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VCE Mathematical Methods ¾
Differentiation I [2.1]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 16
Supplementary Questions	Pg 17 – Pg 29

Section A: Compulsory Questions

Sub-Section [2.1.1]: Find Instantaneous Rate of Change and Average Rate of Change

Question 1

Consider the function $f(x) = x^3 - x^2$.

a. Find the average rate of change of f over the interval $x \in [-2, 2]$.

$$\frac{f(2) - f(-2)}{4} = 4$$

b. Find $f'(x)$.

$$f'(x) = 3x^2 - 2x$$

c. Find the gradient of f when $x = 2$.

$$f'(2) = 12 - 4 = 8$$

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Question 2

a. Let $f(x) = x^2 \sin(3x) + \cos(x^2)$. Find $f'(x)$.

$$f'(x) = 2x\sin(3x) + 3x^2\cos(3x) - 2x\sin(x^2)$$

b. Let $f(x) = \log_e(x)\, e^{x^2}$. Find $f'(1)$.

$$f'(x) = \frac{e^{x^2}}{x} + 2xe^{x^2}\log_e(x)$$
$$f'(1) = e$$

c. Let $f(x) = \frac{x^3+3x}{x^2}$. Find the values of x for which the gradient is -5.

We find $f'(x) = \dfrac{x^2-3}{x^2}$

We solve $\dfrac{x^2-3}{x^2} = -5 \implies x = \pm\dfrac{1}{\sqrt{2}}$

d. Let $f(x) = \frac{\log_e(x^2+3)}{x}$. Find $f'(x)$.

$$f'(x) = \frac{\frac{2x}{x^2+3}x - \log_e(x^2+3)}{x^2} = \frac{2}{x^2+3} - \frac{\log_e(x^2+3)}{x^2} = \frac{2x^2 - (x^2+3)\log_e(x^2+3)}{x^2(x^2+3)}$$

e. Let $f(x) = x^2 e^x$. Find the values of x for which the gradient is $3e$.

We find $f'(x) = x^2e^x + 2xe^x = e^x(x^2+2)$

We solve $e^x(x^2+2) = 3e$ to get $x = 1$ by recognition.

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Question 3

a. Let $f(x) = \frac{\sin(3x)e^{2x}}{\sqrt{x}}$. Find $f'(x)$.

$$f'(x) = \frac{\sqrt{x}(2e^{2x}\sin(3x) + 3e^{2x}\cos(3x)) - \frac{1}{2}x^{-1/2}\sin(3x)e^{2x}}{x}$$
$$= \frac{e^{2x}(4x\sin(3x) - \sin(3x) + 6x\cos(3x))}{2x^{3/2}}$$

b. Let $f(x) = \sqrt{e^{2x} - \cos(x)}$. Find $f'(\pi)$.

$$f(x) = \sqrt{e^{2x} - cos(x)}$$

$$f'(x) = \frac{2e^{2x} + sin(x)}{2\sqrt{(e^{2x} - cos(x))}}$$

$$f'(\pi) = \frac{e^{2\pi}}{\sqrt{e^{2\pi} + 1}}$$

c. Let $f(x) = \frac{x^2 e^x}{x^2+1}$. Find all x values where f has a stationary point.

We find that $f'(x) = \dfrac{xe^x(x^3+x+2)}{(1+x^2)^2}$.
To find stationary points we solve $f'(x) = 0$. Clearly one stationary point occurs when $x = 0$. The other stationary point may be found by solving

$$\begin{aligned} x^3 + x + 2 &= 0 \\ \implies x &= -1 \end{aligned}$$

so stationary points when $x = -1$ and $x = 0$

d. Let $f(x) = \frac{\sin^3(x) + \sin(x)\cos^2(x)}{\cos(x)}$. Show that $f'(x) = \frac{1}{\cos^2(x)}$.

$$\begin{aligned} f'(x) &= \frac{3\cos^2(x)\sin^2(x) + \cos^4(x) - 2\cos^2(x)\sin^2(x) + \sin^4(x) + \sin^2(x)\cos^2(x)}{\cos^2(x)} \\ &= \frac{2\sin^2(x)\cos^2(x) + \cos^4(x) + \sin^4(x)}{\cos^2(x)} \\ &= 2\sin^2(x) + \cos^2(x) + \frac{(1-\cos^2(x))^2}{\cos^2(x)} \\ &= 1 + \sin^2(x) + \frac{\cos^4(x) - 2\cos^2(x) + 1}{\cos^2(x)} \\ &= 1 + \sin^2(x) + \cos^2(x) - 2 + \frac{1}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \end{aligned}$$

e. Let $f(x) = (x^3 + 3)(x^5 + 2)^5$. Find $f'(-1)$.

$f'(x) = 3x^2(x^5 + 2)^5 + 25x^4(x^3 + 3)(x^5 + 2)^4$.
$f'(-1) = 3(1) + 25(2)(-1 + 2)^4 = 53$

Question 4 Tech-Active.

Let $f(x) = \frac{1}{3}x^3 + x^2 - x + 3$. Find when f has a gradient of 3.

We define $f(x)$ on our cas and solve $f'(x) = 3 \implies x = -1 \pm \sqrt{5}$.

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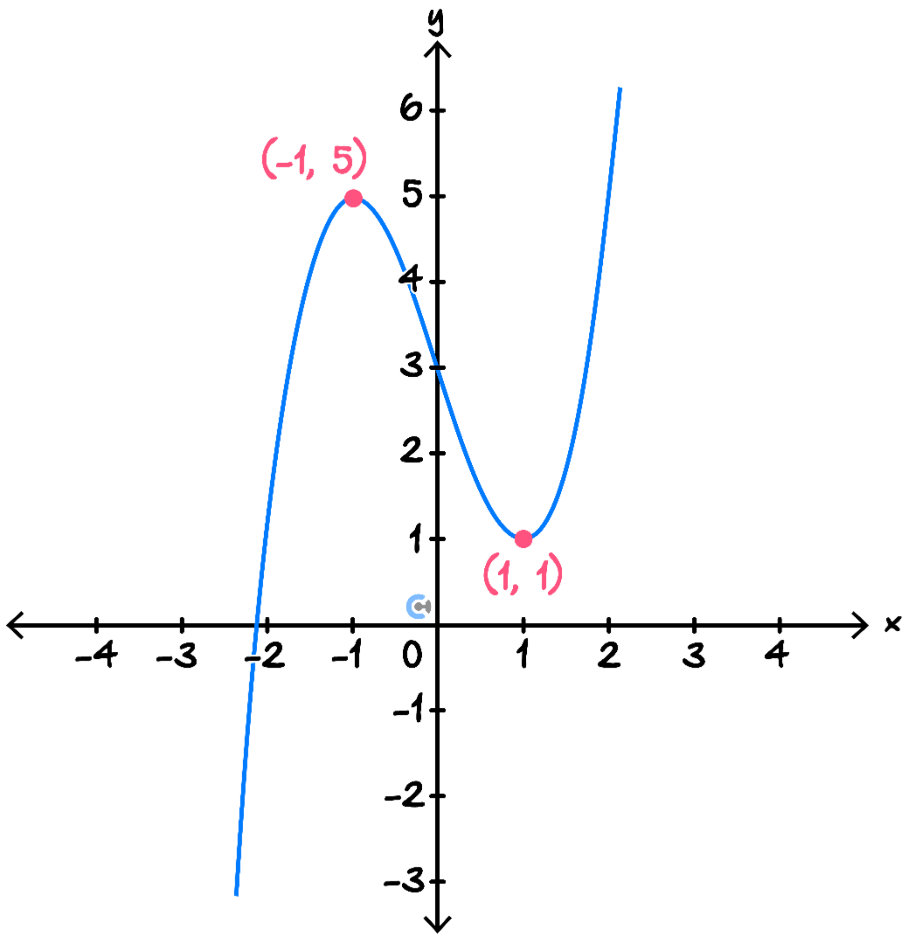
Sub-Section [2.1.2]: Identify the Nature of Stationary Points and Trend (Strictly Increasing and Decreasing)

Question 5

Consider the graph of f shown below.



a. State the nature of the stationary point when $x = 1$.

Local minimum

b. State the values of x for which $f(x)$ is strictly increasing.

$x \in (-\infty, -1] \cup [1, \infty)$

Question 6

Let $f(x) = x^3 - 4x^2 - 3x + 19$.

a. Find the stationary points of f.

$f'(x) = 3x^2 - 8x - 3$. Use the quadratic formula to solve $f'(x) = 0$

$$x = \frac{8 \pm \sqrt{64+36}}{6} = \frac{8 \pm 10}{6} \quad (1)$$

$\implies x = -\frac{1}{3}, 3$

$f\left(-\frac{1}{3}\right) = -\frac{1}{27} - \frac{4}{9} + 1 + 19 = \frac{14}{27} + 19$

$f(3) = 27 - 36 - 9 + 19 = 1$

Stationary points are $\left(-\frac{1}{3}, \frac{527}{27}\right)$ and $(3, 1)$

b. State the nature of the stationary points of $f'(x)$.

Since $f(x)$ is a positive cubic the first stationary point is a local max and the second a local min.

Therefore $\left(-\frac{1}{3}, \frac{527}{27}\right)$ is a local maximum

and $(3, 1)$ is a local minimum.

c. Hence, state the values of x for which $f(x)$ is decreasing.

$-\frac{1}{3} < x < 3$

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Question 7

Let $f(x) = xe^{-x^2+x+2}$.

a. Find the stationary points of f.

$f'(x) = e^{-x^2+x+2} + x(-2x+1)e^{-x^2+x+2} = e^{-x^2+x+2}(-2x^2+x+1)$

Stationary points when $-2x^2+x+1=0$ since $e^{-x^2+x+2} > 0$

$$x = \frac{-1 \pm \sqrt{1+8}}{-4} = \frac{-1 \pm 3}{-4}$$

$\implies x = -\frac{1}{2}, 1$

Stationary points are $\left(-\frac{1}{2}, -\frac{1}{2}e^{5/4}\right)$ and $(1, e^2)$

b. State the nature of the stationary points of $f'(x)$.

We have $f'(-1) = -2$, $f'(0) = e^2$ and $f'(2) = -5$.

From this we conclude that

$$\left(-\frac{1}{2}, -\frac{1}{2}e^{5/4}\right) \text{ is a local minimum}$$

$$(1, e^2) \text{ is a local maximum}$$

c. Hence, state the values of x for which $f(x)$ is strictly increasing.

$$-\frac{1}{2} \le x \le 1$$

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Question 8 Tech-Active.

Let $f(x) = \frac{x^5}{5} - x^4 - \frac{7x^3}{3} + 17x^2 - 24x$.

Find the all values of x for which f is strictly increasing.

We use CAS to solve $f'(x) = 0 \implies x = -3, 1, 2, 4$. Then checking a graph we find Local max when $x = -3$, local min when $x = 1$, local max when $x = 2$ and local min when $x = 4$.

We conclude that f is strictly increasing for

$$x \in (-\infty, -3] \cup [1, 2] \cup [4, \infty)$$

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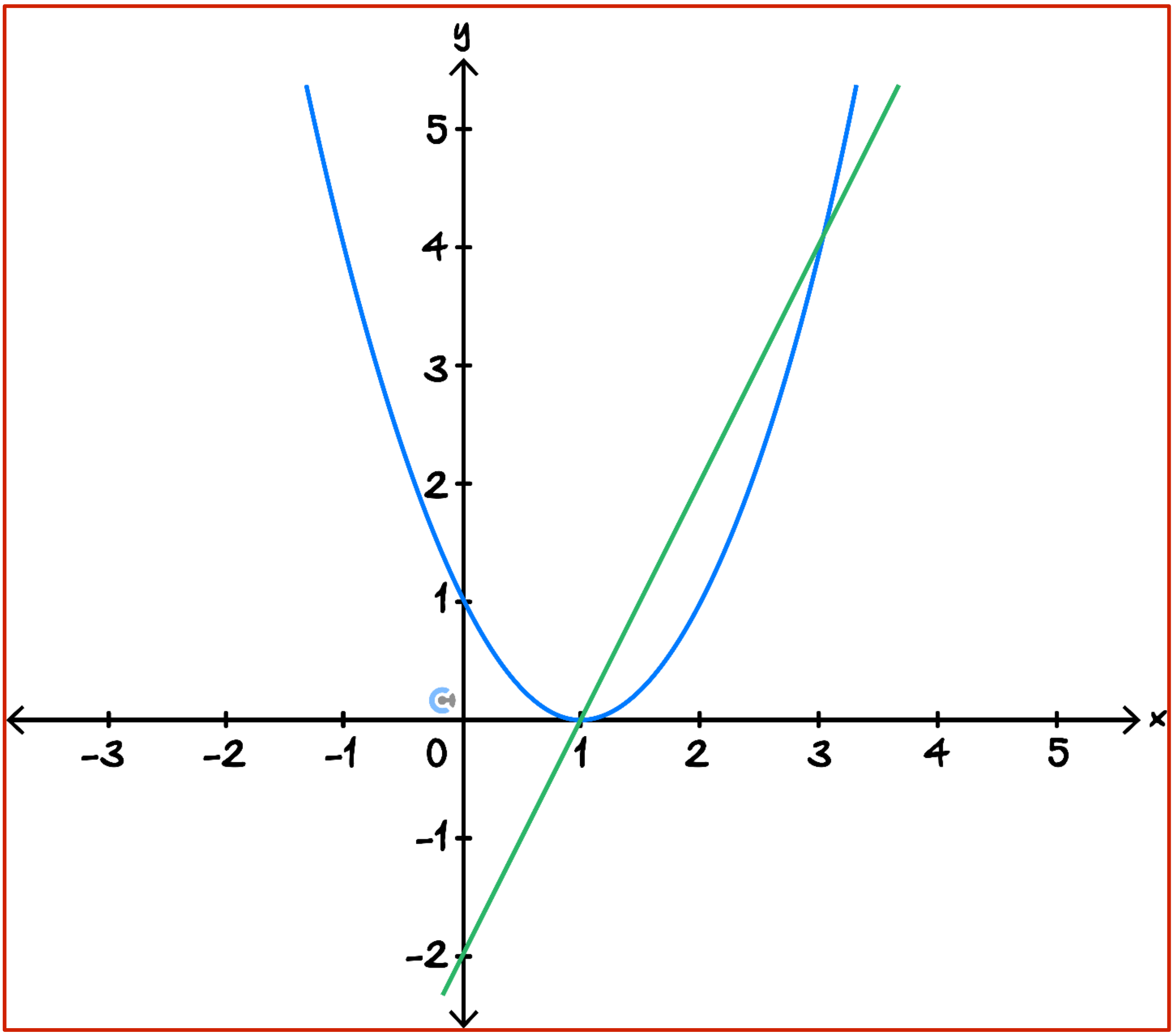
Sub-Section [2.1.3]: Graph Derivative Functions

Question 9

The graph of $f(x)$ is drawn below.

Sketch the graph of $f'(x)$ on the same axes.



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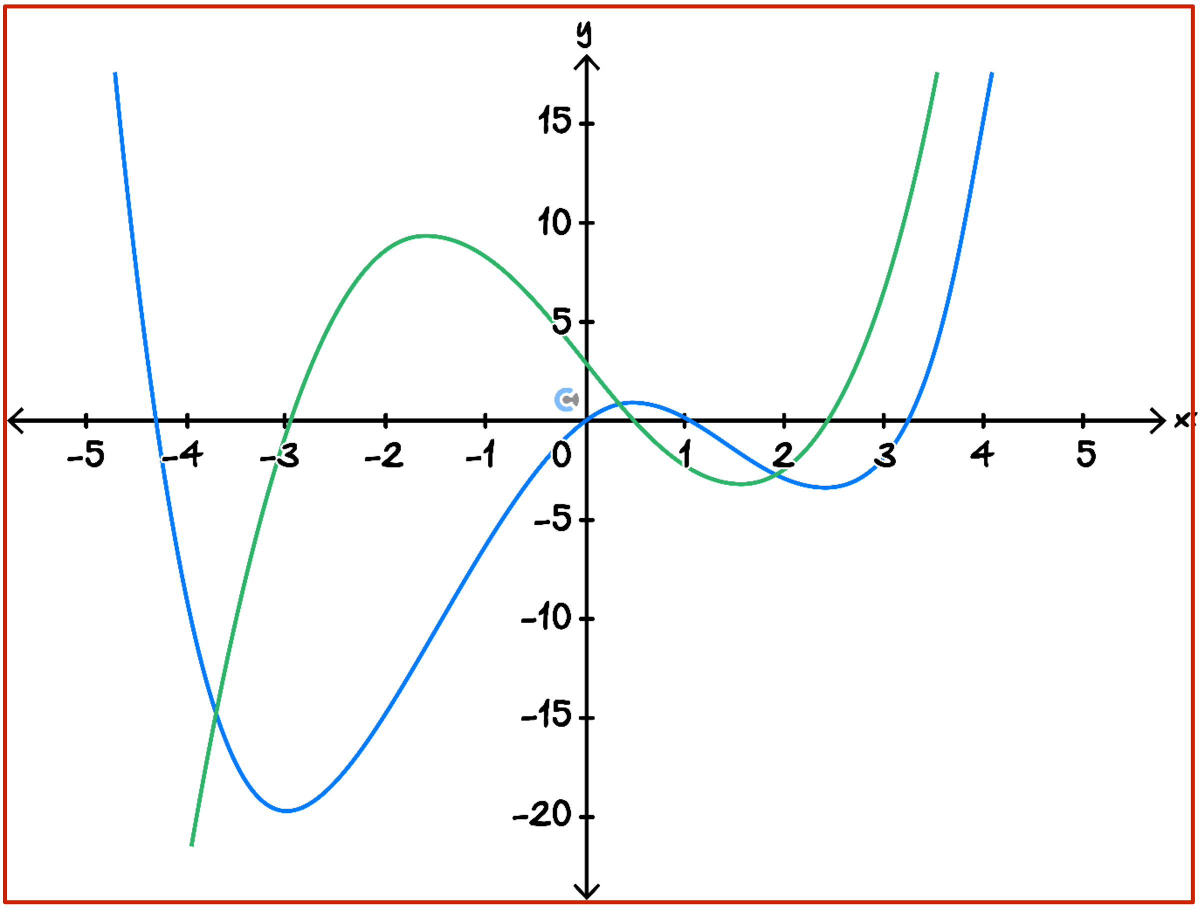
Question 10

The graph of $f(x)$ is drawn below.

Sketch the graph of $f'(x)$ on the same axes.



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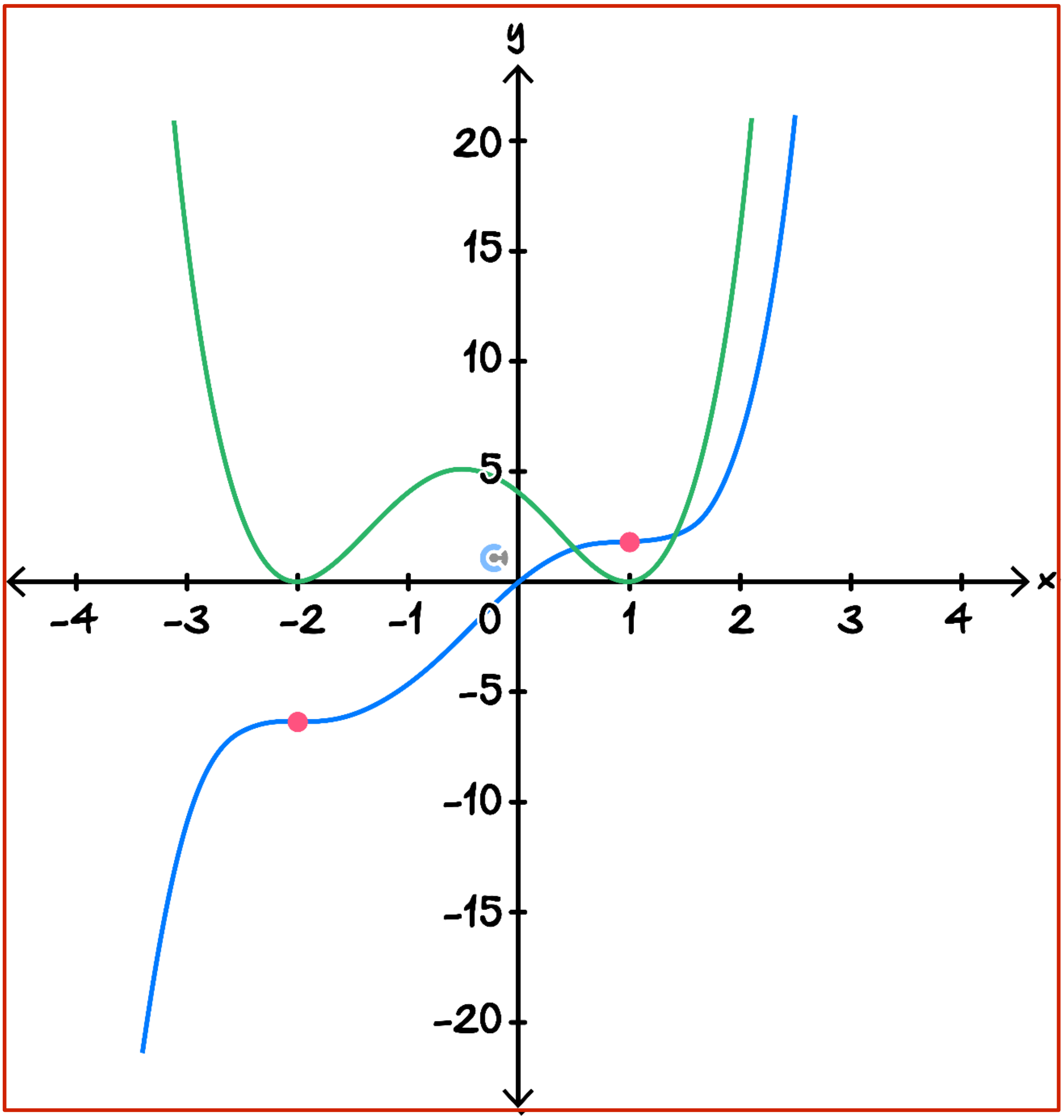
Question 11

The graph of $f(x)$ is drawn below.

Sketch the graph of $f'(x)$ on the same axes.



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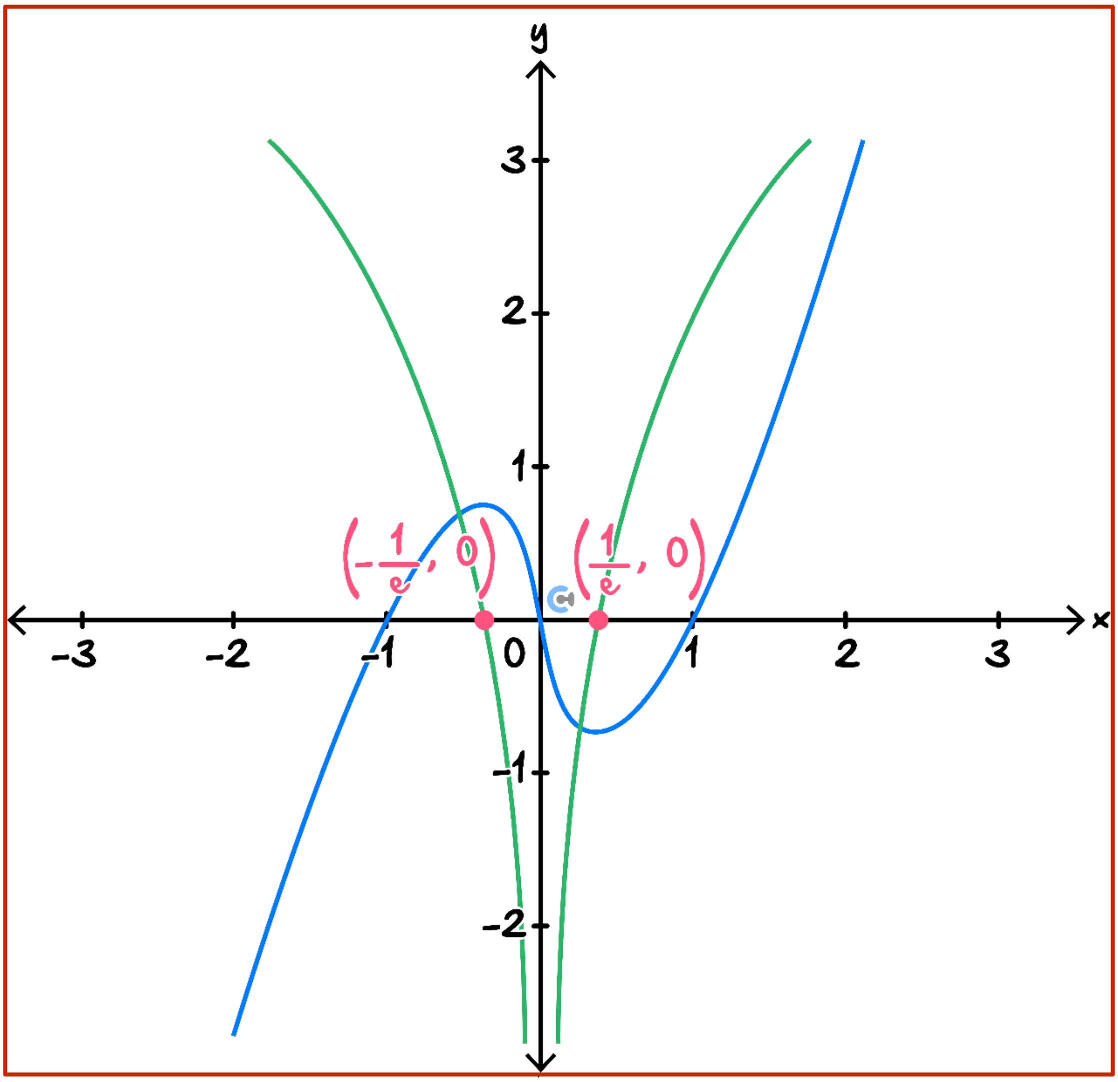
Sub-Section: The 'Final Boss'

Question 12

Let $f(x) = x\log_e(x^2)$. The graph of f is shown on the axes below.



a. Find $f'(x)$.

$$f'(x) = \log_e(x^2) + x\frac{2x}{x^2} = \log_e(x^2) + 2$$

b. Find all stationary points of f and state their nature.

Stationary points occur when $f'(x) = 0$. Solve

$$\log_e(x^2) + 2 = 0$$
$$\log_e(x^2) = -2$$
$$\implies x^2 = e^{-2} = \frac{1}{e^2}$$
$$x = \pm\frac{1}{e}$$

Now $f(e^{-1}) = e^{-1}\log_e(e^{-2}) = -\frac{2}{e}$ and $f(-e^{-1}) = -e^{-1}(-2) = \frac{2}{e}$
Further note (if we were not given a graph) $f(-1) = f(0) = f(1) = 0$ so we conclude that

$$\left(-\frac{1}{e}, \frac{2}{e}\right) \text{ is a local maximum}$$
$$\left(\frac{1}{e}, -\frac{2}{e}\right) \text{ is a local minimum}$$

c. State the values of x for which $f(x)$ is strictly decreasing.

$$-\frac{1}{e} \leq x \leq \frac{1}{e}$$

d. Sketch the graph of $f'(x)$ alongside the graph of f at the start of the question. Label any axes intercepts.

e. Let $g(x) = \cos(x)$. Find $f'(g(x))$.

$$\begin{aligned} f'(g(x)) &= g'(x)f'(g(x)) \\ &= -\sin(x)\left(\log_e(\cos^2(x)) + 2\right) \\ &= -2\sin(x) - \sin(x)\log_e(\cos^2(x)) \end{aligned}$$

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Section B: Supplementary Questions

Sub-Section [2.1.1]: Find Instantaneous Rate of Change and Average Rate of Change

Question 13

a. Find the average rate of change of $f(x) = x^3 + 3x - 2$ over the interval $[0, 2]$.

$$\text{Average rate of change} = \frac{f(2) - f(0)}{2 - 0} = \frac{8 + 6 - 2 + 2}{2} = 7.$$

b. Let $f(x) = \sqrt{x} - e^x$. Find $f'(x)$.

$$f'(x) = \frac{1}{2\sqrt{x}} - e^x.$$

c. Find the gradient of the graph of $y = \sin(x) + 3\cos(x)$ at the point $\left(\frac{\pi}{3}, \frac{3+\sqrt{3}}{2}\right)$.

$$\text{Gradient} = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \cos(x) - 3\sin(x)\Big|_{x=\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) - 3\sin\left(\frac{\pi}{3}\right) = \frac{1 - 3\sqrt{3}}{2}$$

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Question 14

a. Let $y = \tan(x)$, use the quotient rule to show that $\frac{dy}{dx} = \frac{1}{\cos^2(x)}$.

As $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we see that,

$$\frac{dy}{dx} = \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

b. Find the gradient of $y = \sqrt{4 - x^2}$ at the point $(-1, \sqrt{3})$.

$\frac{dy}{dx} = \frac{-x}{\sqrt{4 - x^2}}$.

Thus when $x = -1$, $\frac{dy}{dx} = \frac{1}{\sqrt{4-1}} = \frac{1}{\sqrt{3}}$

c. Let $f(x) = -x \log_e(x)$. At what point is the gradient of f equal to 2?

$f'(x) = -x\frac{1}{x} - \log_e(x) = -1 - \log_e(x)$.

We solve $f'(x) = 2 \implies \log_e(x) = -3 \implies x = e^{-3}$.

Hence the gradient of f is equal to 2 at the point $(e^{-3}, 3e^{-3})$.

d. Let $f(x) = e^{x^2+2}$, find $f'(x)$.

$f'(x) = 2xe^{x^2+2}$.

e. Let $f(x) = \cos^2(x)$. Find $f'\left(\frac{\pi}{3}\right)$.

$f'(x) = -2\sin(x)\cos(x)$.

Hence $f'(\frac{\pi}{3}) = -2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Question 15

a. Let $y = \frac{e^{-x}}{\sin(2x^2)}$. Find and simplify $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-e^{-x}\sin(2x^2) - 4xe^{-x}\cos(2x^2)}{\sin^2(2x^2)} = \frac{-e^{-x}}{\sin(2x^2)}\left(1 + \frac{4x}{\tan(2x^2)}\right)$$

b. Let $f(x) = (x-3)^4(x^3 - 5x^2 + 1)$. Find $f'(2)$.

$f'(x) = 4(x-3)^3(x^3 - 5x^2 + 1) + (3x^2 - 10x)(x-3)^4$.

Hence $f'(2) = 4(-1)^3(8 - 20 + 1) + (12 - 20)(-1)^4 = -4(-11) + (-8) = 36$

c. Let $f(x) = \sqrt{\sin(4x) + 2}$. Find all values of $x \in [0, \pi]$ such that $f'(x) = 0$.

$$f'(x) = \frac{2\cos(4x)}{\sqrt{2 + \sin(4x)}}.$$

Observe that $f'(x) = 0 \iff \cos(4x) = 0 \implies x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

d. Evaluate $\frac{d}{dx}(\log_e(x)\log_e(x^2 + 3x + 4))$.

$$\frac{d}{dx}(\log_e(x)\log_e(x^2 + 3x + 4)) = \frac{(2x + 3)\log_e(x)}{x^2 + 3x + 4} + \frac{\log_e(x^2 + 3x + 4)}{x}.$$

e. Let $f(x) = \frac{(xe^x)^2}{x-1} + 2x$. Solve $f'(x) = 2$ for x.

```
In[1]:= f[x_] := 2 x + 1 / (x - 1) * (x * E ^ x) ^ 2
        f'[x]
```

Out[2]= $2 + \frac{2e^{2x}x}{-1+x} - \frac{e^{2x}x^2}{(-1+x)^2} + \frac{2e^{2x}x^2}{-1+x}$

Solve$\left[2 + \frac{2e^{2x}x}{-1+x} - \frac{e^{2x}x^2}{(-1+x)^2} + \frac{2e^{2x}x^2}{-1+x} == 2, x\right]$

$\left\{\{x \to 0\}, \left\{x \to \frac{1}{4}(1 - \sqrt{17})\right\}, \left\{x \to \frac{1}{4}(1 + \sqrt{17})\right\}\right\}$

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Question 16

Let $f(x) = \frac{\cos(e^{-x}\log_e(x))}{\sin(e^{-x}\log_e(x))}$.

Show that $f'(a) = 0$ implies that $\frac{1}{a} = \log_e(a)$.

Observe that $f(x) = g \circ h(x)$ where $h(x) = e^x \log_e(x)$ and $g(x) = \dfrac{\cos(x)}{\sin(x)}$.

By the chain rule, $f'(x) = h'(x)g'(h(x))$.

As $h'(x) = e^{-x}\left(\dfrac{1}{x} - \log_e(x)\right)$, we observe that if $h(a) = 0$, then $\dfrac{1}{a} = \log_e(a)$.

It is sufficient to show that $g'(h(x)) \neq 0$ for all x.

Since $g'(x) = \dfrac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = -\dfrac{1}{\sin^2(x)} < 0$ for all x, we see that $g'(h(x)) \neq 0$ for all x.

Hence if $f'(a) = 0$, then $h'(a) = 0$ hence $\dfrac{1}{a} = \log_e(a)$.

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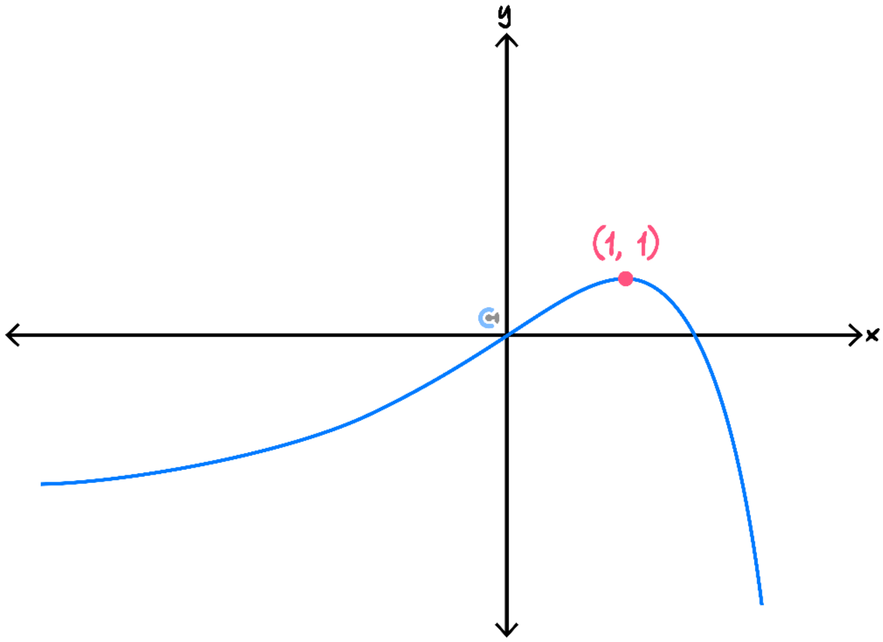
Sub-Section [2.1.2]: Identify the Nature of Stationary Points and Trend

Question 17

The graph of $f(x)$ is drawn below.



a. State the nature of the stationary point when $x = 1$.

Local maximum

b. State the values of x for which $f(x)$ is strictly increasing.

$x \leq 1$

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Question 18

Let $f(x) = 2x^3 + 3x^2 - 12x + 5$.

a. Find the stationary points of f.

$f'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$.
Thus if $f'(x) = 0$ then $x = -2, 1$.
Hence the stationary points of $f(x)$ are $(-2, f(-2)) = (-2, 25)$ and $(1, f(1)) = (1, -2)$.

b. State the nature of the stationary points.

As $f(x)$ is a positive cubic, it's leftmost stationary point $(-2, 25)$ is a local maximum, whilst it's rightmost stationary point $(1, -2)$ is a local minimum.

c. Hence, state the values of x for which $f(x)$ is strictly decreasing.

$-2 \leq x \leq 1$

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Question 19

Let $f(x) = e^{1+4x-3x^2}$.

a. Find the stationary points of $f'(x)$.

$f'(x) = (4-6x)e^{1+4x-3x^2} = g(x)$.
$g'(x) = -6e^{1+4x-3x^2} + (4-6x)^2e^{1+4x-3x^2}$.
If $g'(x) = 0$, then $(4-6x)^2 = 6 \implies x = \dfrac{4 \pm \sqrt{6}}{6}$.
Thus the stationary points of $f'(x)$ are $\left(\dfrac{4-\sqrt{6}}{6}, f'\left(\dfrac{4-\sqrt{6}}{6}\right)\right) = \left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$
and $\left(\dfrac{4+\sqrt{6}}{6}, f'\left(\dfrac{4+\sqrt{6}}{6}\right)\right) = \left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$

b. State the nature of the stationary points of $f'(x)$.

We apply a sign test using 3 observations.

- $\dfrac{4-\sqrt{6}}{6} < \dfrac{2}{3} < \dfrac{4+\sqrt{6}}{6}$, and $f'(\frac{2}{3}) = 0$.
- When $x < \dfrac{3}{2}$ that $f'(x) > 0$ and when $x > \dfrac{3}{2}$ that $f'(x) < 0$.
- When $x \to \pm\infty$, $f'(x) \to 0$.

From these observations we see that the graph of $f'(x)$ graph initially goes up to $\left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$, then comes down through $\left(\dfrac{2}{3}, 0\right)$ to $\left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$, lastly going back up approaching 0.

Hence $\left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$ is a local maximum of the graph of $f'(x)$, and $\left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$ is a local minimum.

c. Hence, state the values of x for which $f'(x)$ is strictly increasing.

$x < \frac{4-\sqrt{6}}{6}$ or $x > \frac{4+\sqrt{6}}{6}$

Question 20

Let $f(x) = x^{\frac{10}{3}}$.

State the values for which $g(x) = f'(x) - f(x)$ is strictly increasing.

Observe that $g(x) = \frac{10}{3}x^{\frac{7}{3}} - x^{\frac{10}{3}}$.

For stationary points we require $g'(x) = \frac{10}{3}\left(\frac{7}{3}x^{\frac{4}{3}} - x^{\frac{7}{3}}\right) = 0$.

Hence $x = 0$ or $x = \frac{7}{3}$.

Since $g(-1) = \frac{-13}{3}$ and $g(1) = \frac{7}{3}$, we see that $(0, 0)$ is a stationary point of inflection.

Since $g(3) = 3^{\frac{4}{3}} < g\left(\frac{7}{3}\right) = \left(\frac{7}{3}\right)^{\frac{7}{3}}$ and $g(0) < g\left(\frac{7}{3}\right)$ we see that $\left(\frac{7}{3}, g\left(\frac{7}{3}\right)\right)$ is a local maximum.

Hence g is strictly increasing for $x \leq \frac{7}{3}$.

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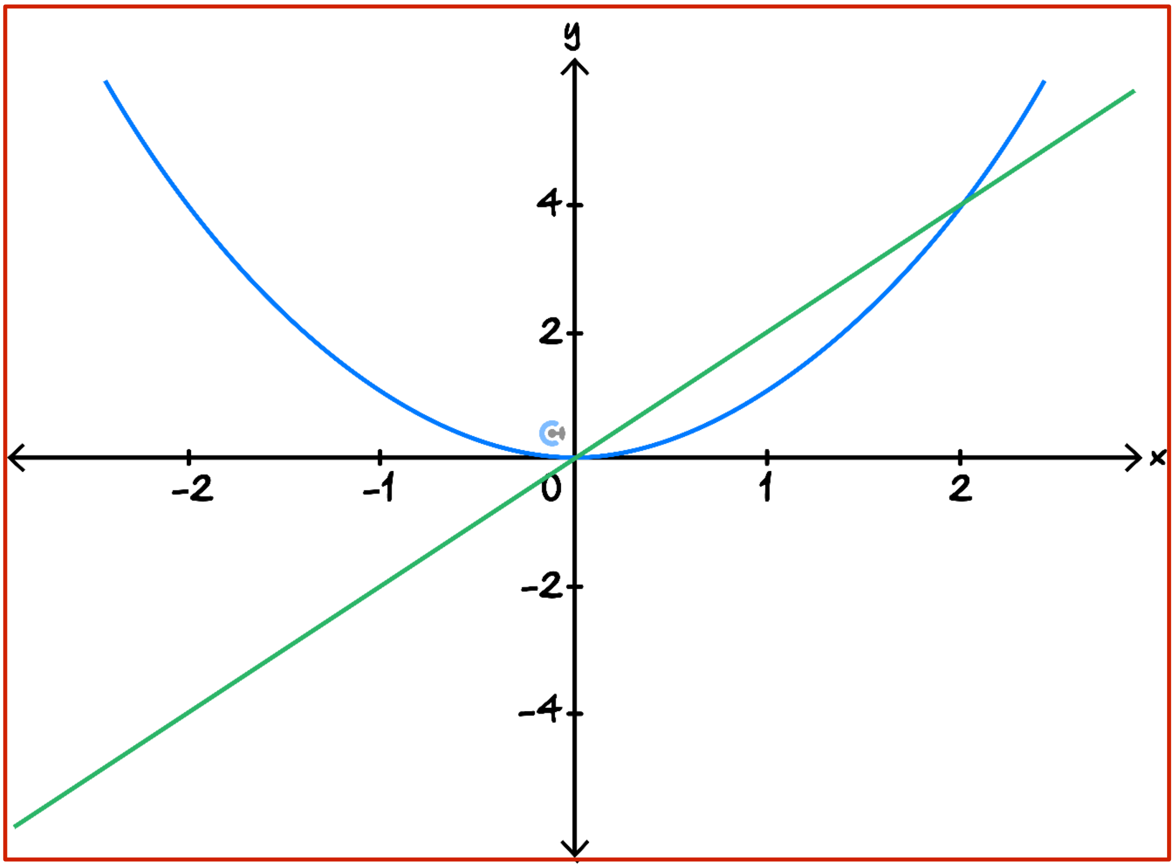
Sub-Section [2.1.3]: Graph Derivative Functions

Question 21

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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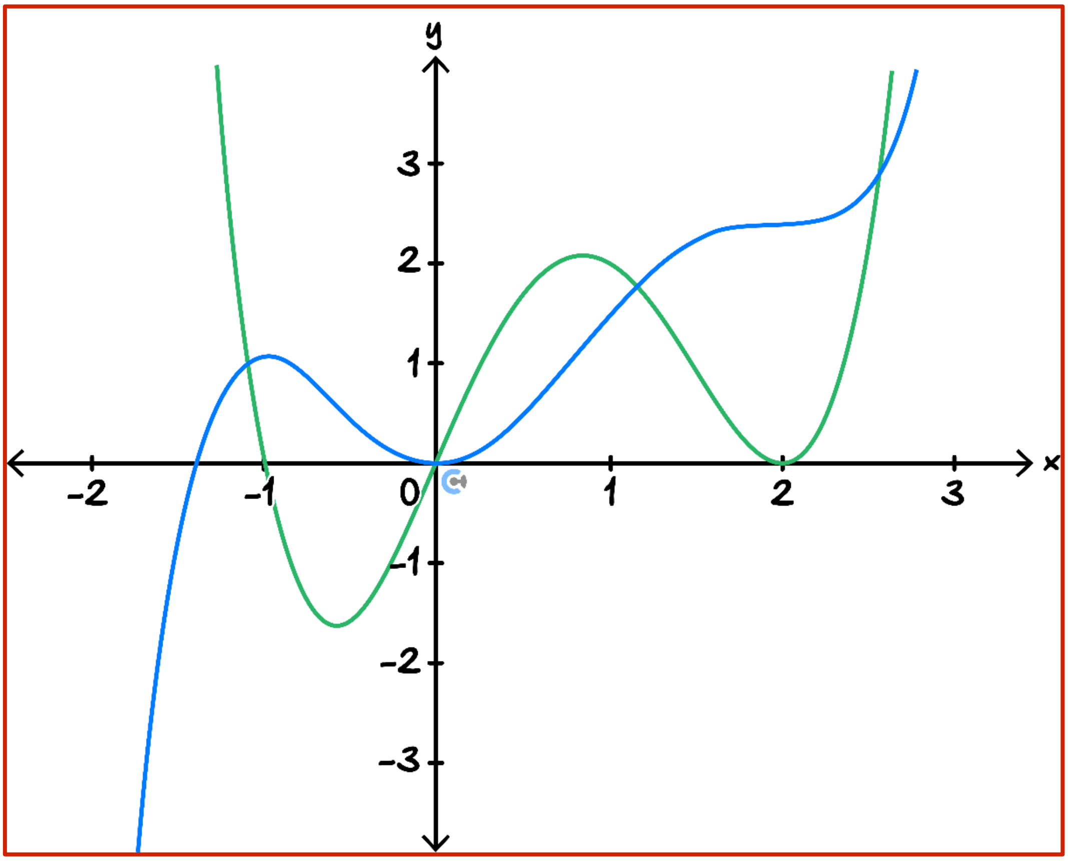
Question 22

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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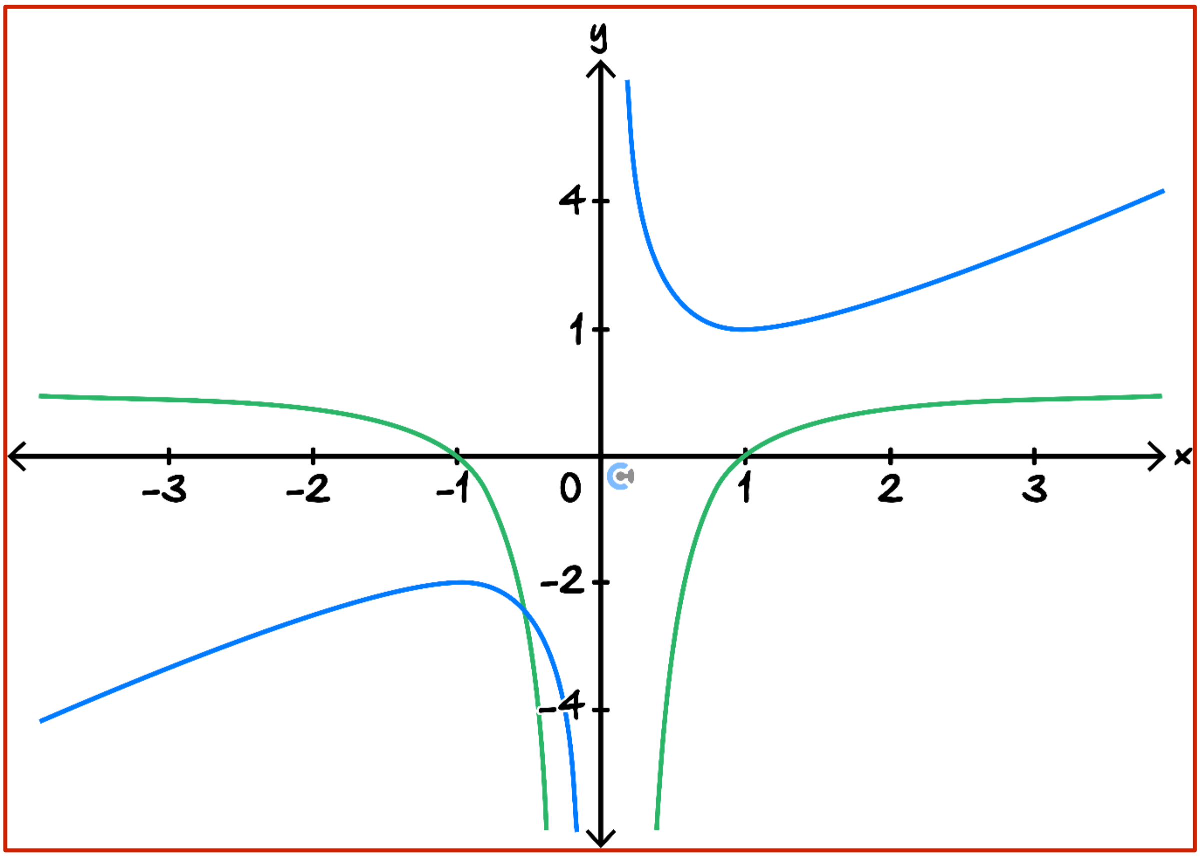
Question 23

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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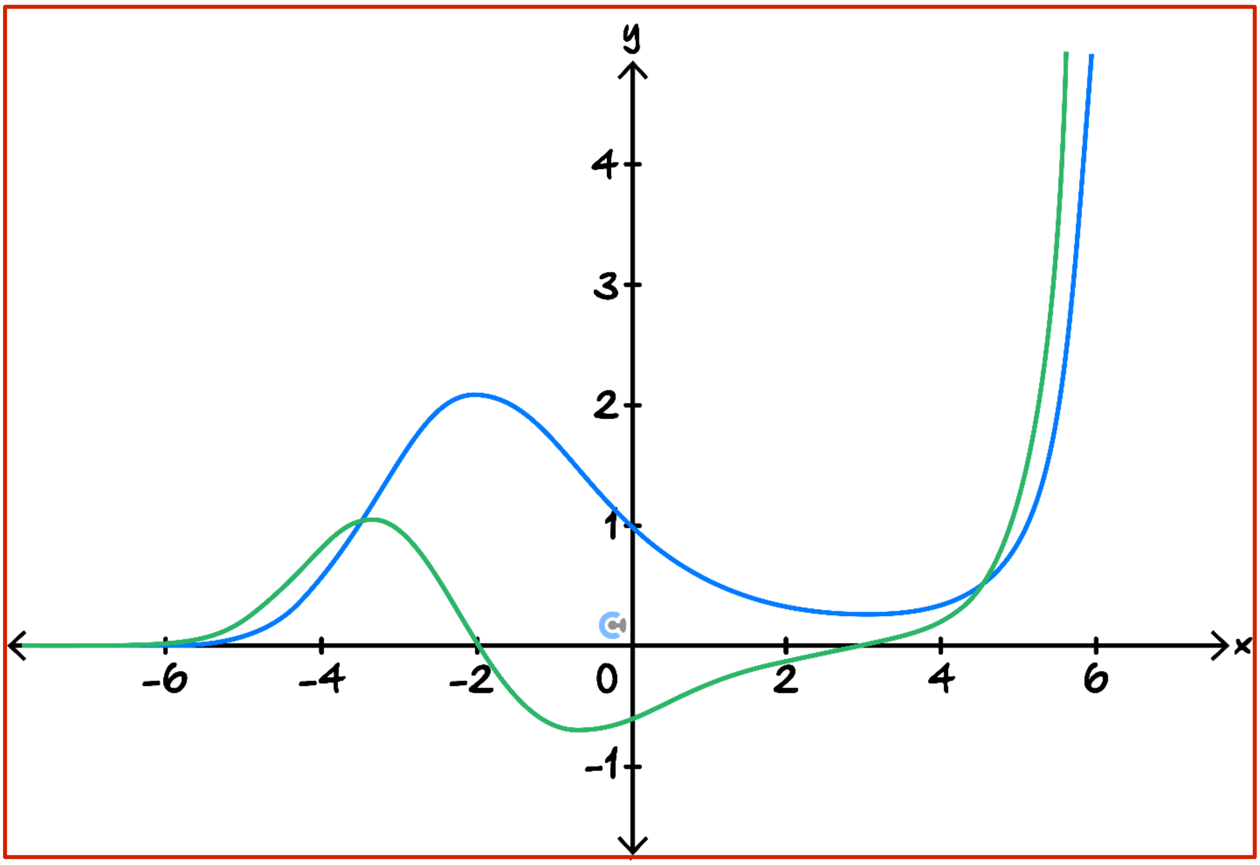
Question 24

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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