow$

$\therefore x = 0$

x	-1	0	1
$f'(x)$	$-\frac{2}{5}$	0	$\frac{2}{5}$
Shape	\	—	/

$\hookrightarrow$ Local min at $x = 0$

$\left.\frac{dy}{dx}\right|_{x=-1} = -\frac{2}{5}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{5}$

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~~**Question 23 Extension.**~~

Consider the function $f(x) = xg(x)$.

It is known that, $g(0) = -5$, $g(1) = -2$ and $g(2) = 1$.

$g'(0) = -4$, $g'(1) = 2$ and $g'(2) = 3$ and that f has only one stationary point.

Show that $f(x)$ has a stationary point when $x = 1$ and identify its nature.

$$f'(x) = 1 \cdot g(x) + x \cdot g'(x)$$

$$\therefore f'(1) = 1 \cdot g(1) + 1 \cdot g'(1) = -2+2$$
$$= 0$$

$$f'(0) = g(0) + g'(0)$$
$$= -5-4$$
$$= -9$$

$$f'(2) = g(2) + g'(2)$$
$$= 1+3$$
$$= 4$$

$\Rightarrow \therefore f$ has a stationary point at $x = 1$

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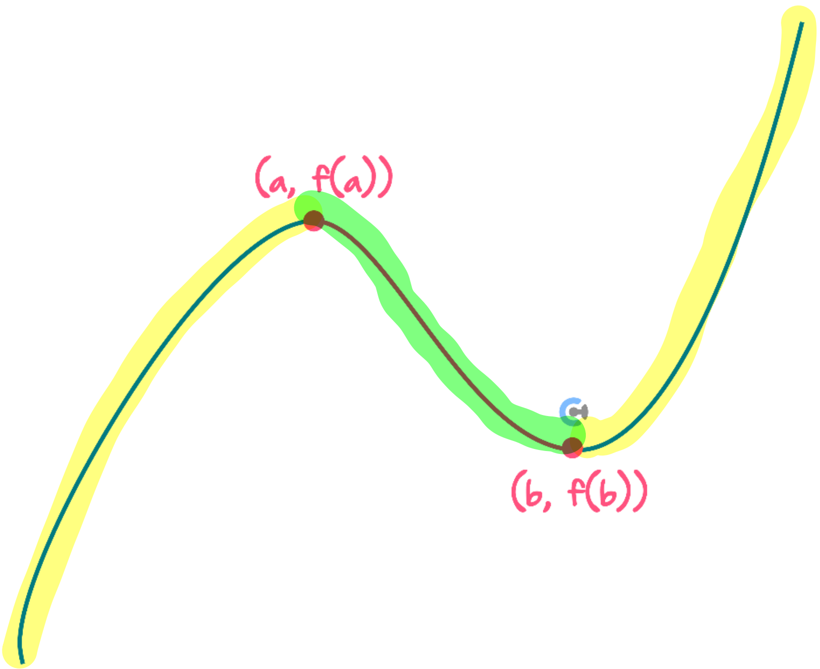
x	0	1	2
$f'(x)$	-9	0	4
Shape	\	—	/

$\therefore$ Local minimum

Sub-Section: Strictly Increasing and Decreasing

Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:

1. Find the turning points.
2. Consider the sign of the derivative between/outside the turning points.

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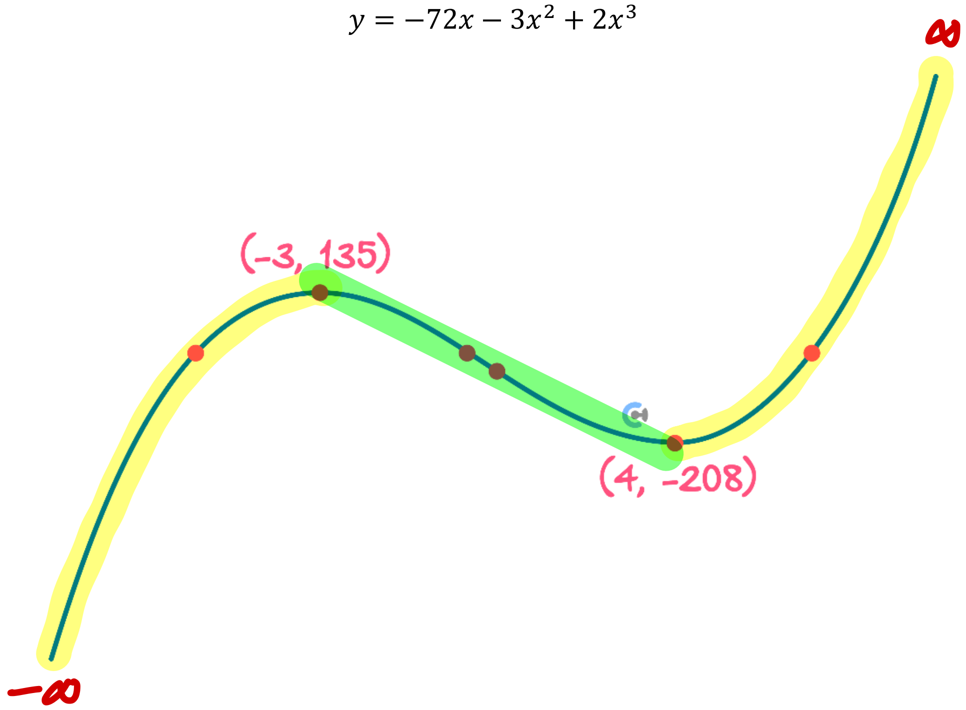
Question 24

State the value(s) of x for the function below for which it is strictly increasing and strictly decreasing.

$$y = -72x - 3x^2 + 2x^3$$



SI : $x \in (-\infty, -3] \cup [4, \infty)$

SD : $x \in [-3, 4]$

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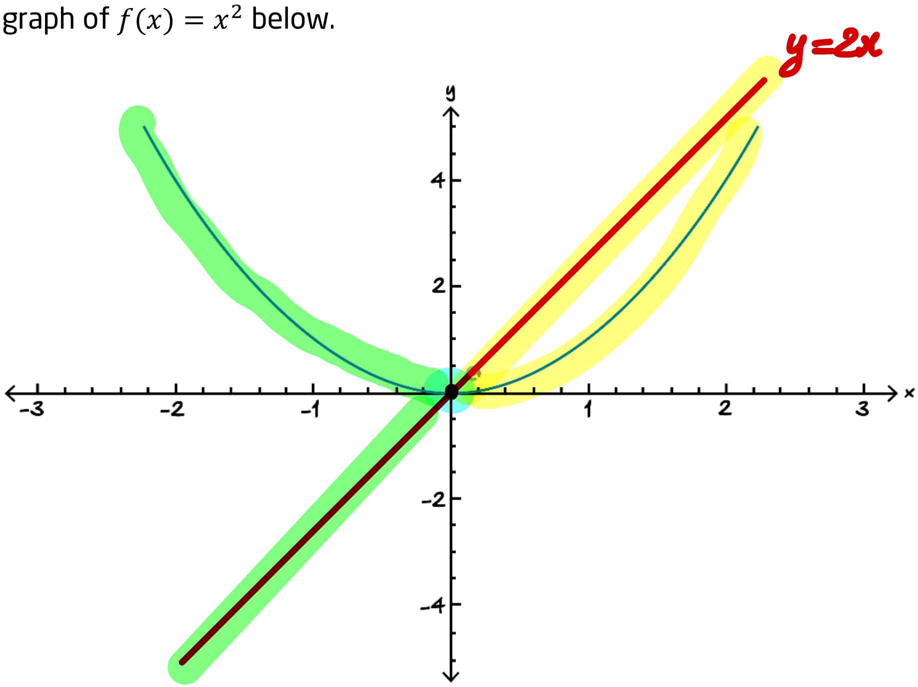
Section D: Graphs of the Derivative Function

Sub-Section: Graphs of Derivative Function

Exploration: Graph of derivative functions

- Consider the graph of $f(x) = x^2$ below.



- What is the derivative of $f(x)$?
- Sketch the derivative above.

$$f'(x) = \underline{\ 2x\ }$$

- What do you notice about $f'(x)$: Derivative when $f(x)$ has a stationary point?

x-intercept ⇒ y=0 = f′(x)=0

- What do you notice about $f'(x)$ when $f(x)$ is increasing?

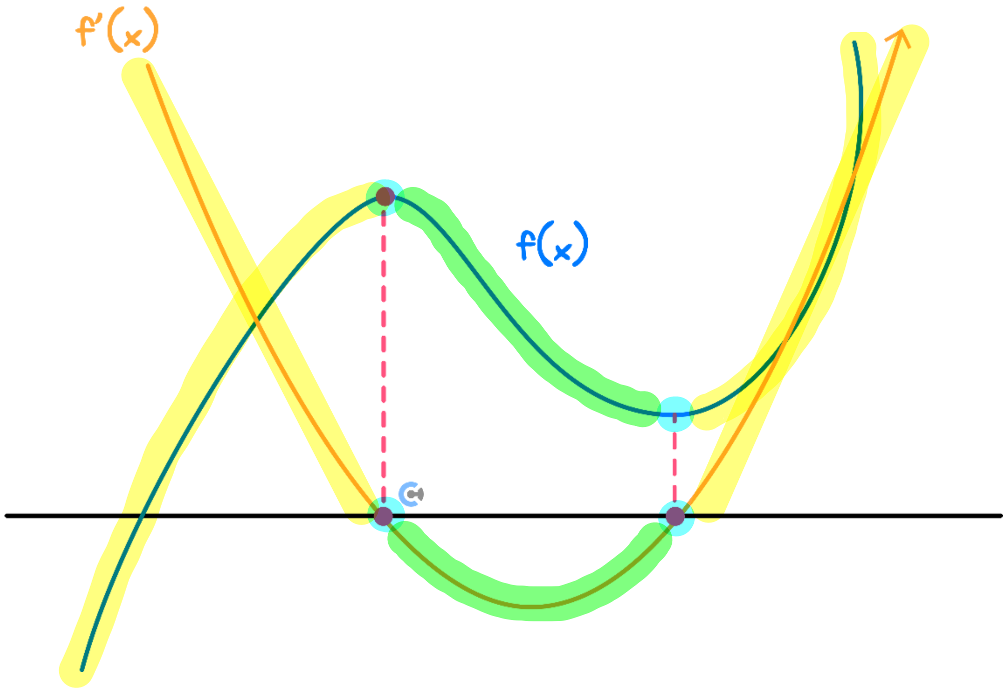
f′(x) ⇒ +ve ⇒ y>0 ⇒ above x-axis

- What do you notice about $f'(x)$ when $f(x)$ is decreasing?

f′(x) ⇒ −ve ⇒ y<0 ⇒ below x-axis

In summary!

<u>Graphs of the Derivative Function</u>



$f(x)$	$f'(x)$
Stationary Point	x-intercept
Increasing	above x-axis
Decreasing	below x-axis

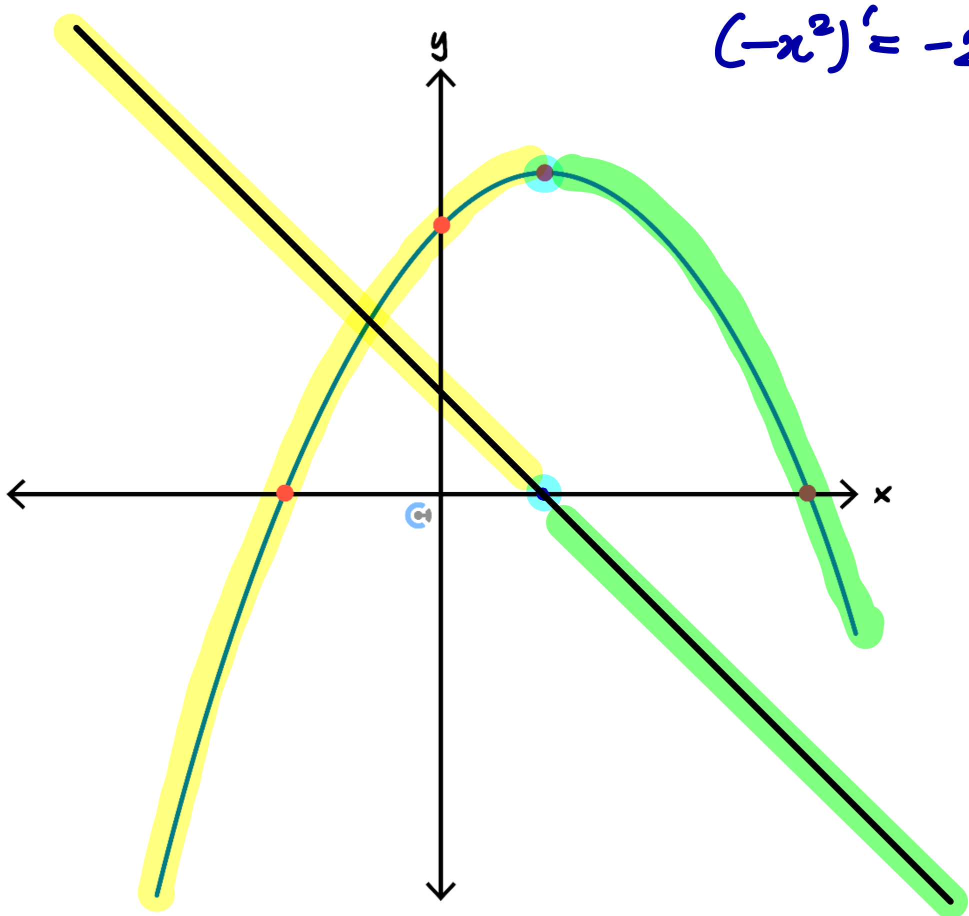
y *value of* $f'(x) =$ gradient of f(x)

- Steps

1. Plot x-intercept at the same x value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.

Question 25 Walkthrough.

Sketch the derivative graph of the function shown below, on the same set of axes.



$(-x^2)' = -2x$ //

Active Recall: Steps on sketching the derivative function.

1. Plot x-intercept at the same x-value as the SPs of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.

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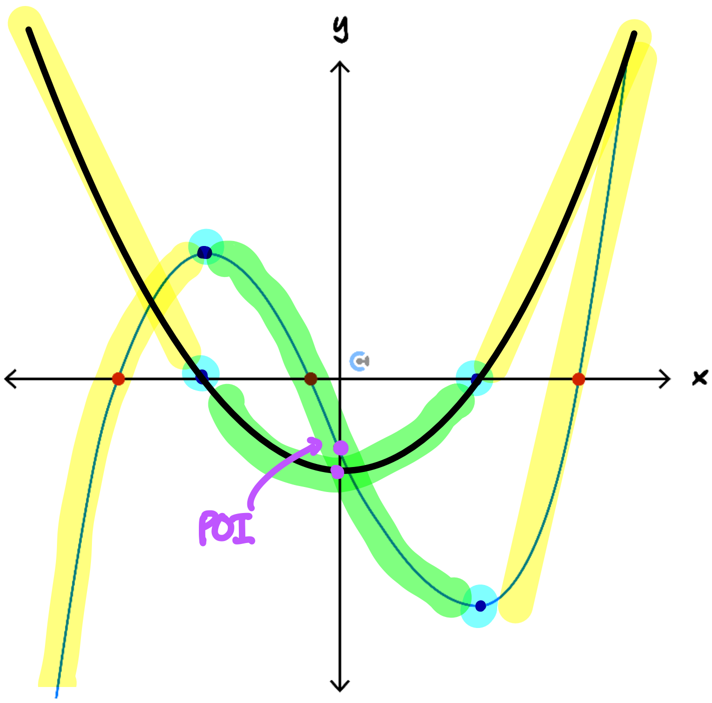
Question 26

Sketch the derivative graph of the function shown below, on the same set of axes.



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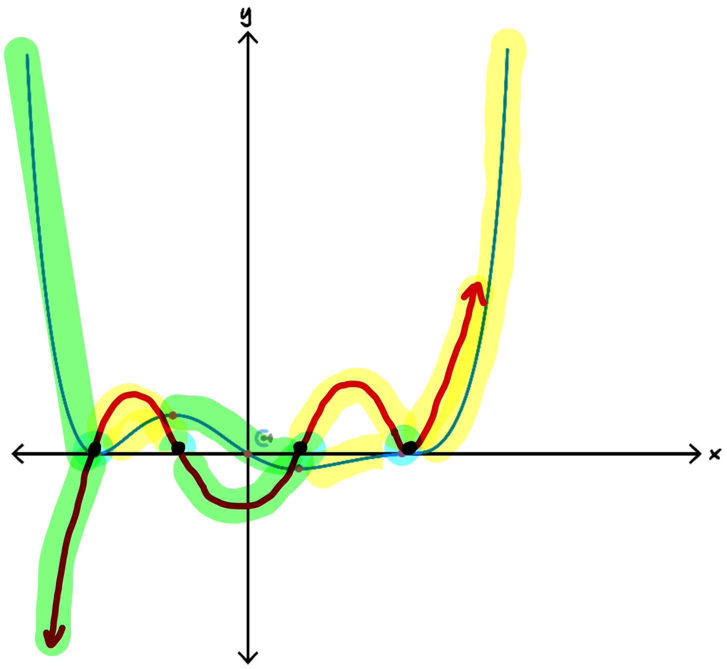
Discord - Sujok

Question 27 Extension.

Sketch the derivative graph of the function shown below, on the same set of axes.



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Contour Checklist

Learning Objective: [2.1.1] – Find the Instantaneous Rate of Change and Average Rate of Change

Key Takeaways

- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \frac{f(b) - f(a)}{b-a}$$

- It is the gradient of the line joining the two points.
- **Instantaneous Rate of Change** is a gradient of a graph at a single point.
- **First Principles** derivative definition:

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

- **The Product Rule**
  - The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

  - Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

- **<u>The Quotient Rule</u>**
  - The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

  - Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

  - Always differentiate the top function first.

- **<u>The Chain Rule</u>**

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

- The process for finding derivatives of **composite functions.**

Learning Objective: [2.1.2] - Identify the Nature of Stationary Points and Trends (Strictly Increasing and Decreasing)

Key Takeaways

- Point where the gradient of the function is zero.

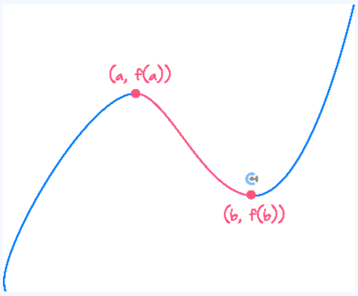
$$f'(x) = 0, \frac{dy}{dx} = 0$$

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighbouring points.

- Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $[a, b]$

- Steps:
  1. Find the TPs.
  2. Consider the sign of the derivative between/outside the turning points.

Learning Objective: [2.1.3] - Graph Derivative Functions

Key Takeaways

Steps on Sketching the Derivative Function:

1. Plot x-intercept at the same x-value as the SP of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is above the x-axis.
   - Original is decreasing → Derivative is below the x-axis.