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VCE Mathematical Methods ¾
AOS 2 Revision [2.0]
SAC 4 Solutions

54 Marks. 15 Minutes Reading. 70 Minutes Writing.

Section A: SAC Questions (54 Marks)

Question 1 (20 marks)

For the following question consider the following cubic function:

$$f(x) = x^3 - 4x^2 + x + 6$$

a.

i. Show that the equation of the tangent at any point $x = a$ is given by: (2 marks)

$$y_t = (3a^2 - 8a + 1)x - 2a^3 + 4a^2 + 6$$

Show every working out.

ii. Find the equation of the normal at any point $x = a$. (2 marks)

$$y_n = \frac{-x}{3a^2 - 8a + 1} + \frac{3a^5 - 20a^4 + 36a^3 + 6a^2 - 46a + 6}{3a^2 - 8a + 1}$$

iii. Find the value of a, for which the tangent line crosses the origin. Provide your answer correct to 3 decimal places. (3 marks)

$$x = 2.486$$

iv. Find the coordinates of the local maxima and minima of $f(x)$. (2 marks)

$$\left(\frac{4+\sqrt{13}}{3}, \frac{70-26\sqrt{13}}{27}\right) \text{ and } \left(\frac{4-\sqrt{13}}{3}, \frac{70+26\sqrt{13}}{27}\right)$$

v. Hence, for what value(s) of a will the normal to the function only intersect the function once, correct to 2 decimal places? (3 marks)

$$a \in (-\infty, -1.05] \cup [3.72, \infty)$$

b. Consider the following family of functions:

$$f(x) = x^3 - 4x^2 + kx + 6$$

where $k \in R$.

i. For what values of k will there be no stationary points? (2 marks)

$$k > \frac{16}{3}$$

ii. Find the coordinates of the stationary points in terms of k. (1 mark)

Solution Pending

iii. Hence, for what values of k will there be one solution for the equation $f(x) = 3$? Provide your answer correct to 1 decimal place. (3 marks)

$$k > 2.6$$

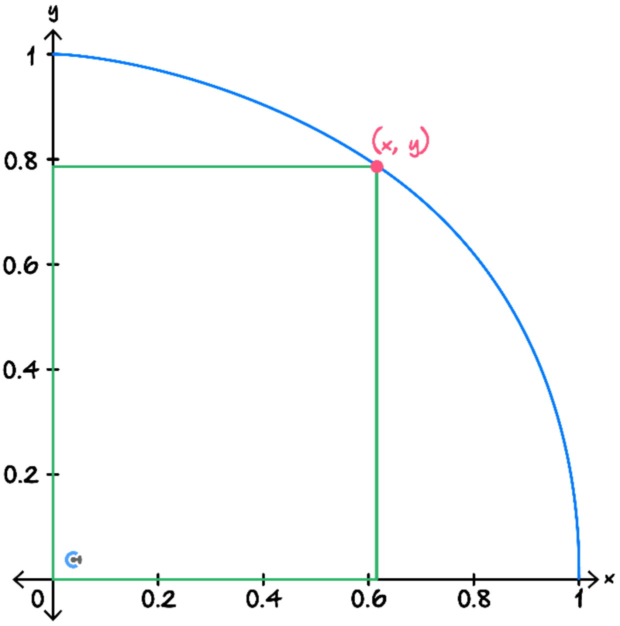
iv. Is it possible to have no real solutions for $f(x) = m$, where $m \in R$? Justify. (2 marks)

For very large m, x^3 dominates. Hence, a cubic function will always initially "start" at negative infinity and "end" at positive infinity. i.e. it will always have a range of R, so there will always be a real solution for $f(x) = m$.

Space for Personal Notes

Question 2 (20 marks)

The figure below depicts a rectangle that is inscribed in a quarter of a unit circle. The equation of the quarter of a unit circle can be described by $f(x) = \sqrt{1 - x^2}$.



Fitting in

a.

i. Express the area of the rectangle in terms of x only. (2 marks)

$$A = x\sqrt{1 - x^2}$$

ii. Express the perimeter of the rectangle in terms of x only. (1 mark)

$$P = 2x + 2\sqrt{1 - x^2}$$

iii. Restrict the domain to model the situation. (1 mark)

$$x \in [0, 1]$$

b. Show that the maximum area and perimeter occur for the same value of x. (3 marks)

$x = \sqrt{2}, A = \frac{1}{2}, P = 2\sqrt{2}$

c. Two parallel lines, with a gradient of -1 form two sides of a rectangle, inscribed in the same portion of a circle in **part a.** One of these lines is a secant line to the quarter-circle (a secant line intersects with a curve at atleast two points).

i. Find the equation of the line passing through the endpoints of the quarter-circle. (1 mark)

$y = 1 - x$

ii. Find the equation of the line that is tangent to the quarter circle. (1 mark)

$y = \sqrt{2} - x$

iii. Hence, what are the possible values of the y-intercept of the secant line? (2 marks)

$c \in [1, \sqrt{2})$

The other two sides of the rectangle originate from the intersection points of the secant line and the circle.

iv. One of the intersection points between the secant line and the circle is given by $(a, f(a))$. State the coordinates of the other intersection point at point $(b, f(b))$, where $b > a$. (1 mark)

$(b, \sqrt{1 - b^2})$

v. Hence or otherwise, find the equation of these two lines in terms of a and b. (2 marks)

```
In[1]:= f[x_] := √(1 - x^2)
In[2]:= Solve[y - f[b] == x - b, y]
Out[2]= {{y → -b + √(1 - b^2) + x}}
In[3]:= Solve[y - f[a] == x - a, y]
Out[3]= {{y → -a + √(1 - a^2) + x}}
```

vi. State the relationship between a and b. Hence, what do you notice about the two lines in **part c. v.**? (2 marks)

$a = \sqrt{1 - b^2}$ or $b = \sqrt{1 - a^2}$
They are inverses.

The area of the rectangle can be generalised in terms of a and b given by the equation below:

Area = Solution Pending

vii. State the area in terms of a. (1 mark)

Solution Pending

viii. Hence, find the maximum area possible for this rectangle. (3 marks)

When the y-intercept is $a = \frac{\sqrt{2(\sqrt{2}+2)}}{2}$.

And the area is $\frac{\sqrt{8}-2}{2}$.

Let it be known that I wanted to make this generalised to a random orientation of the rectangle, but that is gonna be left for the spec kids >:).

Space for Personal Notes

Question 3 (14 marks)

a. For an ideal scenario for a pathogen, it will double its population every day. And hence, only needing 1 pathogen to start a potentially deadly infection. A possible equation to model the scenario can be given in the form $p(t) = 2^t$ where $p(t)$ represents the population of pathogens and t is the number of days.

i. The same equation is rewritten in the form of $p(t) = e^{\log_e(b)t}$. State the value of b. (1 mark)

$p(t) = e^{\ln(2)t}, b = 2$

ii. State $p'(t)$ in terms of $p(t)$. (1 mark)

$p'(t) = \ln(2)p(t)$

iii. Hence, show that: (2 marks)

$$\frac{d}{dx}(a^x) = \ln(a)a^x$$

Basically, a formalised version of the previous two parts.

There are two main ways a pathogen is removed from the body by the immune system or medication.

b. The immune response is dependent on the time after a pathogen is found.The amount of immune cells per 200 ml of blood is given by:

$$i(t_2) = e^{t_2}$$

i. What is the number of immune cells per litre of blood at $t = 0$? (1 mark)

5

ii. Write $i(t_2)$ in terms of t, given that the number of immune cells only starts increasing n number of days after a pathogen is found. Give your answer in terms of n. (1 mark)

$i(t - n)$

Taking into account the immune response, the population of the pathogen is given by: $p^*(t) = p(t) - i(t_2)$.

iii. Given that the pathogen is eliminated no later than a month after the initial infection, how many days did it take to detect it initially? (2 marks)

```
In[18]:= Solve[i[t] == p[30], t, Reals]
Out[18]= {{t → 30 Log[2]}}

In[21]:= 30 Log[2] // N
Out[21]= 20.7944

In[22]:= 30 - 30 Log[2] // N
Out[22]= 9.20558
```

$30(1 - \ln(2))$

iv. The sickest day corresponds to the day with the highest number of pathogens. For what value of t does this occur, given that it took 5 days to detect the pathogen? Provide your answer upto 2 decimal places. (2 marks)

15.1 days

c. Now, consider the effects of medication. The amount of medicine present in blood for one dosage is given by the following:

$$m(t_3) = e^{125t_3(a-t_3)}$$

i. Given that it takes 6 days to reach the peak effect of the medication, find the value of a. (2 marks)

$a = 12$

ii. Express t_3 in terms of t and the time the patient took the medication, b. (2 marks)

$m(t - b)$

Space for Personal Notes

Section B: TI Solutions

Question Number	Solutions
1(a)(i)	Define $f(x)=x^3-4\cdot x^2+x+6$ *Done* *methods_diffcalc\tangent_line*$(f(x),x,a)$ ▸ Derivative: $3\cdot x^2-8\cdot x+1$ ▸ Gradient: $3\cdot a^2-8\cdot a+1$ ▸ Passes Through: $\begin{bmatrix} a & a^3-4\cdot a^2+a+6 \end{bmatrix}$ ▸ x −Intercept: $\begin{bmatrix} \frac{2\cdot(a^3-2\cdot a^2-3)}{3\cdot a^2-8\cdot a+1} & 0 \end{bmatrix}$ ▸ Vertical Intercept: $\begin{bmatrix} 0 & -2\cdot(a^3-2\cdot a^2-3) \end{bmatrix}$ ▸ Tangent Line: $(3\cdot a^2-8\cdot a+1)\cdot x-2\cdot(a^3-2\cdot a^2-3)$
1(a)(ii)	$\text{normalLine}(f(x),x,a)$ $\frac{3\cdot a^5-20\cdot a^4+36\cdot a^3+6\cdot a^2-46\cdot a+6}{3\cdot a^2-8\cdot a+1}-\frac{x}{3\cdot a^2-8\cdot a+1}$
1(a)(iii)	Solve vertical intercept equal 0 $\text{solve}(-2\cdot(a^3-2\cdot a^2-3)=0,a)$ $a=2.48558$
1(a)(iv)	*methods_func\analyse*$(f(x),x)$ ▸ Start Point: $\begin{bmatrix} -\infty & -\infty \end{bmatrix}$ ▸ End Point: $\begin{bmatrix} \infty & \infty \end{bmatrix}$ ▸ Maximal Domain: $-\infty<x<\infty$ ▸ x −Intercepts: (3) $\begin{bmatrix} -1 & 0 \end{bmatrix},\begin{bmatrix} 2 & 0 \end{bmatrix},$ $\begin{bmatrix} 3 & 0 \end{bmatrix}$

	▶ Vertical Intercept: $[0 \quad 6]$ ▶ Derivative: $3\cdot x^2-8\cdot x+1$ ▶ Inflection Point: $\left[\frac{4}{3} \quad \frac{70}{27}\right]$ (Decreasing) ▶ Stationary Points: (2) $\left[\frac{-(\sqrt{13}-4)}{3} \quad \frac{26\cdot\sqrt{13}}{27}+\frac{70}{27}\right]$ (Local max.) $\left[\frac{\sqrt{13}+4}{3} \quad \frac{70}{27}-\frac{26\cdot\sqrt{13}}{27}\right]$ (Local min.)
1(a)(v)	Define $n(x)$=normalLine$(f(x),x,a)$ *Done* *methods_diffcalc\solve_touch*$(n(x),f(x),x,a)$ ▶ Derivative 1: $\frac{-1}{3\cdot a^2-8\cdot a+1}$ ▶ Derivative 2: $3\cdot x^2-8\cdot x+1$ ▶ Equating functions and derivatives. ▶ Solutions: $x=2.52420$ and $a=-1.04840$ $x=1.95574$ and $a=0.08852$ $x=0.71093$ and $a=2.57815$ $x=0.14247$ and $a=3.71507$ solve$\left(\frac{d}{dx}(f(x))=0,x\right)$ $x=0.13148$ or $x=2.53518$ Use sliders to see when one point of intersection occurs. $a \in (-\infty,-1.05) \cup (0.09,0.13] \cup [2.54,2.58) \cup (3.71,\infty)$
1(b)(i)	Define $f(x)=x^3-4\cdot x^2+k\cdot x+6$ *Done* Define $df(x)=\frac{d}{dx}(f(x))$ *Done* solve$(df(x)=0,x)$ $x=\frac{\sqrt{16-3\cdot k}+4}{3}$ or $x=\frac{-(\sqrt{16-3\cdot k}-4)}{3}$ solve$(16-3\cdot k<0,k)$ $k>\frac{16}{3}$

1(b)(ii)	$\text{methods\_func\textbackslash analyse}(f(x),x)$ ▸ Start Point: $[-\infty \quad -\infty]$ ▸ End Point: $[\infty \quad \infty]$ ▸ Maximal Domain: $-\infty<x<\infty$ ▸ x −Intercepts: (3) ▸ Vertical Intercept: $[0 \quad 6]$ ▸ Derivative: $3\cdot x^2-8\cdot x+k$ ▸ Inflection Point: $\left[\frac{4}{3} \quad \frac{4\cdot k}{3}+\frac{34}{27}\right]$ ▸ Stationary Points: (2) $\left[\frac{\sqrt{16-3\cdot k}+4}{3} \quad \frac{-2\cdot(16-3\cdot k)^{\frac{3}{2}}}{27}+\frac{4\cdot k}{3}+\frac{34}{27}\right]$ $\left[\frac{-(\sqrt{16-3\cdot k}-4)}{3} \quad \frac{2\cdot(16-3\cdot k)^{\frac{3}{2}}}{27}+\frac{4\cdot k}{3}+\frac{34}{27}\right]$
1(b)(iii)	Using a slider, we see that both stationary points must be above 3. Note that if a cubic has no stationary point, then it is one-to-one, so there is always a unique solution. Hence we must also include our answer to **part i.** $\text{solve}\left(\frac{2\cdot(16-3\cdot k)^{\frac{3}{2}}}{27}+\frac{4\cdot k}{3}+\frac{34}{27}>3,k\right)$ $k\le\frac{16}{3}$ $\text{solve}\left(\frac{-2\cdot(16-3\cdot k)^{\frac{3}{2}}}{27}+\frac{4\cdot k}{3}+\frac{34}{27}>3,k\right)$ $2.60587<k\le\frac{16}{3}$
2(b)	Define $f(x)=\sqrt{1-x^2}$ *Done* Define $a(x)=x\cdot f(x)$ *Done* Define $p(x)=2\cdot x+2\cdot f(x)$ *Done*

$methods\_func\backslash analysed(a(x),x,0,1)$

- Start Point: $\begin{bmatrix} 0 & 0 \end{bmatrix}$
- End Point: $\begin{bmatrix} 1 & 0 \end{bmatrix}$
- Maximal Domain: $0 \le x \le 1$
- x –Intercepts: (2)
  $\begin{bmatrix} 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}$
- Vertical Intercept: $\begin{bmatrix} 0 & 0 \end{bmatrix}$
- Derivative:
  $$\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$
- Inflection Point:
  $\begin{bmatrix} 0 & 0 \end{bmatrix}$ (Increasing)
- Stationary Point:
  $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}$ (Local max.)

$methods\_func\backslash analysed(p(x),x,0,1)$

- Start Point: $\begin{bmatrix} 0 & 2 \end{bmatrix}$
- End Point: $\begin{bmatrix} 1 & 2 \end{bmatrix}$
- Maximal Domain: $0 \le x \le 1$
- No x –Intercepts Found
- Vertical Intercept: $\begin{bmatrix} 0 & 2 \end{bmatrix}$
- Derivative: $2 - \frac{2 \cdot x}{\sqrt{1-x^2}}$
- No Inflection Points Found
- Stationary Point:
  $\begin{bmatrix} \frac{\sqrt{2}}{2} & 2 \cdot \sqrt{2} \end{bmatrix}$ (Local max.)

2(c)(ii)	methods_diffcalc\solve_gradd$(f(x),x,-1,0,1)$ ▶ Maximal Domain: $0\le x\le 1$ ▶ Derivative: $\frac{-x}{\sqrt{1-x^2}}$ ▶ Found point with gradient -1 : $\begin{bmatrix} \text{"x"} & \text{"y"} & \text{"Eqn."} & \text{"x-Int."} & \text{"y-Int."} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2}-x & \sqrt{2} & \sqrt{2} \end{bmatrix}$
2(c)(v)	DelVar a *Done* methods_misc\linear_info$([a \quad f(a)],1)$ ▶ Point: $[a \quad \sqrt{1-a^2}]$ Gradient: 1 ▶ Linear Equation: $y=x+\sqrt{1-a^2}-a$ ▶ x−Intercept: $[a-\sqrt{1-a^2} \quad 0]$ ▶ y−Intercept: $[0 \quad \sqrt{1-a^2}-a]$ methods_misc\linear_info$([b \quad f(b)],1)$ ▶ Point: $[b \quad \sqrt{1-b^2}]$ Gradient: 1 ▶ Linear Equation: $y=x+\sqrt{1-b^2}-b$ ▶ x−Intercept: $[b-\sqrt{1-b^2} \quad 0]$ ▶ y−Intercept: $[0 \quad \sqrt{1-b^2}-b]$
2(c)(vii)	Height of the diagonal rectangle is distance between two points on the secant intersecting function methods_misc\linear_info$([a \quad f(a)],[b \quad f(b)])$ Point 1: $[a \quad \sqrt{1-a^2}]$ Point 2: $[b \quad \sqrt{1-b^2}]$ Midpoint: $\left[\frac{a+b}{2} \quad \frac{\sqrt{1-a^2}+\sqrt{1-b^2}}{2}\right]$ Distance: $\sqrt{-2\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}+a\cdot b-1\right)}$

	Width found by taking y intercept of perpendicular at (a,f(a)) then taking distance to (a,f(a)). $methods\_misc\backslash linear\_info([a \;\; f(a)],1)$ ▸ Point: $[a \;\; \sqrt{1-a^2}]$ Gradient: 1 ▸ Linear Equation: $y=x+\sqrt{1-a^2}-a$ ▸ x−Intercept: $[a-\sqrt{1-a^2} \;\; 0]$ ▸ y−Intercept: $[0 \;\; \sqrt{1-a^2}-a]$ $s\_misc\backslash linear\_info([a \;\; f(a)],[0 \;\; \sqrt{1-a^2}-a])$ ▸ Point 1: $[a \;\; \sqrt{1-a^2}]$ ▸ Point 2: $[0 \;\; \sqrt{1-a^2}-a]$ ▸ Midpoint: $\left[\frac{a}{2} \;\; \frac{2\cdot\sqrt{1-a^2}-a}{2}\right]$ ▸ Distance: $\lvert a\rvert\cdot\sqrt{2}$ Define $b=\sqrt{1-a^2}$ *Done* ©get height in terms of a $\sqrt{-2\cdot\left(\sqrt{1-a^2}\cdot\sqrt{1-b^2}+a\cdot b-1\right)}$ → $\sqrt{-2\cdot\left(\sqrt{1-a^2}\cdot\lvert a\rvert+a\cdot\sqrt{1-a^2}-1\right)}$ ©define area Define $area(a)=\sqrt{-2\cdot\left(\sqrt{1-a^2}\cdot\lvert a\rvert+a\cdot\sqrt{1-a^2}-1\right)}\cdot a\cdot\sqrt{2}$ *Done* $area(a)\vert a>0$ → $2\cdot a\cdot\sqrt{1-2\cdot a\cdot\sqrt{1-a^2}}$
2(c)(viii)	*Note: The analyse program might freeze with this question. Hold [esc] or [home] to exit if required.* $\text{solve}\left(\frac{d}{da}(area(a))=0,a\right)$ → $a=\frac{\sqrt{2-\sqrt{2}}}{2}$ $area\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)$ → $\frac{\sqrt{8}-2}{2}$

3(a)	Define $p(t)=2^t$ — *Done* Define $dp(t)=\frac{d}{dt}(p(t))$ — *Done* $dp(t)$ — $\ln(2)\cdot 2^t$
3(b)(iii)	Define $i(t)=\mathbf{e}^t$ — *Done* $\text{solve}(i(t)=p(30),t)$ — $t=30\cdot\ln(2)$ $30-30\cdot\ln(2)$ — $30-30\cdot\ln(2)$
3(b)(iv)	$methods\_func\backslash analyse(p2(t),t)$ ▸ Start Point: $[-\infty \quad 0.00000]$ ▸ End Point: $[\infty \quad -\infty]$ ▸ Maximal Domain: $-\infty<t<\infty$ ▸ Asymptote: $y=0.00000$ (Horizontal) ▸ t−Intercept: $[16.29350 \quad 0.00000]$ ▸ Vertical Intercept: $[0.00000 \quad 0.99326]$ ▸ Derivative: $0.69315\cdot(2.00000)^t-0.00674\cdot(2.71828)^t$ ▸ Inflection Point: $[13.90465 \quad 7967.83240]$ (Increasing) ▸ Stationary Point: $[15.09907 \quad 10769.69680]$ (Local max.)
3(c)	Define $m(t)=\mathbf{e}^{125\cdot t\cdot(a-t)}$ — *Done* Define $dm(t)=\frac{d}{dt}(m(t))$ — *Done* $\text{solve}(dm(6)=0,a)$ — $a=12$

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