CONTOUREDUCTION

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# VCE Mathematical Methods ¾
## AOS 2 Revision [2.0]
## SAC 3 Solutions

45 Marks. 10 Minutes Reading. 65 Minutes Writing.

## Section A: SAC Questions (45 Marks)

**Question 1** (10 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 3x^2 + 2$.

**a.** Find the coordinates of the stationary points of $f(x)$. (2 marks)

$(0, 2), (2, -2)$

**b.** Using the turning points found in **part a.**, find the equation of the line segment joining these two points. State the length of this line segment. (3 marks)

$y = -2x + 2$

$\text{Distance} = 2\sqrt{5}$

Another function $g(x)$ exists such that $g(x) = f(x) + kx$, where $k$ is a real constant.

**c.** Find the $x$-values of the stationary points of $g(x)$ in terms of $k$ as well as the coordinate of the point of inflection. (3 marks)

$\text{S.P.: } x = 1 \pm \sqrt{1 - \frac{k}{3}}$

$\text{P.O.I: } (1, k)$

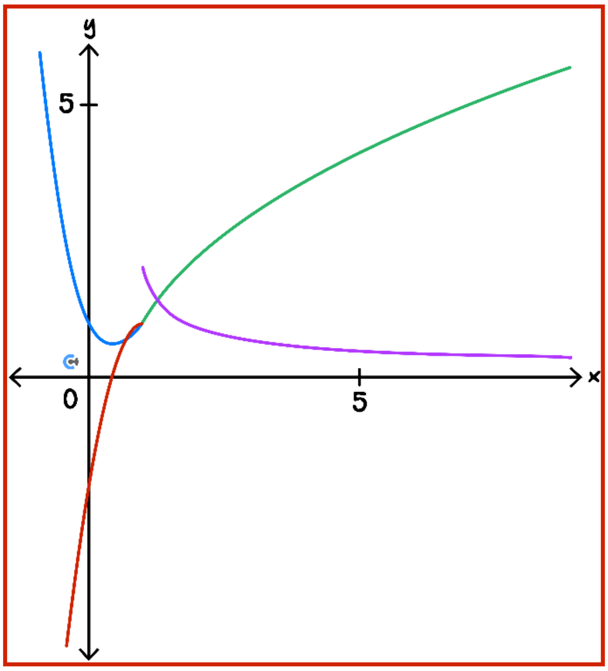
A tangent line is made at the inflection point found in **part c**.

**d.** Given that this tangent line passes through the coordinate $(5, 2)$, solve for the value(s) of $k$. (2 marks)

$$k = \frac{14}{5}$$

**Question 2** (7 marks)

The hybrid function $f(x) = \begin{cases} -x^3 + 3x^2 - 2x + 1, & x \leq 1 \\ \sqrt{4x - 3}, & x > 1 \end{cases}$ is sketched on the graph below:




**a.** Define the derivative function $f'(x)$. (2 marks)

$$f'(x) = \begin{cases} -3x^2 + 6x - 2, & x < 1 \\ \dfrac{2}{\sqrt{4x - 3}}, & x > 1 \end{cases}$$

**b.** On the same axes, sketch the graph of the derivative function. (2 marks)

Open circle at discontinuous points.

**c.** Find the values of $x$ when $f(x)$ is strictly increasing. (1 mark)

$$x \geq -\frac{\sqrt{3}-3}{3}$$

**d.** Let $g(x) = \begin{cases} -x^3 + 3x^2 - 2x + 1 + kx, & x \leq 1 \\ \sqrt{4x-3}, & x > 1 \end{cases}$ with $k > 0$.

Find the values of $x$ when $g(x)$ is strictly increasing in terms of $k$. (2 marks)

$$-\frac{\sqrt{3(k+1)}-3}{3} \leq x < 1 \text{ or } x > 1$$

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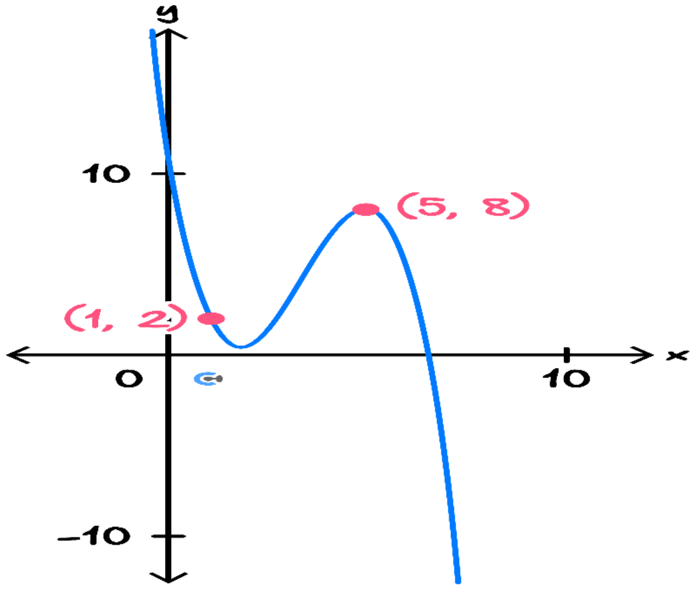
**Question 3** (8 marks)

James is standing outside his house in Doreen at the point $(1,2)$ and decides to take his dog Mango to the park located at $(5, 8)$. A map is shown below which depicts James current whereabouts as well as the local park. Mango is a rather impatient dog and instead of walking on the footpath he decides to walk in a straight line directly to the park.




**a.** Calculate the distance travelled if Mango walks in a straight line from James' house to the park. (1 mark)

$$d = 2\sqrt{13}$$

Once James and Mango arrive at the park, they decide to follow the exact path of the park given by:

$$f(x) = -\frac{1}{2}x^3 + 5x^2 - 13x + 10.5$$

Assume that they follow the path when they return home after a long walk.

**b.** You could say that their path travelled is a relationship that is tangential to the park at every single point. If James and Mango are currently standing at $x = a$, state the equation of the path they are travelling in terms of $a$. (1 mark)

$$y = \frac{2a^3 - 10a^2 + 21}{2} - \frac{(3a^2 - 20a + 26)x}{2}$$

Whenever the gradient of the path is equal to 2, Mango's bladder kicks in and he suddenly needs to take a leak.

**c.**

**i.** State the values of $x$ whenever Mango uses the toilet. (2 marks)

$$x = \frac{\pm\sqrt{10} + 10}{3}$$

**ii.** Hence, state the equations of the tangent lines whenever Mango takes a leak. (2 marks)

$$y = 2x - \frac{20\sqrt{10} + 133}{54}$$
$$y = 2x + \frac{20\sqrt{10} - 133}{54}$$

After a while, Mango needs to do a number 2. The points at which Mango needs to do this occur whenever the gradient equals zero.

**d.** Find the coordinates of the points at which Mango does a number 2. (2 marks)

$$\left(\frac{10 - \sqrt{22}}{3}, \frac{-44\sqrt{22} + 227}{54}\right)$$
$$\left(\frac{10 + \sqrt{22}}{3}, \frac{44\sqrt{22} + 227}{54}\right)$$

**Space for Personal Notes**

**Question 4** (10 marks)

Consider the functions:

$$f(x) = \frac{x+2}{x+3}$$

$$g(x) = \sqrt{5-x}$$

where both functions are defined over their maximal domains.

**a.** State the domains of $f(x)$ and $g(x)$. (2 marks)

$\text{dom } f = \mathbb{R} \setminus \{-3\}$
$\text{dom } g = (-\infty, 5]$

**b.** Determine whether $f(g(x))$ or $g(f(x))$ is defined. Give a reason why. (3 marks)

$f(g(x))$ is defined as $\text{ran } g = [0, \infty)$ which is a subset of $\text{dom } f = \mathbb{R} \setminus \{-3\}$.
$g(f(x))$ is not defined as $\text{ran } f = \mathbb{R} \setminus \{1\}$ which is a subset of $\text{dom } g = (-\infty, 5]$.

**c.** For the function that was defined, find the range of that function. (2 marks)

$\left[\frac{2}{3}, \infty\right)$

**d.** For the undefined function, state the maximal domain of that function such that the composite function exists. (3 marks)

$x \leq -\frac{13}{4}$ or $x > -3$

**Question 5** (10 marks)

A piecewise function is defined by:

$$f(x) = \begin{cases} ax^3 + x, & x \leq 1, \\ (3a + b)x - 2, & x > 1, \end{cases}$$

where $a$ and $b$ are constants with $a > 0$.

**a.** State the values of $a$ and $b$ such that the function joins smoothly at $x = 1$. Hence, state the domain of the derivative function. (3 marks)

$a = 1$ and $b = 1$
dom $f' = \mathbb{R}$

**b.**

**i.** Find the coordinates of the $x$-intercept of $f(x)$. (1 mark)

(0,0)

**ii.** Hence, state the angle in degrees that $f(x)$ makes with the $x$-axis at the $x$-intercept. (2 marks)

$45°$

**c.** Find the coordinates of the intersection point(s) between $f(x)$ and $f^{-1}(x)$. (2 marks)

(0,0)

**d.** Hence, state the possible axis(es) of symmetry that exist for $f(x)$ and $f^{-1}(x)$. Explain why there is only one axis of symmetry or why there are multiple axes of symmetry. (2 marks)

$y = x$
The function is strictly increasing over its domain.

Space for Personal Notes

## Section B: TI Solutions

| Question Number | Solutions |
|---|---|
| 1(a) | Define $f(x)=x^3-3\cdot x^2+2$ *Done*<br>*methods_func\analyse*$(f(x),x)$<br>▸ Start Point: $[-\infty \quad -\infty]$<br>▸ End Point: $[\infty \quad \infty]$<br>▸ Maximal Domain: $-\infty<x<\infty$<br>▸ $x$ −Intercepts: (3)<br>$[-(\sqrt{3}-1) \quad 0], [1 \quad 0],$<br>$[\sqrt{3}+1 \quad 0]$<br>▸ Vertical Intercept: $[0 \quad 2]$<br>▸ Derivative: $3\cdot x^2-6\cdot x$<br>▸ Inflection Point:<br>$[1 \quad 0]$ (Decreasing)<br>▸ Stationary Points: (2)<br>$[0 \quad 2]$ (Local max.)<br>$[2 \quad -2]$ (Local min.) |
| 1(b) | *methods_misc\linear_info*$([0 \quad 2],[2 \quad -2])$<br>▸ Point 1: $[0 \quad 2]$<br>▸ Point 2: $[2 \quad -2]$<br>▸ Midpoint: $[1 \quad 0]$<br>▸ Distance: $2\cdot\sqrt{5}$<br>▸ Gradient: $-2$<br>▸ Perp. Bisector: $y=\frac{x}{2}-\frac{1}{2}$<br>▸ Linear Equation: $y=2-2\cdot x$<br>▸ x−Intercept: $[1 \quad 0]$<br>▸ y−Intercept: $[0 \quad 2]$ |

| | |
|---|---|
| 1(c) | $methods\_func\backslash analyse(g(x),x)$<br>▶ Start Point: $[-\infty \quad -\infty]$<br>▶ End Point: $[\infty \quad \infty]$<br>▶ Maximal Domain: $-\infty<x<\infty$<br>▶ $x$ –Intercepts: (3)<br>▶ Vertical Intercept: $[0 \quad 2]$<br>▶ Derivative: $3\cdot x^2-6\cdot x+k$<br>▶ Inflection Point:<br>$[1 \quad k]$<br>▶ Stationary Points: (2)<br>$\left[\frac{\sqrt{-3\cdot(k-3)}+3}{3} \quad \frac{2\cdot(k-3)\cdot\sqrt{-3\cdot(k-3)}}{9}+k\right]$<br>$\left[\frac{-(\sqrt{-3\cdot(k-3)}-3)}{3} \quad k-\frac{2\cdot(k-3)\cdot\sqrt{-3\cdot(k-3)}}{9}\right]$ |
| 1(d) | tangentLine$(g(x),x,1)$ $\quad (k-3)\cdot x+3$<br>Define $l(x)=(k-3)\cdot x+3$ $\quad$ *Done*<br>solve$(l(5)=2,k)$ $\quad k=\frac{14}{5}$ |
| 2(a) | Define $f(x)=\begin{cases}-x^3+3\cdot x^2-2\cdot x+1, & x\le 1\\ \sqrt{4\cdot x-3}, & x>1\end{cases}$ $\quad$ *Done*<br>Define $df(x)=\frac{d}{dx}(f(x))$ $\quad$ *Done*<br>$df(x)$ $\quad \begin{cases}-3\cdot x^2+6\cdot x-2, & x<1\\ \frac{2}{\sqrt{4\cdot x-3}}, & x>1\end{cases}$ |
| 2(c) | Check the answer with the graph to make sure you are only including appropriate stationary points.<br>solve$(df(x)\ge 0,x)$ $\quad x\ge\frac{-(\sqrt{3}-3)}{3}$ and $x\ne 1$ |

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| 2(d) | Define $g(x)=\begin{cases}-x^3+3\cdot x^2-2\cdot x+1+k\cdot x, & x\le 1\\ \sqrt{4\cdot x-3}, & x>1\end{cases}$ *Done*<br>Define $dg(x)=\frac{d}{dx}(g(x))$ *Done*<br>$dg(x)$ $\begin{cases}-3\cdot x^2+6\cdot x+k-2, & x<1\\ \frac{2}{\sqrt{4\cdot x-3}}, & x>1\end{cases}$<br>solve$(dg(x)\ge 0,x)\vert k>0$<br>$x\ge\frac{-(\sqrt{3\cdot(k+1)}-3)}{3}$ and $x\ne 1$ and $k>0$<br>Note: We can rewrite this as the union of two intervals |
| 3(a) | *methods_misc\linear_info*$([1\ \ 2],[5\ \ 8])$<br>▸ Point 1: $[1\ \ 2]$<br>▸ Point 2: $[5\ \ 8]$<br>▸ Midpoint: $[3\ \ 5]$<br>▸ Distance: $2\cdot\sqrt{13}$<br>▸ Gradient: $\frac{3}{2}$<br>▸ Perp. Bisector: $y=7-\frac{2\cdot x}{3}$<br>▸ Linear Equation: $y=\frac{3\cdot x}{2}+\frac{1}{2}$<br>▸ x−Intercept: $\left[\frac{-1}{3}\ \ 0\right]$<br>▸ y−Intercept: $\left[0\ \ \frac{1}{2}\right]$ |

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| 3(b) | Note: Convert decimal to fraction to obtain exact value of tangent.<br>Define $f(x)=\frac{-1}{2}\cdot x^3+5\cdot x^2-13\cdot x+\frac{21}{2}$ *Done*<br>tangentLine$(f(x),x,a)$<br>$\frac{2\cdot a^3-10\cdot a^2+21}{2}-\frac{(3\cdot a^2-20\cdot a+26)\cdot x}{2}$ |
| 3(c) | Compute Tangent Lines? 1 OK Cancel<br>*methods_diffcalc\solve_grad*$(f(x),x,2)$<br>▸ Maximal Domain: $-\infty<x<\infty$<br>▸ Derivative: $\frac{-3\cdot x^2}{2}+10\cdot x-13$<br>▸ Found points with gradient 2 : (2)<br>$\begin{bmatrix} \text{"x"} & \text{"y"} & \text{"Eqn."} & \text{"x-Int."} & \text{"y-Int."} \\ \frac{-(\sqrt{10}-10)}{3} & \frac{227}{54}-\frac{28\cdot\sqrt{10}}{27} & 2\cdot x-\frac{20\cdot\sqrt{10}+133}{54} & \frac{20\cdot\sqrt{10}+133}{108} & \frac{-(20\cdot\sqrt{10}+133)}{54} \\ \frac{\sqrt{10}+10}{3} & \frac{28\cdot\sqrt{10}}{27}+\frac{227}{54} & 2\cdot x+\frac{20\cdot\sqrt{10}-133}{54} & \frac{-(20\cdot\sqrt{10}-133)}{108} & \frac{20\cdot\sqrt{10}-133}{54} \end{bmatrix}$ |
| 3(d) | *methods_diffcalc\solve_grad*$(f(x),x,0)$<br>▸ Maximal Domain: $-\infty<x<\infty$<br>▸ Derivative: $\frac{-3\cdot x^2}{2}+10\cdot x-13$<br>▸ Found points with gradient 0 : (2)<br>$\begin{bmatrix} \frac{-(\sqrt{22}-10)}{3} & \frac{227}{54}-\frac{22\cdot\sqrt{22}}{27} \end{bmatrix}$<br>$\begin{bmatrix} \frac{\sqrt{22}+10}{3} & \frac{22\cdot\sqrt{22}}{27}+\frac{227}{54} \end{bmatrix}$ |

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| 4(a) | Define $f(x)=\frac{x+2}{x+3}$ *Done*<br>Define $g(x)=\sqrt{5-x}$ *Done*<br>domain$(f(x),x)$ $x\neq -3$<br>domain$(g(x),x)$ $-\infty<x\leq 5$ |
| 4(c) | Note: function is monotonic so we can substitute in the range of $g$ to obtain the range of $f(g(x))$.<br>$f(0)$ $\frac{2}{3}$<br>$f(\infty)$ undef |
| 4(d) | solve$(f(x)\leq 5,x)$ $x\leq \frac{-13}{4}$ or $x>-3$ |
| 5(a) | _diffcalc\solve_smooth$(f1(x),f2(x),x,\{a,b\},1)$<br>Left Derivative: $3\cdot a\cdot x^2+1$<br>Right Derivative: $3\cdot a+b$<br>"At x=1:" "Left Func." "Right Func."<br>"Value:" $a+1$ $3\cdot a+b-2$<br>"Gradient:" $3\cdot a+1$ $3\cdot a+b$<br>Solutions:<br>$a=1$ and $b=1$ |
| 5(b)(i) | Define $a=1$ *Done*<br>Define $b=1$ *Done*<br>zeros$(f1(x),x)\|x\leq 1$ $\{0\}$<br>zeros$(f2(x),x)\|x\geq 1$ $\{\ \}$ |
| 5(b)(ii) | $\frac{d}{dx}(f1(x))\|x=0$ 1 |