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VCE Mathematical Methods ¾
AOS 2 Revision [2.0]
SAC 7 Solutions

54 Marks. 5 Minute Reading. 71 Minutes Writing.

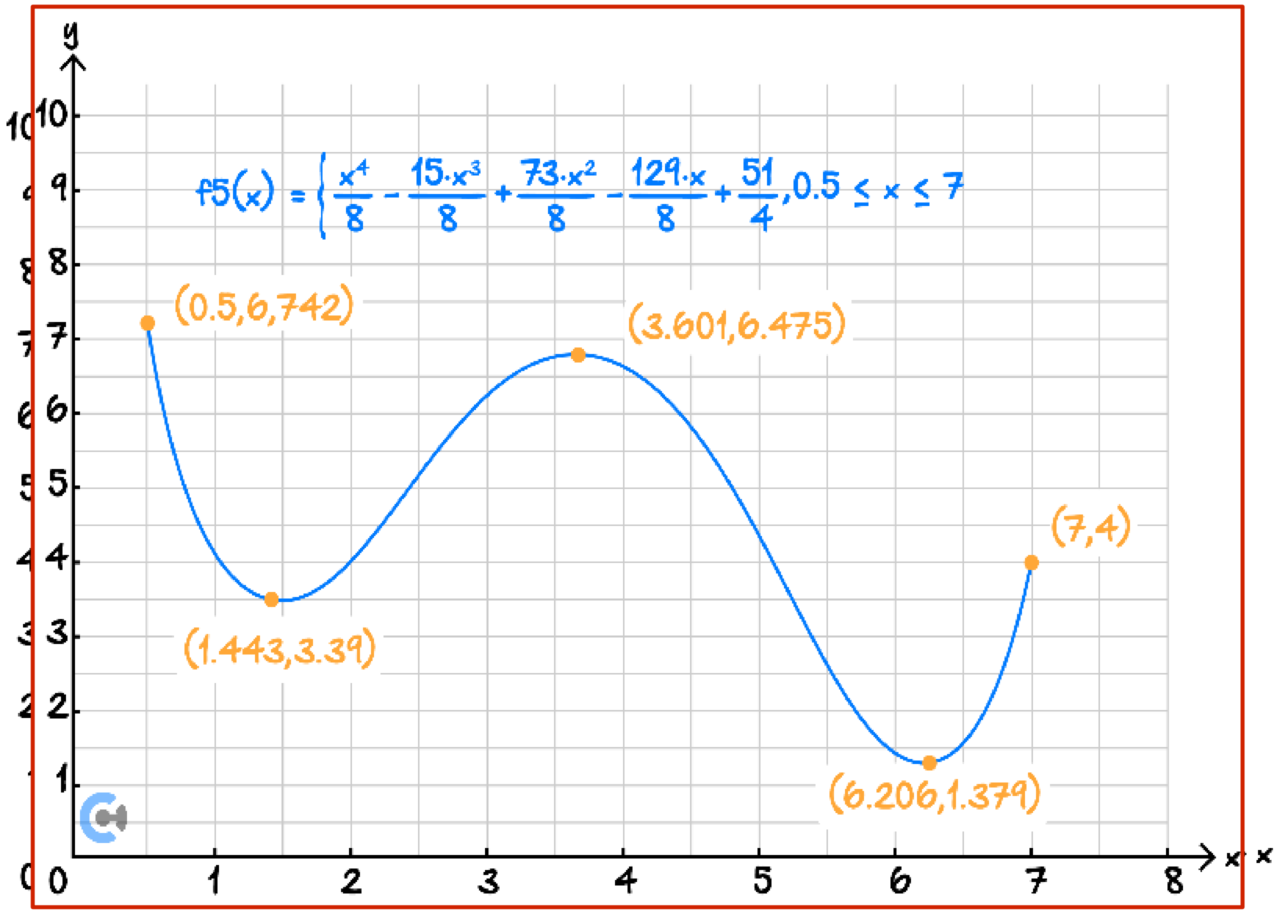
Section A: Tangents and Normals SAC Questions (54 Marks)

Question 1 (12 marks)

In Perfect Physics Land, there is a unique stream that flows between some mountains that has become quite a tourist attraction. The river can be modelled by the quartic:

$$r(x) = \frac{x^4}{8} - \frac{15x^3}{8} + \frac{73x^2}{8} - \frac{129x}{8} + \frac{51}{4}$$

a. Graph the river on the below axis for $x \in \left[\frac{1}{2}, 7\right]$, labelling key features correct to 3 decimal places. (3 marks)



Since it has become such a tourist attraction, the council have begun constructing lookouts that look onto some of the most beautiful parts of the river from a distance. To do this, they have begun modelling lines of sight with tangents.

b. Find the derivative of $r(x)$. (1 mark)

Derivative: $r'(x) = \frac{x^3}{2} - \frac{45x^2}{8} + \frac{73x}{4} - \frac{129}{8}$

c. Hence, find the tangent at the point of inflection on the graph of $r(x)$ between $x = 4$ and $x = 6$, a location considered to be one of the best spots on the river. Give your answer to 2 decimal places. (2 marks)

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In[116]:= r[x_] := x^4/8 - 15 x^3/8 + 73 x^2/8 - 129 x/8 + 51/4

In[118]:= Solve[r''[x] == 0, x] // N
Out[118]= {{x → 2.37311}, {x → 5.12689}}

In[120]:= Solve[y - r[5.12689] == r'[5.12689] (x - 5.12689),
            y] // Expand
Out[120]= {{y → 19.1625 - 3.03223 x}}
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d. Find the acute angle the tangent found in **part c.** makes with the x-axis, to the nearest whole degree. (1 mark)

Angle: 72°

To get a better sense of the geography of the river to aid them in building outlooks, further investigation regarding the river is done with tangents and lines perpendicular to the flow of the stream.

e. Find a line that is perpendicular to $r(x)$ between its 2 points of inflection, given that it is parallel to the tangent found in **part c.**, with values given correct to 3 decimal places. (3 marks)

Normal: $y = 17.015 - 3.032x$

f. A new tangent is made at $x = 4$ as they try to come up with other possible locations of interest that a lookout should be placed in line with to view. Find the angle between this tangent and the one in **part c.**, in degrees correct to 1 decimal place. (2 marks)

23.4°

Question 2 (9 marks)

Another stream is used to host a canoeing competition, with one of the major competitors being Perfect Physics Land Secondary School, who decide to model the stream to help in the setup. The stream is modelled with the below equation:

$$l(x) = 2\cos\left(\frac{x}{2} + \pi\right) + 3$$

Where both axes are given as factors of 50 metres (i.e., $1x = 50\ m$, $2x = 100\ m$, etc.). The finish line is located at the first point of inflection on the graph with a negative gradient.

a. Find the coordinate of the finish line. (2 marks)

Co-ordinate: $(3\pi, 3)$

A flag is to be hung directly above the stream to indicate the finish line. To do this, supports are being put into place that use tangents and lines perpendicular to the stream's motion.

b. Find the tangent of the stream at the finish line and the acute angle, this tangent makes with the x-axis in degrees. (2 marks)

Tangent: $y = 3(\pi + 1) - x$; Angle: $45°$.

c. Find the line perpendicular to the tangent that intersects at the finish line and the acute angle, this line makes with the x-axis in degrees. (2 marks)

Tangent: $y = x - 3(\pi - 1)$; Angle: $45°$.

d. State the type of triangle formed by the tangent, the perpendicular line and the x-axis. (1 mark)

Isosceles

The support wires are to form an X shape with the finish line being in the middle of each wire and 1 end of the wire being fixed to the x-axis.

e. Given this setup, determine the length of wire needed to create the X-pattern to hold up the flag, correct to the nearest 10 metres. (2 marks)

$850\ m$

Space for Personal Notes

Question 3 (11 marks)

Romina, a teacher at Perfect Physics Land Secondary School, wants to model the average increase in her student's grades over the course of her teaching. She decides to use a composite function based on the functions below:

$$f(x) = e^{\frac{x}{2}}, g(x) = (x-2)^3 + 1$$

Where x is months of teaching, and the composite function is the average increase in marks since the start of the teaching period.

a. Find $f \circ g$, stating its domain and range. (2 marks)

$$f \circ g(x) = e^{\frac{1}{2}((x-2)^3+1)}$$
$$\text{dom } f \circ g = [0, \infty)$$
$$\text{ran } f \circ g = [0, \infty)$$

To better understand the trends in student grade increase, Romina decides to do more investigation into the rate at which the marks increase.

b. Show that the derivative of $f \circ g$ is equal to $\frac{3(x^2-4x+4)}{2} e^{\frac{x^3}{2} - 3x^2 + 6x - \frac{7}{2}}$. (2 marks)

$$\frac{d}{dx}\left(e^{\frac{(x-2)^3+1}{2}}\right) = \underbrace{\frac{1}{2} e^{\frac{(x-2)^3+1}{2}}}_{\text{outside derivative}} \times \underbrace{3(x-2)^2}_{\text{inside derivative}} = \frac{3}{2}(x-2)^2 e^{\frac{(x-2)^3+1}{2}}.$$
$$= \frac{3(x^2-4x+4)}{2} e^{\frac{x^3}{2} - 3x^2 + 6x - \frac{7}{2}}$$

c. Hence or otherwise, find the points of inflection of $f \circ g$, giving answers to 3 decimal places. (2 marks)

$(0.899, 0.846), (2, 1.649)$

Romina wants to see when she was teaching the best and what would have happened if her teaching kept that rate of improvement.

d. Find the greatest rate of mark increase that occurs before the stationary point of $f\ o\ g$ and state the coordinates of where this occurs, giving answers to 3 decimal places. (2 marks)

1.538, (0.899,0.846)

To model what would have happened if her teaching remained at its peak before the stationary point, Romina decides to draw in a tangent.

e. Find the tangent of the point found in **part d.**, giving values to 3 decimal places. (1 mark)

$y = 1.538x - 0.537$

f. Another tangent is drawn on the graph of $f\ o\ g$ such that it is parallel to the tangent found in **part e.**, find the coordinate at which this tangent was drawn, to 3 decimal places. (2 marks)

(2.719,1.985)

Space for Personal Notes

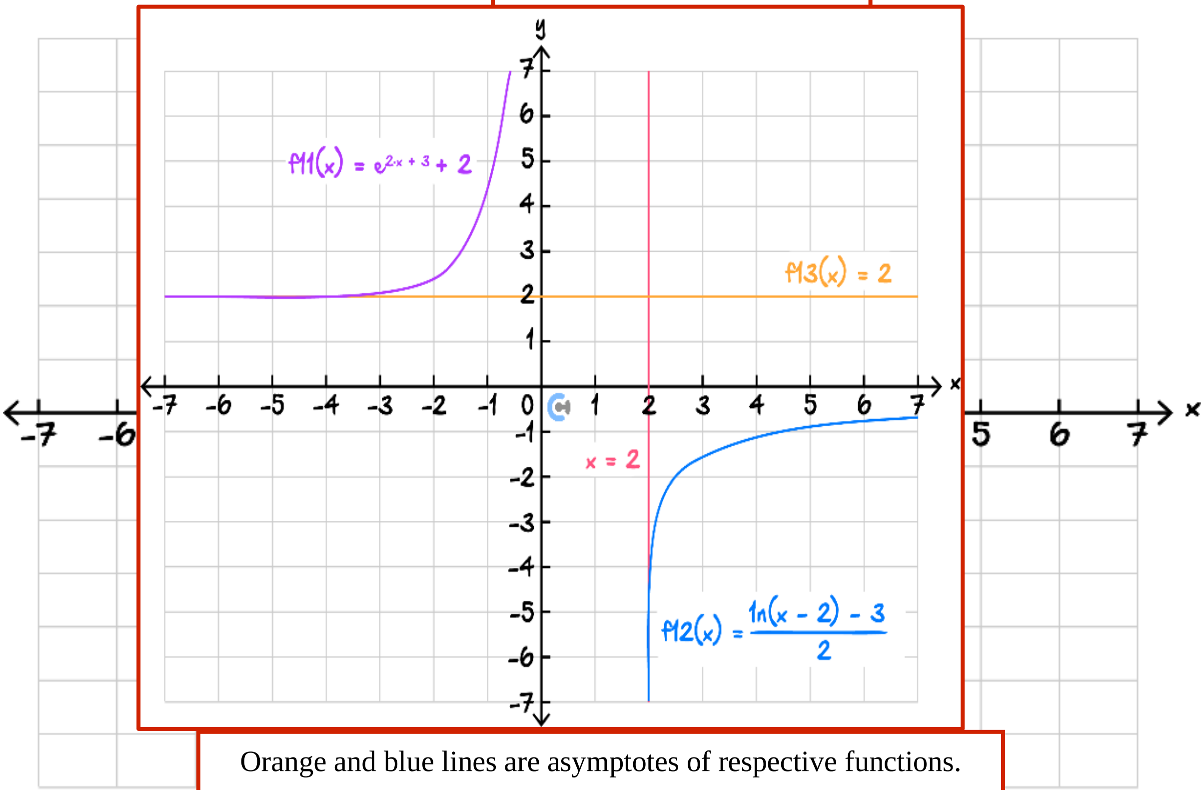
Question 4 (11 marks)

John is a student at Perfect Physics Land Secondary School and trying to get ahead of his classmates, decides to try learning about decay models. To do this, he realises he needs an understanding of exponentials so begins by investigating the equation below:

$$p(x) = e^{2x+3} + 2$$

a. Graph $p(x)$ for John, labelling all asymptotes. (2 marks)

Answer below with **part c.**



Orange and blue lines are asymptotes of respective functions.

John's curiosity is making him continue his investigation. He decides he also wants to learn about log functions and their relation to exponentials.

b. Show that the inverse of $p(x)$ is equal to $p^{-1}(x) = \frac{\ln(x-2)-3}{2}$. (2 marks)

$$y = e^{2x+3} + 2$$

Swap x and y for inverse:

$$x = e^{2y+3} + 2$$

$$x - 2 = e^{2y+3}$$

$$\ln(x-2) = 2y + 3$$

$$2y = \ln(x-2) - 3 \Rightarrow y = \frac{1}{2}\ln(x-2) - \frac{3}{2}$$

$$\boxed{p^{-1}(x) = \frac{1}{2}\ln(x-2) - \frac{3}{2}.}$$

c. Graph $p^{-1}(x)$ on the same axis above, labelling asymptotes. (2 marks)

To further investigate the relationships between exponentials and logarithms, John decides to draw in some tangents.

d. Find the tangent of $p(x)$ at $x = \frac{-\ln(2)-3}{2}$. (2 marks)

Tangent: $y = x + \frac{\ln(2)+8}{2}$

e. Find the tangent of $p^{-1}(x)$ at $x = \frac{5}{2}$. (2 marks)

Tangent: $y = x - \frac{\ln(2)+8}{2}$

f. State the coordinates of the intersection points (if any) between the lines in **part d.** and **part e**. (1 mark)

No intersections.

Space for Personal Notes

Question 5 (11 marks)

Scott, one of the best athletes at Perfect Physics Land Secondary School, is competing in the discus at the school's Athletics Day. He swings around several times in a circle, to build up momentum, with his arm stretched out holding the discus, such that the discus follows the circle equation given below from a bird's eye view:

$$x^2 + (y-2)^2 = 4$$

a. By rearranging the relation above, show that the upper half of the circle is given by $y = \sqrt{4-x^2} + 2$ and the lower half is given by $y = -\sqrt{4-x^2} + 2$. (3 marks)

$$x^2 + (y-2)^2 = 4$$
$$(y-2)^2 = 4 - x^2$$
$$y - 2 = \pm\sqrt{4-x^2}$$
$$y = 2 \pm \sqrt{4-x^2}$$

Upper half of the circle corresponds to the positive square root:
$$y = \sqrt{4-x^2} + 2$$

Lower half of the circle corresponds to the negative square Root: $y = -\sqrt{4-x^2} + 2$

Scott spins anticlockwise and releases the discus from the bottom half such that the throw is tangential to the circle at the point of release. To ensure he beats his old record, he needs to the discus to reach the point $(\sqrt{(35)} + 12, 3)$.

b. Find the coordinate of the point at which Scott releases the discus and tangent of this coordinate such that it passes through the point $(\sqrt{(35)} + 12, 3)$. (4 marks)

Co-ordinate: $\left(\frac{1}{3}, 2 - \frac{\sqrt{35}}{3}\right)$, Equation of tangent: $y = \frac{\sqrt{35}\cdot x}{35} - \frac{2(6\sqrt{35}-35)}{35}$.

c. Scott thinks of channelling the momentum from his spinning such that he can throw it directly from where he is standing at $(0,2)$, causing it to be perpendicular to his circular spin.

Find the point at which Scott's new throw crosses the circle and the equation of the line perpendicular to the circle that represents the new throw such that the throw perfectly passes through the point $(\sqrt{(35)} + 12, 3)$, with all values being given correct to 4 decimal places. (**Hint:** Use the equation of the top of the circle.) (4 marks)

Co-ordinate: $(1.9969, 2.1115)$, Equation of tangent: $y = 0.0558x + 2.0000$
(Additional zeros for 2 not required.)

Space for Personal Notes

Section B: TI Solutions

Question Number	Solutions
1(a)	Define $r(x)=\frac{x^4}{8}-\frac{15\cdot x^3}{8}+\frac{73\cdot x^2}{8}-\frac{129\cdot x}{8}+\frac{5}{}$ ▸ $methods\_func\backslash analysed\left(r(x),x,\frac{1}{2},7\right)$ ▸ Start Point: $\begin{bmatrix}0.50000 & 6.74219\end{bmatrix}$ ▸ End Point: $\begin{bmatrix}7.00000 & 4.00000\end{bmatrix}$ ▸ Maximal Domain: $0.50000\le x\le 7.00000$ ▸ No x –Intercepts Found ▸ Vertical Intercept: $\begin{bmatrix}0.00000 & 12.75000\end{bmatrix}$ ▸ Derivative: $0.50000\cdot x^3-5.62500\cdot x^2+18.25000\cdot x-16.1$ ▸ ▸ Inflection Points: (2.00000) $\begin{bmatrix}2.37311 & 4.77836\end{bmatrix}$ (Increasing) $\begin{bmatrix}5.12689 & 3.61661\end{bmatrix}$ (Decreasing) ▸ Stationary Points: (3.00000) $\begin{bmatrix}1.44311 & 3.39033\end{bmatrix}$ (Local min.) $\begin{bmatrix}3.60107 & 6.47520\end{bmatrix}$ (Local max.) $\begin{bmatrix}6.20582 & 1.37910\end{bmatrix}$ (Local min.)
1(b)	$\frac{d}{dx}(r(x))$ $\quad\frac{x^3}{2}-\frac{45\cdot x^2}{8}+\frac{73\cdot x}{4}-\frac{129}{8}$
1(c)	tangentLine$(r(x),x,5.12689)$ $19.16254-3.03223\cdot x$
1(d)	$\frac{\tan^{-1}(3.03223)\cdot 180}{\pi}$ $\quad 71.74795$

1(e)	$\text{ds\_diffcalc\textbackslash solve\_gradd}\left(r(x),x,\frac{1}{3.03223},2,5\right)$ ▸ Maximal Domain: $2\le x\le 5$ ▸ Derivative: $\frac{x^3}{2}-\frac{45\cdot x^2}{8}+\frac{73\cdot x}{4}-\frac{129}{8}$ ▸ Found point with gradient 0.32979 : $[3.48231 \quad 6.45553]$ $\text{normalLine}(r(x),x,3.48231)$ $17.015-3.032\cdot x$
1(f)	$\text{tangentLine}(r(x),x,4)$ $\frac{43}{4}-\frac{9\cdot x}{8}$ $\frac{\tan^{-1}\left(\frac{-9}{8}\right)\cdot 180}{\pi}$ -48.366 $72-48.4$ 23.600
2(a)	$\text{Define } l(x)=2\cdot\cos\left(\frac{x}{2}+\pi\right)+3$ *Done* $\text{methods\_func\textbackslash analysed}(l(x),x,0,4\cdot\pi)$ ▸ Start Point: $[0 \quad 1]$ ▸ End Point: $[4\cdot\pi \quad 1]$ ▸ Maximal Domain: $0\le x\le 4\cdot\pi$ ▸ No x –Intercepts Found ▸ Vertical Intercept: $[0 \quad 1]$ ▸ Derivative: $\sin\left(\frac{x}{2}\right)$ ▸ Inflection Points: (2) $[\pi \quad 3]$ (Increasing) $[3\cdot\pi \quad 3]$ (Decreasing) ▸ Stationary Points: (3) $[0 \quad 1]$ (Local min.) $[2\cdot\pi \quad 5]$ (Local max.) $[4\cdot\pi \quad 1]$ (Local min.)

2(b), (c)	methods_diffcalc\function_lines$(l(x),x,3\cdot\pi)$ ▸ Derivative: $\sin\left(\frac{x}{2}\right)$ ▸ Gradient: -1 ▸ Perpendicular Gradient: 1 ▸ Passes Through: $[3\cdot\pi \quad 3]$ $\begin{bmatrix} \text{"Tangent Line"} & \text{"x-Int."} & \text{"Vert.-Int."} \\ y=3\cdot(\pi+1)-x & 3\cdot(\pi+1) & 3\cdot(\pi+1) \end{bmatrix}$ $\begin{bmatrix} \text{"Normal Line"} & \text{"x-Int."} & \text{"Vert.-Int."} \\ y=x-3\cdot(\pi-1) & 3\cdot(\pi-1) & -3\cdot(\pi-1) \end{bmatrix}$
2(e)	◂ods_misc\linear_info$([3\cdot\pi \quad 3],[3\cdot(\pi+1) \quad 0])$ ▸ Point 1: $[9.425 \quad 3.000]$ ▸ Point 2: $[12.425 \quad 0.000]$ ▸ Midpoint: $[10.925 \quad 1.500]$ ▸ Distance: 4.243 $4\cdot 4.243$ → 16.972 $16.972\cdot 50$ → 848.600
3(a)	Define $f(x)=e^{\frac{x}{2}}$ → *Done* Define $g(x)=(x-2)^3+1$ → *Done* $f(g(x))$ → $e^{\frac{x^3}{2}-3\cdot x^2+6\cdot x-\frac{7}{2}}$
3(b)	Define $dfg(x)=\frac{d}{dx}(f(g(x)))$ → *Done* factor$(dfg(x))$ → $\dfrac{3\cdot(x-2)^2\cdot e^{\frac{x^3}{2}-3\cdot x^2+6\cdot x-\frac{7}{2}}}{2}$

3(c)	*methods_func\analyse*$(f(g(x)),x)$ ▸ Start Point: $[-\infty \quad 0.000]$ ▸ End Point: $[\infty \quad \infty]$ ▸ Maximal Domain: $-\infty<x<\infty$ ▸ Asymptote: $y=0.000$ (Horizontal) ▸ No x –Intercepts Found ▸ Vertical Intercept: $[0.000 \quad 0.030]$ ▸ Derivative: $(0.045\cdot x^2-0.180\cdot x+0.180)\cdot(2.718)^{0.500\cdot x}$ ▸ Inflection Points: (2.000) $[0.899 \quad 0.841]$ (Increasing) $[2.000 \quad 1.637]$ (Increasing) ▸ Stationary Point: $[2.000 \quad 1.637]$ (Inflection)
3(d)	$dfg(0.899)$ 1.538 $dfg(2)$ 0
3(e)	$\text{tangentLine}(f(g(x)),x,0.899)$ $1.538\cdot x-0.537$
3(f)	*methods_diffcalc\solve_grad*$(f(g(x)),x,1.538)$ ▸ Maximal Domain: $-\infty<x<\infty$ ▸ Derivative: $\left(\frac{3\cdot e^{\frac{-7}{2}}\cdot x^2}{2}-6\cdot e^{\frac{-7}{2}}\cdot x+6\cdot e^{\frac{-7}{2}}\right)\cdot e^{\frac{x^3}{2}-3\cdot x}$ ▸ Found points with gradient 1.538 : (3) $[0.893 \quad 0.837]$ $[0.906 \quad 0.856]$ $[2.719 \quad 1.985]$

4(d)	Define $p(x)=\mathbf{e}^{2\cdot x+3}+2$ Done $\text{tangentLine}\left(p(x),x,\frac{-\ln(2)-3}{2}\right)$ $x+\frac{\ln(2)+8}{2}$
4(e)	Define $p1(x)=\frac{1}{2}\cdot\ln(x-2)-\frac{3}{2}$ Done $\text{tangentLine}\left(p1(x),x,\frac{5}{2}\right)$ $x-\frac{\ln(2)+8}{2}$
5(b)	Define $f(x)=-\sqrt{4-x^2}+2$ Done Define $t(x)=\text{tangentLine}(f(x),x,a)$ Done $t(x)$ $\frac{a\cdot x}{\sqrt{4-a^2}}+\frac{2\cdot\left(\sqrt{4-a^2}-2\right)}{\sqrt{4-a^2}}$ $\text{solve}\left(t\left(\sqrt{35}+12\right)=3,a\right)$ $a=\frac{1}{3}$ $f\left(\frac{1}{3}\right)$ $2-\frac{\sqrt{35}}{3}$ $t(x)\vert a=\frac{1}{3}$ $\frac{\sqrt{35}\cdot x}{35}-\frac{2\cdot\left(6\cdot\sqrt{35}-35\right)}{35}$
5(c)	◂thods_misc\linear_info([0 2],[$\sqrt{35}$+12 3]) ▸ Point 1: [0. 2.] ▸ Point 2: [17.9161 3.] ▸ Midpoint: [8.95805 2.5] ▸ Distance: 17.944 ▸ Gradient: 0.055816 ▸ Perp. Bisector: $y=162.993-17.9161\cdot x$ ▸ Linear Equation: $y=0.055816\cdot x+2.$ Define $l(x)=0.055816\cdot x+2.$ Done methods_func\intersect($\sqrt{4-x^2}+2,l(x),x$) ▸ Intersection Points: (1) [1.99689 2.11146]