CONTOUREDUCATION

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VCE Mathematical Methods ¾
Differentiation I [2.1]
Workbook

Outline:

Learning Objectives:

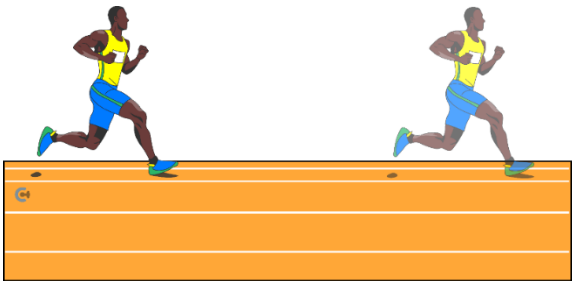
- ☐ **MM34 [2.1.1]** - Find the instantaneous rate of change and average rate of change
- ☐ **MM34 [2.1.2]** - Identify the nature of stationary points and trends (strictly increasing and decreasing)
- ☐ **MM34 [2.1.3]** - Graph Derivative Functions

Section A: Introduction to Differentiation

Sub-Section: Average Rate of Change

Context: Average Rate of Change

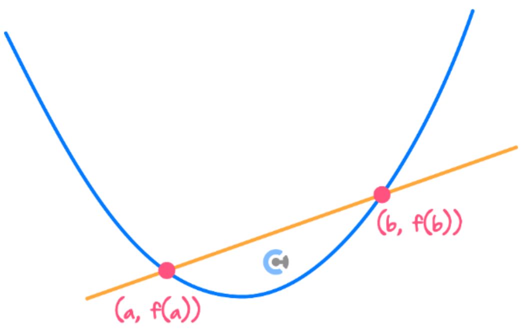
- Usain Bolt ran the world record $100\ m$ race in 9.58 seconds.
- Let's say we take a photo of Usain Bolt at the start and end of the race.



- How would the average speed between two instances be calculated?

$$\textit{Average Speed} = \frac{\textit{Change in } \underline{\hspace{4cm}} \textit{ from start to end}}{\textit{Change in } \underline{\hspace{4cm}}}$$

Average Rate of Change



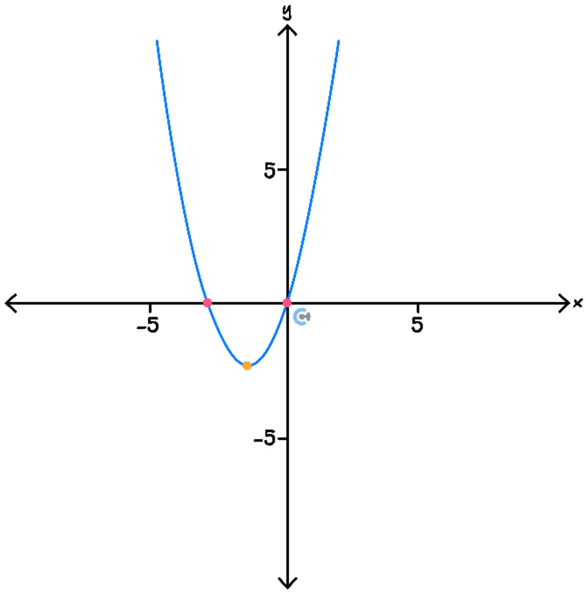
- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \underline{\hspace{6cm}}$$

- It is the ____________________ of the line joining the two points.

Question 1

Find the average rate of change of $y = x^2 + 3x$ over the interval $x \in [-1,2]$.



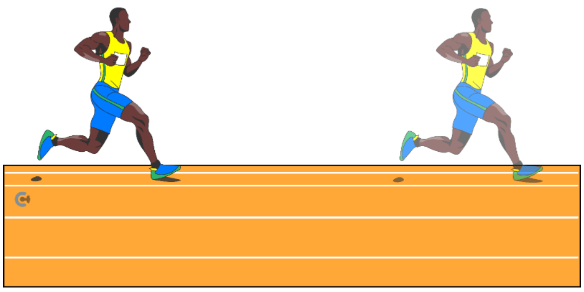
<u>Discussion:</u> Do we differentiate for average rate of change?

Sub-Section: Instantaneous Rate of Change

How can we now find Usain Bolt's speed at a single moment?

Context: Instantaneous rate of change

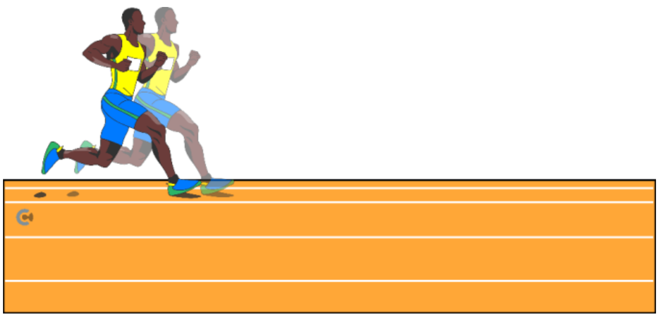
- Usain Bolt ran the world record $100\ m$ race in 9.58 seconds.



- We took two photos of Usain Bolt at the ____________ and ____________ of the race.
- We calculated the average speed over the ________________________ using these two photos.

$$\textit{Average Speed} = \frac{\textit{Distance}}{\textit{Time Taken}}$$

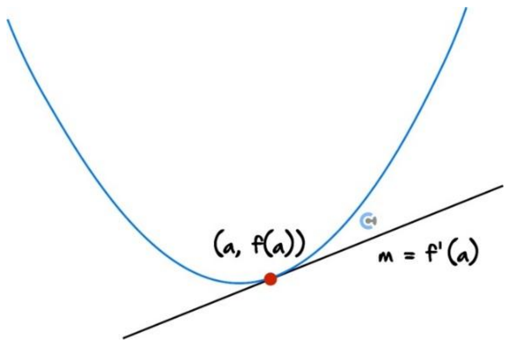
- How can we find his speed at the **start of the race?**
- Where should the two photos be taken? [At the start/At the end/Start and End]



- How closely should the two photos be taken to find the speed at a single moment? [Super Close/Super Far]
- This is the instantaneous rate of change!

Definition

<u>Instantaneous Rate of Change</u>



- **Instantaneous Rate of Change** is a gradient of a graph at a single ______________________.

$$\textit{Instantaneous Rate of Change} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- **Differentiation** is the process of finding the derivative of a function.

Question 2

Consider the function $f(x)$ and its derivative $f'(x)$. It is known that $f(2) = 4$, $f(3) = 9$, $f'(2) = 3$ and $f'(3) = -4$.

Find the gradient of the function $f(x)$ at $x = 3$.

Definition

<u>Alternative Notation for Derivative</u>

$$f'(x) = \frac{dy}{dx}$$

<u>Discussion:</u> How does the notation $\frac{dy}{dx}$ make sense?

Sub-Section: Differentiation

How do we find derivative functions?

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	
$\sin(x)$	
$\cos(x)$	
$\tan(x)$	
e^x	
$\log_e(x)$	

Question 3

Consider the function $f(x) = x^3 - 4x$.

Find the gradient of the function at $x = 2$.

Question 4

Consider the function $f(x) = 2e^x - 4$.

Find the gradient of the function at $x = 3$.

Question 5

Consider the function $f(x) = 2\log_e(x)$.

Find the gradient of the function at $x = 2e$.

Question 6

Consider the function $f(x) = \cos(x) + \sin(x)$.

Find the gradient of the function at $x = \frac{\pi}{4}$.

Calculator Commands: Finding Derivatives

- **Mathematica**

```
f'[x]
```

- **TI**
  - Shift Minus

$$\frac{d}{dx}(f(x))$$

- **Casio**
  - Math 2

$$\frac{d}{dx}(f(x))$$

Question 7 Tech-Active.

Consider the function $f(x) = \tan(x)$.

Find the gradient of the function at $x = \frac{\pi}{6}$.

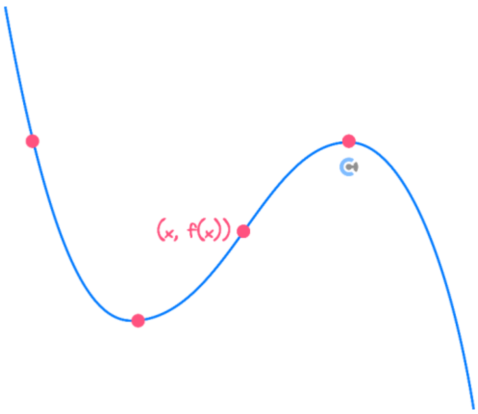
NOTE: You must substitute the x-value **after** finding the derivative function first.

<u>Discussion:</u> What would happen if you derived $f\left(\frac{\pi}{6}\right)$ instead of CAS?

Sub-Section: First Principle

Where do all the derivative rules come from?

First Principle



$$f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- The fundamental method of ____________________.

Exploration: Visualisation of First Principle

- Desmos link

Question 8

Consider the function $f(x) = x^3$.

Find the derivative using the first principle.

NOTE: It's the same as the table above!

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Section B: Advanced Differentiation

Sub-Section: Product Rule

How do we find the derivative when two functions are multiplied?
For example: $x^2 \sin(x)$.

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = \_\_\_\_\_\_\_\_\_\_$$

NOTE: We never differentiate two functions at once!

Question 9 Walkthrough.

Find the derivative of $f(x) = x^3 \tan(x)$.

NOTE: We **never** differentiate **both** functions at the same time!

Your turn!

Question 10

Find the derivatives of:

a. $f(x) = x^2 e^x$

b. $y = 3\sin(x)\cos(x)$

c. $g(x) = \log_e(x) \cdot x$

Question 11 Extension.

Find the derivative of $f(x) = x^3 \log_e(x) \sin(x)$.

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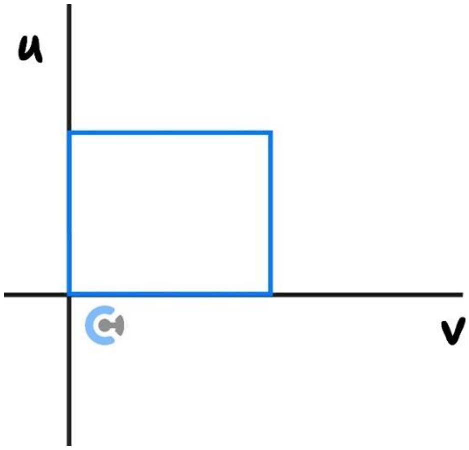
How does this work?

Extension: Understanding Product Rule

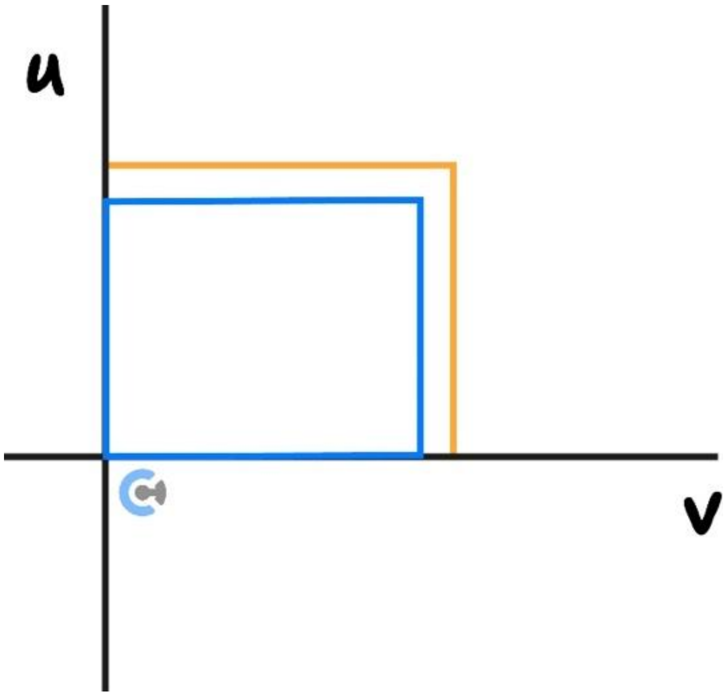
- Consider the rectangle in the diagram below.



- What would the area of the rectangle be?

$$\textit{Area of the rectangle} = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Now let's say the rectangle grew in size!
- Let's take a very close photo of this rectangle while it is growing (similar to Usain Bolt)



- How can we find the instantaneous **change of** $\boldsymbol{u} \cdot \boldsymbol{v}$?

$$\boldsymbol{d(u \cdot v)} = \underline{\hspace{4cm}}$$

- Divide all sides by dx. What do you notice?

$$\frac{\boldsymbol{d(u \cdot v)}}{\boldsymbol{dx}} = \underline{\hspace{4cm}}$$

  - This is a product rule!

<u>Discussion:</u> What shape do we use for proving product rule for three functions?

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Sub-Section: Quotient Rule

The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \rule{6cm}{0.4pt}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \rule{4cm}{0.4pt}$$

Always differentiate the top function first.

Question 12 Walkthrough.

Find the derivative of $y = \frac{x^2}{\sin(x)}$.

NOTE: The order **matters** for the quotient rule! We differentiate the **top function first**.

Question 13

Find the derivatives of:

a. $\frac{e^x}{4x^3}$

b. $\frac{\log_e(x)}{x}$

c. $g(x) = \frac{\sin(x)}{\cos(x)}$

NOTE: The last question is a derivative of tan.

Question 14 Extension.

Find the derivative of $y = \frac{x^2 e^x}{\log_e(x)}$.

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How does the quotient rule work? (Extended)

Exploration: Proving Quotient Rule

- Consider the rule:

$$y = \frac{u}{v}$$

- Instead of using the quotient rule, we can cross multiply and use product!

$$\_\_\_\_\_\_ = u$$

$$\_\_\_\_\_\_ = u'$$

- What happens now when we make y' the subject?

$$\_\_\_\_\_\_ = u' - \_\_\_\_\_\_$$

$$\_\_\_\_ = \_\_\_\_\_\_\_\_$$

- Now substitute $y = \frac{u}{v}$ back again!

$$\_\_\_\_ = \_\_\_\_\_\_\_\_$$

$$\_\_\_\_ = \_\_\_\_\_\_\_\_$$

- What does this look like?

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Sub-Section: Chain Rule

The Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- The process for finding derivatives of **composite functions.**

How does the chain rule work?

Exploration: Understanding chain rule.

- Consider the function we want to differentiate with respect to x is:

$$y = f(g(x))$$

- We can remove the composition by letting the inside function equal to u.

$$\text{Let } u = \_\_\_\_\_\_\_\_$$

$$\text{Then } y = \_\_\_\_\_\_\_\_$$

- We can now derive y ____________________.

$$\frac{dy}{du} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Note that we have $\frac{dy}{du}$ instead of $\frac{dy}{dx}$ as we derived in terms of u.

- To find $\frac{dy}{dx}$, we simply multiply by $\frac{dy}{du}$ with $\frac{du}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Finally, we can substitute $u = g(x)$.

$$\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Question 15 Walkthrough.

Find the derivative of $f(x) = \sin(x^2)$.

NOTE: Always let the inside function equal to u.

Your turn!

Question 16

Find the derivatives of:

a. $e^{x^2+\frac{1}{2}x}$

b. $\left(4x+\frac{1}{x}\right)^3$

c. $\log_e(x^2-4)$

Question 17 Extension.

Find the derivative of $f(x) = x^3 \log_e(x^2)\sin^2(x)$.

Is there a quicker way to do a chain rule?

Definition

Shortcut for Chain Rule

$$y = f(g(x))$$

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

- Derive the outside function only.
- Multiply the function by the derivative of the inside.

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Question 18 Walkthrough.

Using the quick method of the chain rule, find the derivative of $f(x) = \cos(x^3)$.

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Your turn!

Question 19

Using the quick method of chain rule, find the derivative of:

a. e^{3x^2-x}

b. $\log_e(x^2+9x+6)$

c. $g(x)=\tan(x^2)$

Question 20 Extension.

Find the derivative of $g(t) = \log_e\left(\cos\left(\sqrt{t+1}\right)\right)$.

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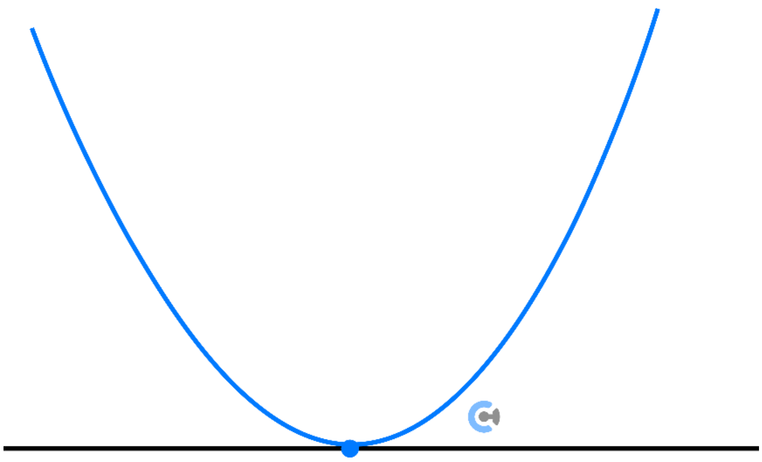
Section C: Stationary Points and Strictly Increasing

Sub-Section: Stationary Points

What would be the gradient of a point that is neither increasing nor decreasing?

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

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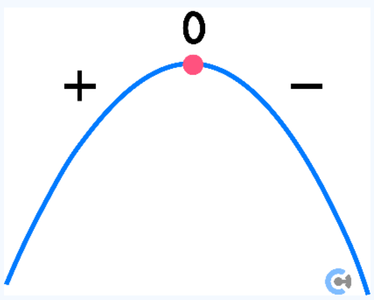
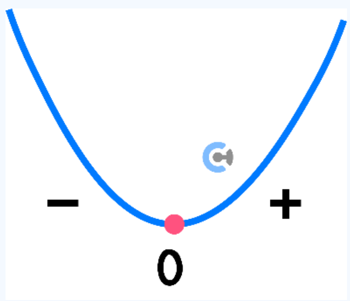
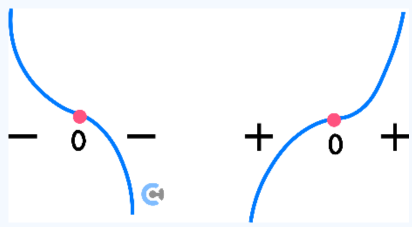
What are the types of stationary points?

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

Sign Test

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the ____________________ points.

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Question 21 Walkthrough.

Find and identify the nature of the stationary points of $y = -e^{x^2+4}$.

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Question 22

Find and identify the nature of the stationary points of $y = \log_e(x^2 + 4)$.

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Question 23 Extension.

Consider the function $f(x) = xg(x)$.

It is known that, $g(0) = -5$, $g(1) = -2$ and $g(2) = 1$.

$g'(0) = -4$, $g'(1) = 2$ and $g'(2) = 3$ and that f has only one stationary point.

Show that $f(x)$ has a stationary point when $x = 1$ and identify its nature.

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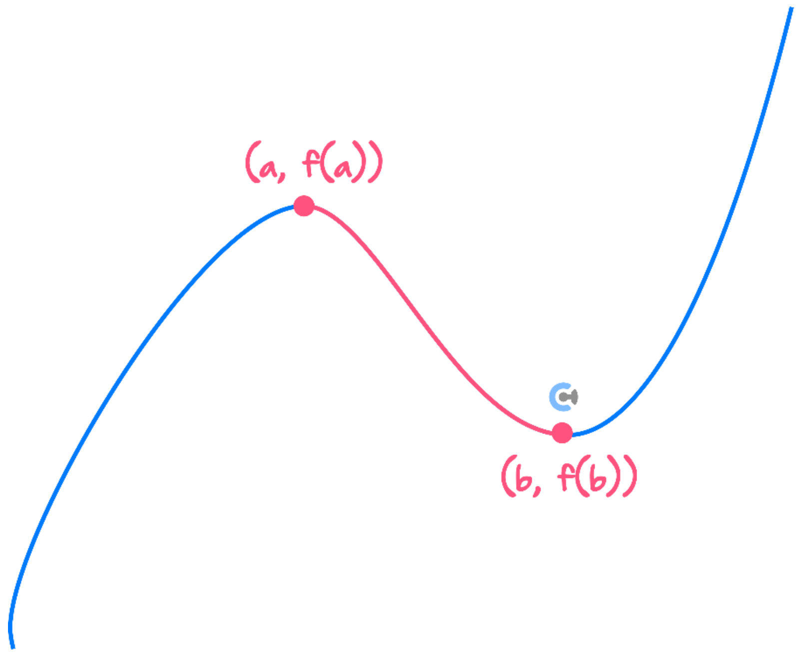
Sub-Section: Strictly Increasing and Decreasing

Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:

1. Find the turning points.
2. Consider the sign of the derivative between/outside the turning points.

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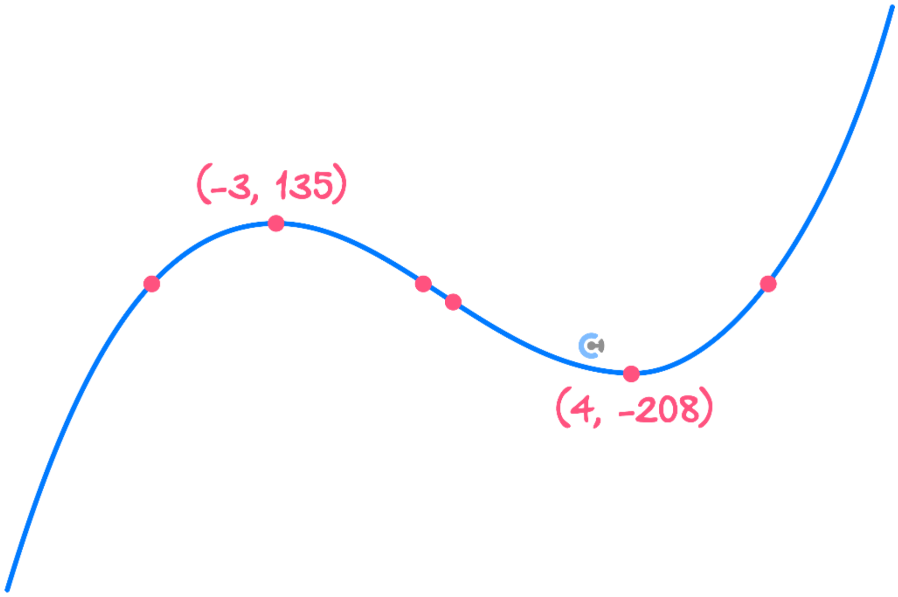
Question 24

State the value(s) of x for the function below for which it is strictly increasing and strictly decreasing.

$$y = -72x - 3x^2 + 2x^3$$



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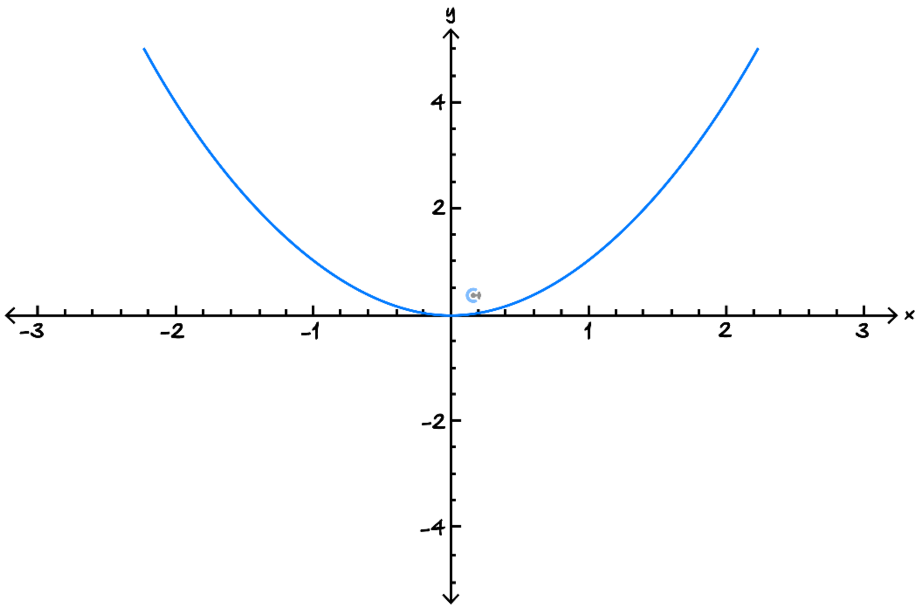
Section D: Graphs of the Derivative Function

Sub-Section: Graphs of Derivative Function

Exploration: Graph of derivative functions

- Consider the graph of $f(x) = x^2$ below.



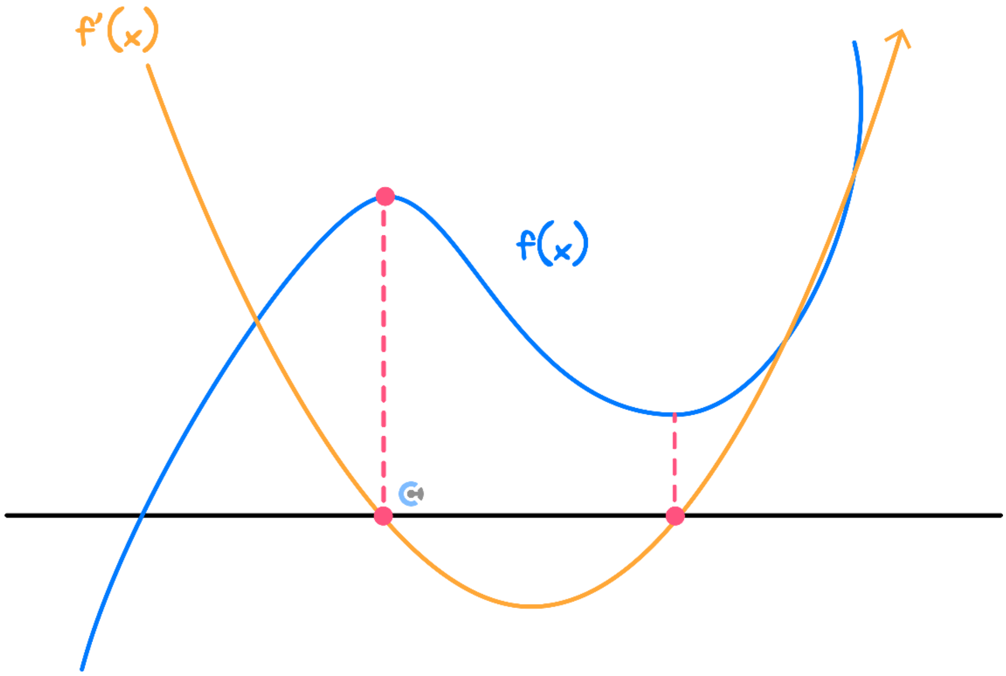
- What is the derivative of $f(x)$?
- Sketch the derivative above.

$$f'(x) = \_\_\_\_\_\_\_$$

- What do you notice about $f'(x)$: Derivative when $f(x)$ has a stationary point?
- What do you notice about $f'(x)$ when $f(x)$ is increasing?
- What do you notice about $f'(x)$ when $f(x)$ is decreasing?

In summary!

Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	
Increasing	
Decreasing	

$$y\ value\ of\ f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- **Steps**

1. Plot x-intercept at the same x value as the stationary point of the original.

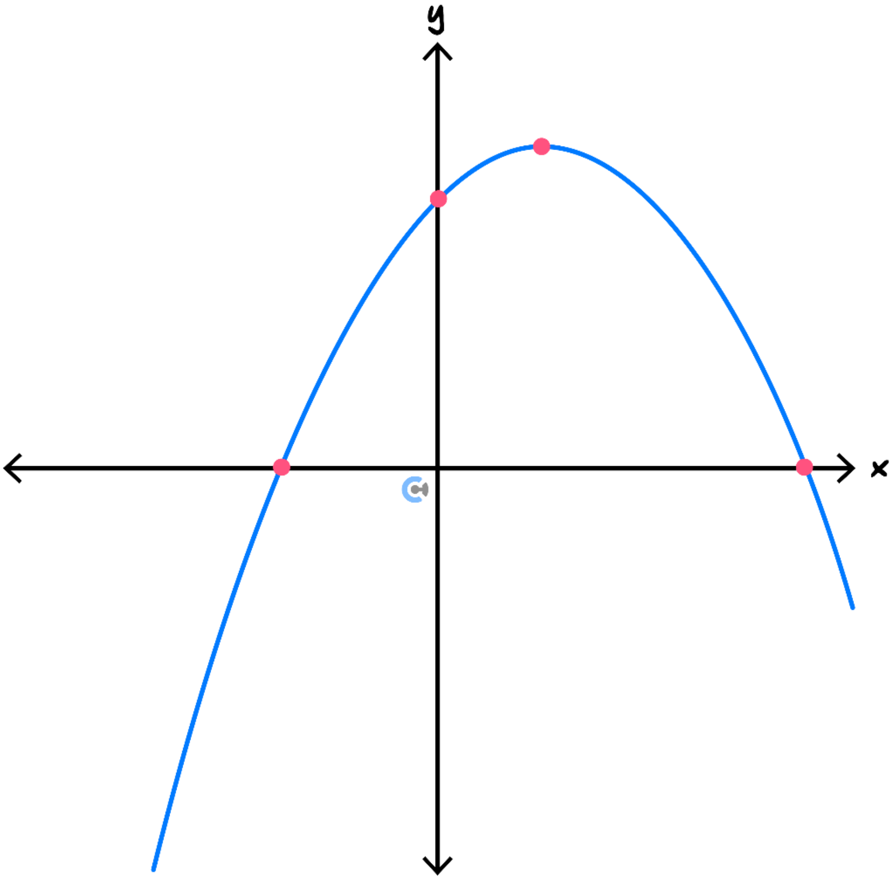
2. Consider the trend of the original function and sketch the derivative.

   - Original is increasing → Derivative is above the x-axis.

   - Original is decreasing → Derivative is below the x-axis.

Question 25 Walkthrough.

Sketch the derivative graph of the function shown below, on the same set of axes.



Active Recall: Steps on sketching the derivative function.

1. Plot x-intercept at the same x-value as the ________________________ of the original.

2. Consider the trend of the original function and sketch the derivative.

   - Original is increasing → Derivative is ______________ the x-axis.
   - Original is decreasing → Derivative is _____________ the x-axis.

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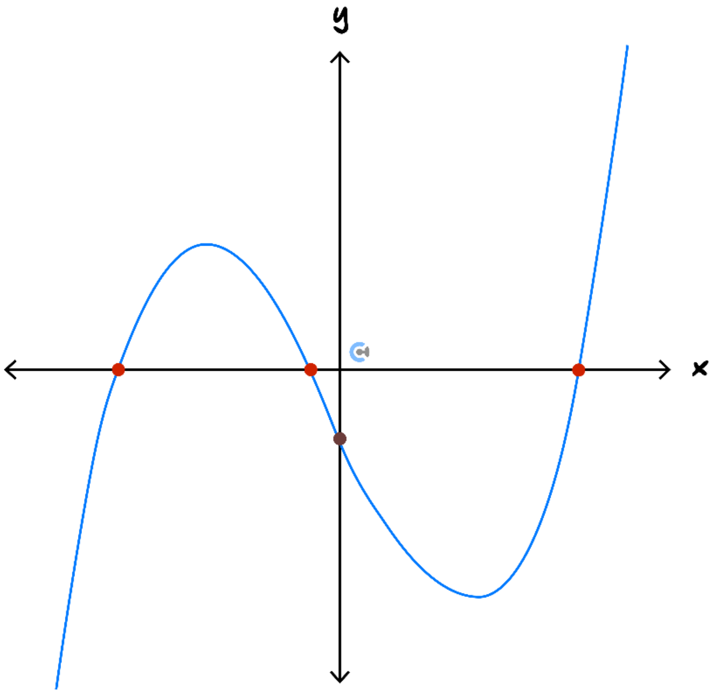
Question 26

Sketch the derivative graph of the function shown below, on the same set of axes.



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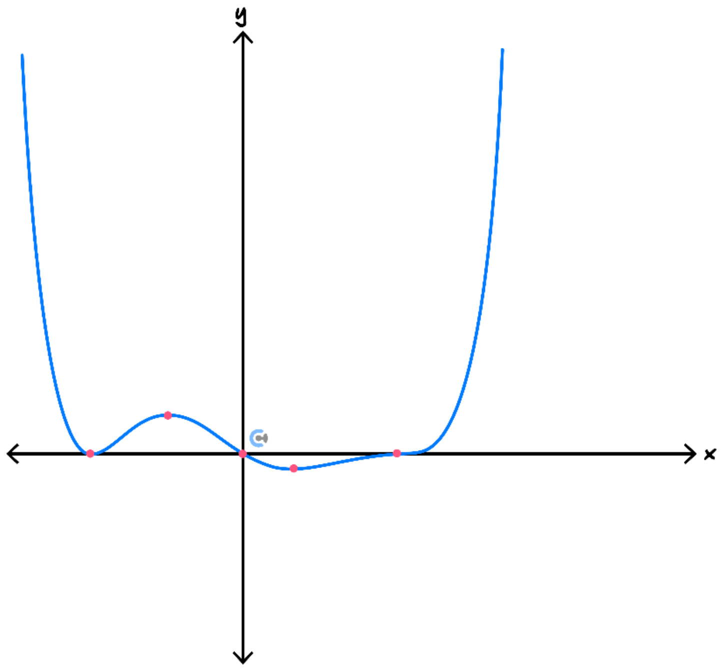
Question 27 Extension.

Sketch the derivative graph of the function shown below, on the same set of axes.



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Contour Checklist

☐ Learning Objective: [2.1.1] – Find the Instantaneous Rate of Change and Average Rate of Change

Key Takeaways

- ☐ The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- ☐ It is the ____________________of the line joining the two points.

- ☐ **Instantaneous Rate of Change** is a gradient of a graph at a single __________________________.

- ☐ First Principles derivative definition:

$$f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- ☐ The Product Rule

  - The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

  - Or, in another form:

$$\frac{d}{dx}(u \cdot v) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- **The Quotient Rule**
  - The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \rule{6cm}{0.4pt}$$

  - Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \rule{4cm}{0.4pt}$$

  - Always differentiate the top function first.

- **The Chain Rule**

$$y = f(g(x))$$

$$\frac{dy}{dx} = \rule{5cm}{0.4pt}$$

- The process for finding derivatives of **composite functions.**

Learning Objective: [2.1.2] - Identify the Nature of Stationary Points and Trends (Strictly Increasing and Decreasing)

Key Takeaways

- Point where the ____________________ of the function is zero.

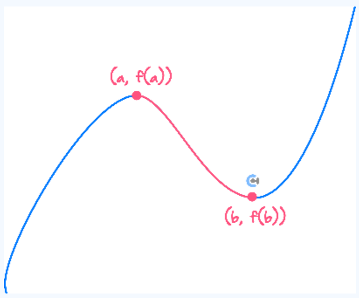
$$f'(x) = 0, \frac{dy}{dx} = 0$$

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the ___________________ points.

- Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in$ _________________________

Strictly Decreasing: ___________________

- Steps:

  1. Find the _____________________.
  2. Consider the sign of the _____________________ between/outside the turning points.

Learning Objective: [2.1.3] - Graph Derivative Functions

Key Takeaways

Steps on Sketching the Derivative Function:

1. Plot x-intercept at the same x-value as the ____________________ of the original.
2. Consider the trend of the original function and sketch the derivative.
   - Original is increasing → Derivative is ____________ the x-axis.
   - Original is decreasing → Derivative is ____________ the x-axis.