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VCE Mathematical Methods ¾
AOS 2 Revision [2.0]
Contour Check (Part 1) Solutions

Contour Check

Section A: [2.1] - Differentiation I (Checkpoints)

Sub-Section [2.1.1]: Find Instantaneous Rate of Change and Average Rate of Change

Question 1

a. Find the average rate of change of $f(x) = x^3 + 3x - 2$ over the interval $[0, 2]$.

$$\text{Average rate of change} = \frac{f(2) - f(0)}{2 - 0} = \frac{8 + 6 - 2 + 2}{2} = 7.$$

b. Let $f(x) = \sqrt{x} - e^x$. Find $f'(x)$.

$$f'(x) = \frac{1}{2\sqrt{x}} - e^x.$$

c. Find the gradient of the graph of $y = \sin(x) + 3\cos(x)$ at the point $\left(\frac{\pi}{3}, \frac{3+\sqrt{3}}{2}\right)$.

$$\text{Gradient} = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \cos(x) - 3\sin(x)\Big|_{x=\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) - 3\sin\left(\frac{\pi}{3}\right) = \frac{1 - 3\sqrt{3}}{2}$$

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Question 2

a. Let $y = \tan(x)$, use the quotient rule to show that $\frac{dy}{dx} = \frac{1}{\cos^2(x)}$.

As $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we see that,

$$\frac{dy}{dx} = \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

b. Find the gradient of $y = \sqrt{4 - x^2}$ at the point $(-1, \sqrt{3})$.

$$\frac{dy}{dx} = \frac{-x}{\sqrt{4 - x^2}}.$$

Thus when $x = -1$, $\frac{dy}{dx} = \frac{1}{\sqrt{4-1}} = \frac{1}{\sqrt{3}}$

c. Let $f(x) = -x\log_e(x)$. At what point is the gradient of f equal to 2?

$f'(x) = -x\frac{1}{x} - \log_e(x) = -1 - \log_e(x)$.

We solve $f'(x) = 2 \implies \log_e(x) = -3 \implies x = e^{-3}$.

Hence the gradient of f is equal to 2 at the point $(e^{-3}, 3e^{-3})$.

d. Let $f(x) = e^{x^2+2}$, find $f'(x)$.

$$f'(x) = 2xe^{x^2+2}.$$

e. Let $f(x) = \cos^2(x)$. Find $f'\left(\frac{\pi}{3}\right)$.

$f'(x) = -2\sin(x)\cos(x)$.

Hence $f'(\frac{\pi}{3}) = -2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Question 3

a. Let $y = \frac{e^{-x}}{\sin(2x^2)}$. Find and simplify $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-e^{-x}\sin(2x^2) - 4xe^{-x}\cos(2x^2)}{\sin^2(2x^2)} = \frac{-e^{-x}}{\sin(2x^2)}\left(1 + \frac{4x}{\tan(2x^2)}\right)$$

b. Let $f(x) = (x-3)^4(x^3 - 5x^2 + 1)$. Find $f'(2)$.

$f'(x) = 4(x-3)^3(x^3 - 5x^2 + 1) + (3x^2 - 10x)(x-3)^4$.

Hence $f'(2) = 4(-1)^3(8 - 20 + 1) + (12 - 20)(-1)^4 = -4(-11) + (-8) = 36$

c. Let $f(x) = \sqrt{\sin(4x) + 2}$. Find all values of $x \in [0, \pi]$ such that $f'(x) = 0$.

$$f'(x) = \frac{2\cos(4x)}{\sqrt{2 + \sin(4x)}}.$$

Observe that $f'(x) = 0 \iff \cos(4x) = 0 \implies x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

d. Evaluate $\frac{d}{dx}(\log_e(x)\log_e(x^2 + 3x + 4))$.

$$\frac{d}{dx}\left(\log_e(x)\log_e(x^2 + 3x + 4)\right) = \frac{(2x+3)\log_e(x)}{x^2 + 3x + 4} + \frac{\log_e(x^2 + 3x + 4)}{x}.$$

e. Let $f(x) = \frac{(xe^x)^2}{x-1} + 2x$. Solve $f'(x) = 2$ for x.

```
In[1]:= f[x_] := 2 x + 1 / (x - 1) * (x * E ^ x) ^ 2
        f'[x]
```

$$\text{Out[2]=}\ 2 + \frac{2e^{2x}x}{-1+x} - \frac{e^{2x}x^2}{(-1+x)^2} + \frac{2e^{2x}x^2}{-1+x}$$

$$\text{Solve}\left[2 + \frac{2e^{2x}x}{-1+x} - \frac{e^{2x}x^2}{(-1+x)^2} + \frac{2e^{2x}x^2}{-1+x} == 2,\ x\right]$$

$$\left\{\{x \to 0\}, \left\{x \to \frac{1}{4}(1 - \sqrt{17})\right\}, \left\{x \to \frac{1}{4}(1 + \sqrt{17})\right\}\right\}$$

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Question 4

Let $f(x) = \frac{\cos(e^{-x}\log_e(x))}{\sin(e^{-x}\log_e(x))}$.

Show that $f'(a) = 0$ implies that $\frac{1}{a} = \log_e(a)$.

Observe that $f(x) = g \circ h(x)$ where $h(x) = e^{x}\log_e(x)$ and $g(x) = \dfrac{\cos(x)}{\sin(x)}$.
By the chain rule, $f'(x) = h'(x)g'(h(x))$.
As $h'(x) = e^{-x}\left(\dfrac{1}{x} - \log_e(x)\right)$, we observe that if $h(a) = 0$, then $\dfrac{1}{a} = \log_e(a)$.
It is sufficient to show that $g'(h(x)) \neq 0$ for all x.
Since $g'(x) = \dfrac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = -\dfrac{1}{\sin^2(x)} < 0$ for all x, we see that $g'(h(x)) \neq 0$ for all x.
Hence if $f'(a) = 0$, then $h'(a) = 0$ hence $\dfrac{1}{a} = \log_e(a)$.

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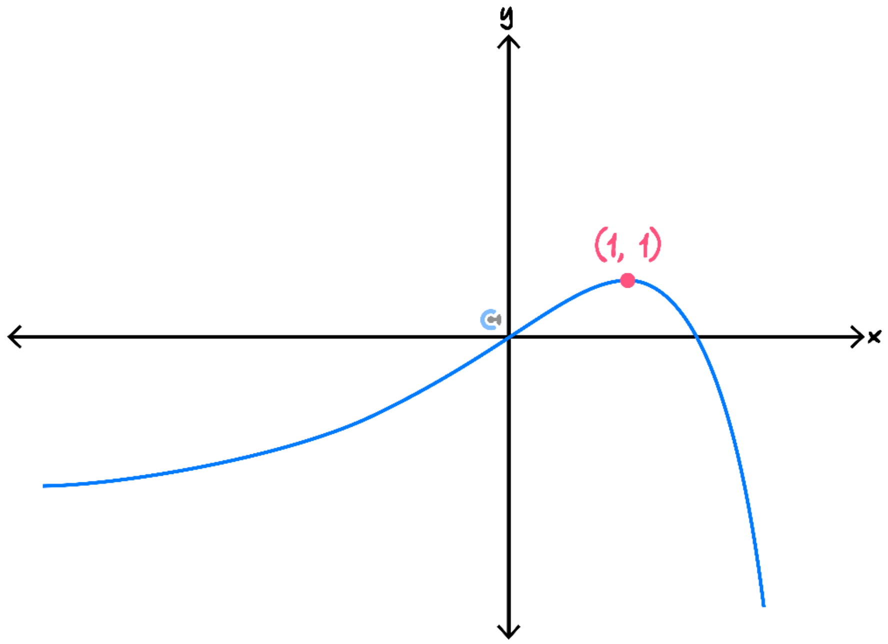
Sub-Section [2.1.2]: Identify the Nature of Stationary Points and Trend

Question 5

The graph of $f(x)$ is drawn below.



a. State the nature of the stationary point when $x = 1$.

Local maximum.

b. State the values of x for which $f(x)$ is strictly increasing.

$x \leq 1$

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Question 6

Let $f(x) = 2x^3 + 3x^2 - 12x + 5$.

a. Find the stationary points of f.

$f'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$.
Thus if $f'(x) = 0$ then $x = -2, 1$.
Hence the stationary points of $f(x)$ are $(-2, f(-2)) = (-2, 25)$ and $(1, f(1)) = (1, -2)$.

b. State the nature of the stationary points.

As $f(x)$ is a positive cubic, it's leftmost stationary point $(-2, 25)$ is a local maximum, whilst it's rightmost stationary point $(1, -2)$ is a local minimum.

c. Hence, state the values of x for which $f(x)$ is strictly decreasing.

$-2 \leq x \leq 1$

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Question 7

Let $f(x) = e^{1+4x-3x^2}$.

a. Find the stationary points of $f'(x)$.

$f'(x) = (4-6x)e^{1+4x-3x^2} = g(x)$.
$g'(x) = -6e^{1+4x-3x^2} + (4-6x)^2 e^{1+4x-3x^2}$.
If $g'(x) = 0$, then $(4-6x)^2 = 6 \implies x = \dfrac{4 \pm \sqrt{6}}{6}$.
Thus the stationary points of $f'(x)$ are $\left(\dfrac{4-\sqrt{6}}{6}, f'\left(\dfrac{4-\sqrt{6}}{6}\right)\right) = \left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$ and $\left(\dfrac{4+\sqrt{6}}{6}, f'\left(\dfrac{4+\sqrt{6}}{6}\right)\right) = \left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$

b. State the nature of the stationary points of $f'(x)$.

We apply a sign test using 3 observations.

- $\dfrac{4-\sqrt{6}}{6} < \dfrac{2}{3} < \dfrac{4+\sqrt{6}}{6}$, and $f'(\frac{2}{3}) = 0$.
- When $x < \dfrac{3}{2}$ that $f'(x) > 0$ and when $x > \dfrac{3}{2}$ that $f'(x) < 0$.
- When $x \to \pm\infty$, $f'(x) \to 0$.

From these observations we see that the graph of $f'(x)$ graph initially goes up to $\left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$, then comes down through $\left(\dfrac{2}{3}, 0\right)$ to $\left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$, lastly going back up approaching 0.

Hence $\left(\dfrac{4-\sqrt{6}}{6}, \sqrt{6}e^{\frac{23}{6}}\right)$ is a local maximum of the graph of $f'(x)$, and $\left(\dfrac{4+\sqrt{6}}{6}, -\sqrt{6}e^{\frac{23}{6}}\right)$ is a local minimum.

c. Hence, state the values of x for which $f'(x)$ is strictly increasing.

$x < \frac{4-\sqrt{6}}{6}$ or $x > \frac{4+\sqrt{6}}{6}$.

Question 8

Let $f(x) = x^{\frac{10}{3}}$.

State the values for which $g(x) = f'(x) - f(x)$ is strictly increasing.

Observe that $g(x) = \frac{10}{3}x^{\frac{7}{3}} - x^{\frac{10}{3}}$.

For stationary points we require $g'(x) = \frac{10}{3}\left(\frac{7}{3}x^{\frac{4}{3}} - x^{\frac{7}{3}}\right) = 0$.

Hence $x = 0$ or $x = \frac{7}{3}$.

Since $g(-1) = \frac{-13}{3}$ and $g(1) = \frac{7}{3}$, we see that $(0, 0)$ is a stationary point of inflection.

Since $g(3) = 3^{\frac{4}{3}} < g\left(\frac{7}{3}\right) = \left(\frac{7}{3}\right)^{\frac{7}{3}}$ and $g(0) < g\left(\frac{7}{3}\right)$ we see that $\left(\frac{7}{3}, g\left(\frac{7}{3}\right)\right)$ is a local maximum.

Hence g is strictly increasing for $x \leq \frac{7}{3}$.

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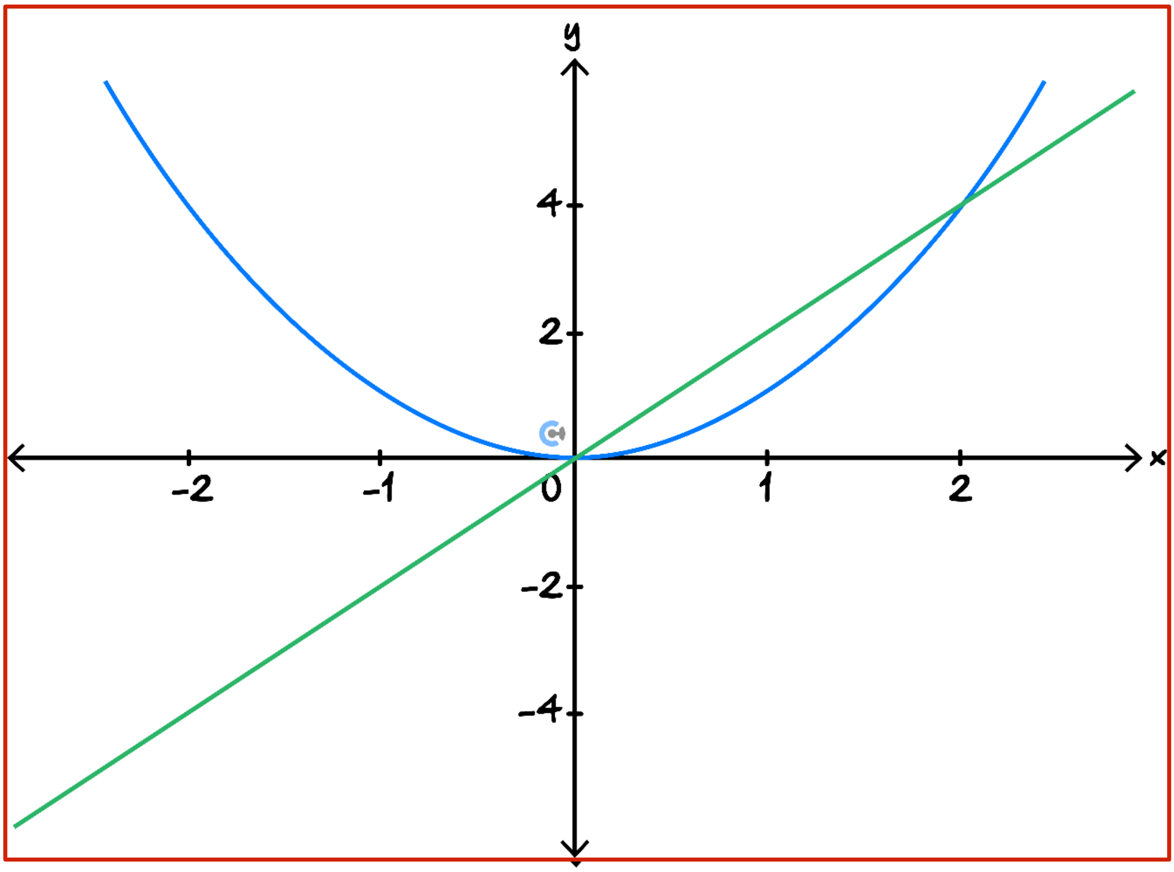
Sub-Section [2.1.3]: Graph Derivative Functions

Question 9

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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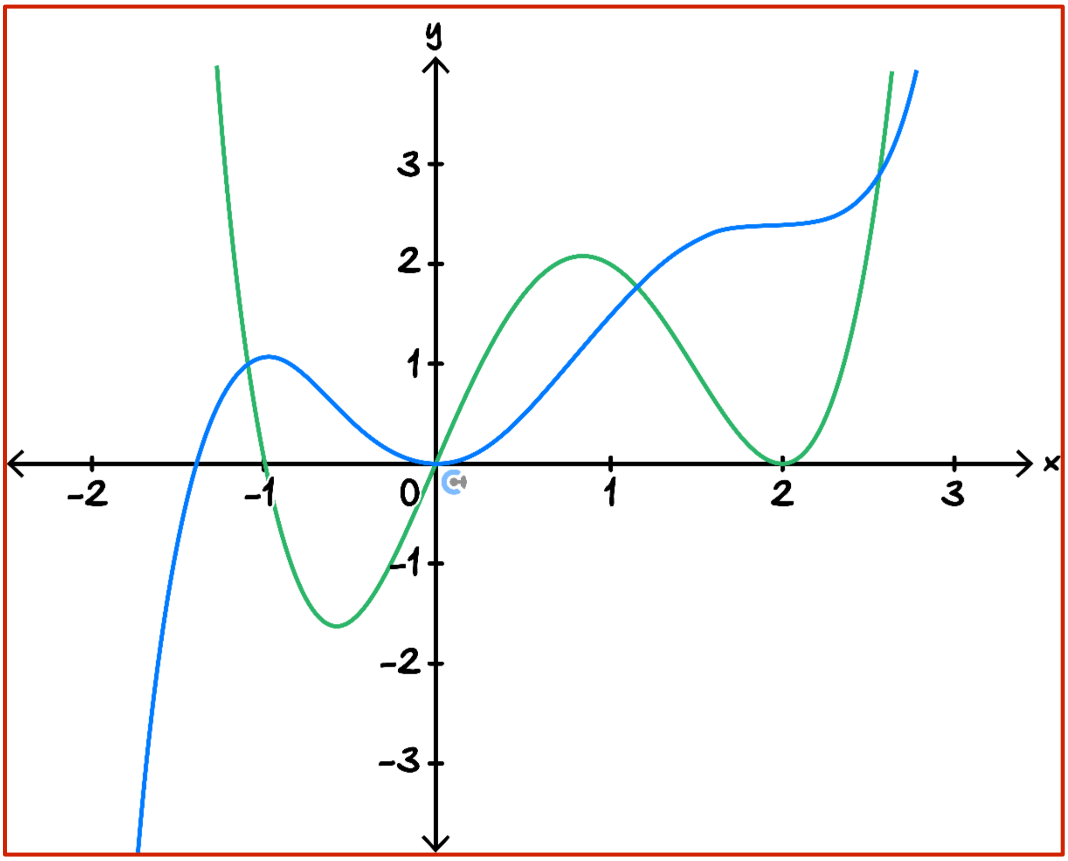
Question 10

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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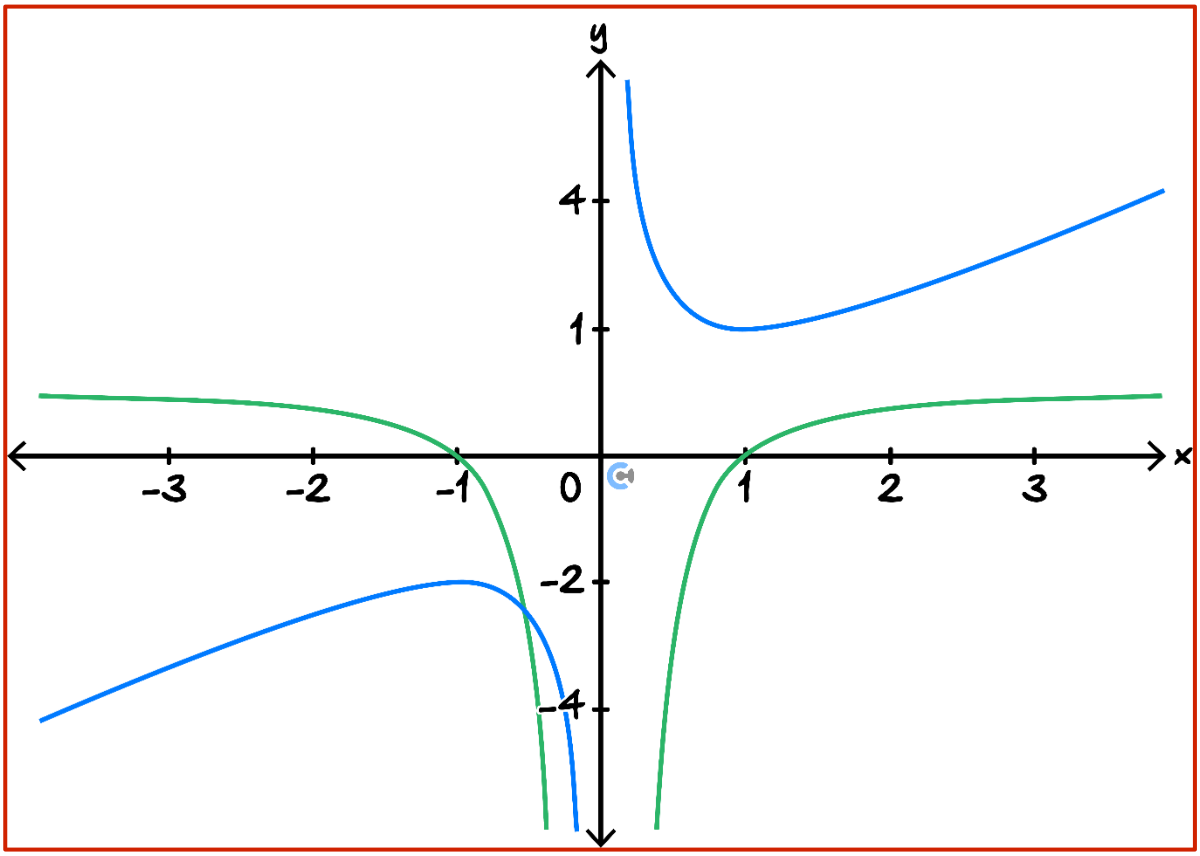
Question 11

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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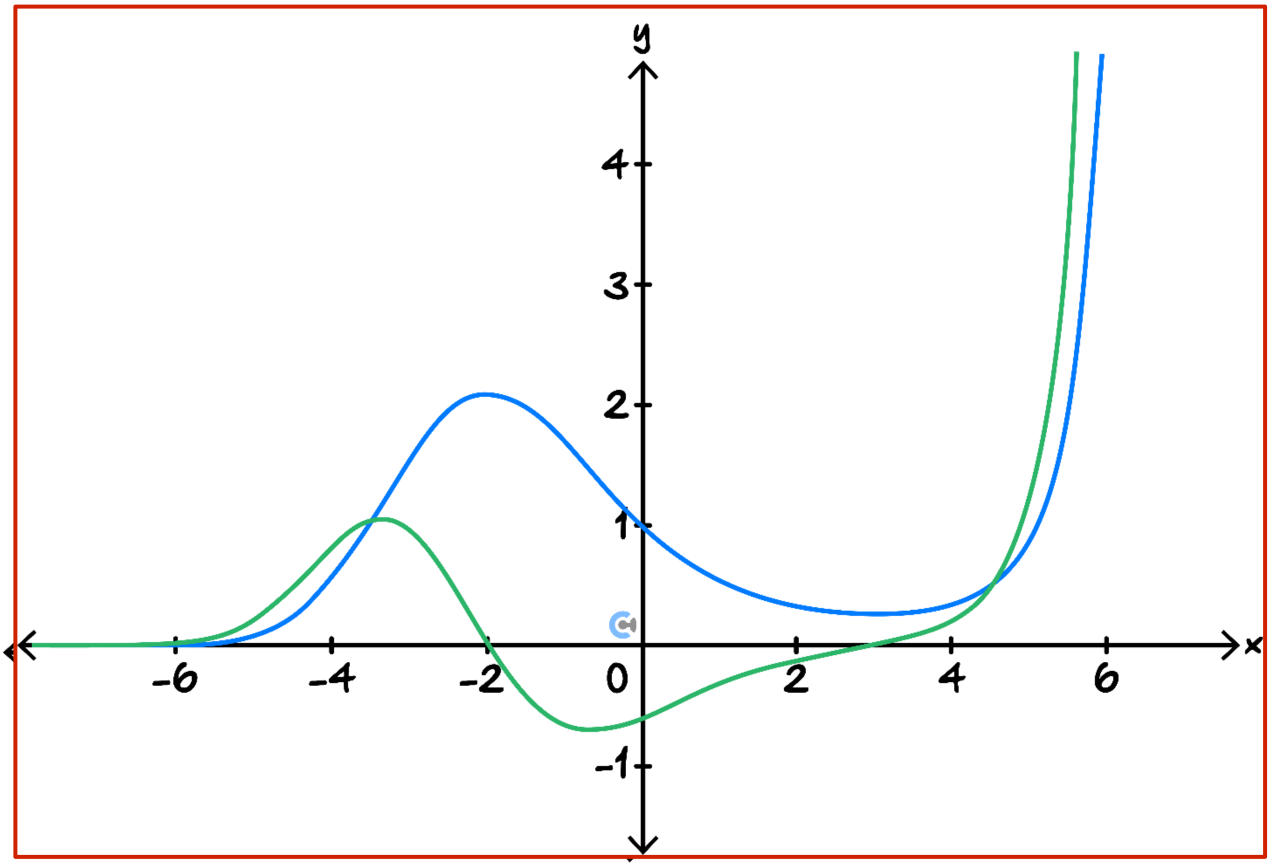
Question 12

The graph of $f(x)$ is drawn below.

Draw the graph of $f'(x)$ on the same axes.



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Section B: [2.2] - Differentiation II (Checkpoints)

Sub-Section [2.2.1]: Evaluate Limits and Find Points Where the Function is not Continuous

Question 13

Evaluate the following limits:

a. $\lim_{x \to 3}(x^3 - 2x^2 + 5)$

As $x^3 - 2x^2 + 5$ is continuous at $x = 3$, we may substitute $x = 3$ to obtain $\lim_{x \to 3}\left(x^3 - 2x^2 + 5\right) = 14$.

b. $\lim_{x \to 4}(2^{\sqrt{x}} + \log_3(x^3 + 2x))$

As $2^{\sqrt{x}} + \log_3\left(x^3 + 2x\right)$ is continuous at $x = 4$, we may substitute $x = 4$ to obtain $\lim_{x \to 4}\left(2^{\sqrt{x}} + \log_3\left(x^3 + 2x\right)\right) = 4 + \log_3(72)$

c. $\lim_{x \to 3}(f(x))$, where:

$$f(x) = \begin{cases} 2x + 1, & x < 3 \\ 3x - 2, & x \geq 3 \end{cases}$$

At $x = 3$, we can check that $\lim_{x \to 3^-} f(x) = 2(3) + 1 = 7$ and $\lim_{x \to 3^+} f(x) = 3(3) - 2 = 7$. As the left and right limits are equal, we may conclude that $\lim_{x \to 3} f(x) = 7$.

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Question 14

Find the points x for which the following functions are **discontinuous** and state a reason as to why they are discontinuous.

a. $f(x) = \begin{cases} x, & x < 1 \\ x+1, & x \geq 1 \end{cases}$

Note that $\lim_{x \to 0^-} f(x) = 1$ but $\lim_{x \to 0^+} f(x) = 2$. Since the left and right limits are not equal at $x = 1$, we conclude that $f(x)$ is discontinuous at $x = 1$.

b. $f(x) = \frac{50}{x^2-7x+6}$

We solve $x^2 - 7x + 6 = 0$ to find that $x = 1$ or $x = 6$. Thus, $f(x)$ is discontinuous at $x = 1$ and $x = 6$ as the function is not defined at these points.

c. $f(x) = \frac{x^2-4x+3}{x-3}$

We solve $x - 3 = 0$ to find that $x = 3$. Thus, $f(x)$ is not continuous at $x = 3$ as it is not defined at this point.
Note that $\lim_{x \to 3} f(x)$ exists, but the function is not defined at $x = 3$.

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Question 15

Consider the following function $f(x)$ with the rule:

$$f(x) = \begin{cases} 3^{x-2} + 5x, & x < 2 \\ ax + 6, & x \geq 2 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Note that $f(x)$ is already continuous for all $x \neq 2$, and all that remains is to make $f(x)$ continuous at $x = 2$. Thus, we need $3^{2-2} + 5 \cdot 2 = 2a + 6$, so that $a = 5/2$.

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Question 16

Consider the following function $f(x)$ with the rule:

$$f(x) = \begin{cases} x^2 - 4x - 12, & x < 7 \\ a^2 - ax + 1, & 7 \le x < 10 \\ -x - 5, & x \ge 10 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Since $f(x)$ is continuous, we require that the right and left limits for $x = 7$ should be equal – i.e. $f(7^+) = f(7^-)$. Hence, we obtain the equation $9 = a^2 - 7a + 1$, i.e. $a^2 - 7a - 8 = 0$. Thus, $a = -1$ or $a = 8$.

Furthermore, we require that the right and left limits for $x = 10$ should be equal – i.e. $f(10^+) = f(10^-)$. Hence, we obtain a new equation $-15 = a^2 - 10a + 1$, i.e. $a^2 - 10a + 16 = 0$. Therefore, $a = 2$ or $a = 8$.

Since we require that $f(x)$ is continuous for all $x \in \mathbb{R}$, we need the value of a which satisfies the equations at both $x = 7$ and $x = 10$, so the final answer is $a = 8$.

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Sub-Section [2.2.2]: Apply Differentiability to Find Points Where Functions are not Differentiable, Domain of the Derivative and Unknowns of a Function

Question 17

Find the values of x such that the following functions are not differentiable.

a. $f(x) = \begin{cases} -x+5, & x < 2 \\ x+1, & x \geq 2 \end{cases}$

By inspection of the sketch, there is a sharp point $x = 2$. Hence, the function is not differentiable at $x = 2$.

b. $f(x) = \frac{1}{x^2-4x+3}$

By inspection of the sketch, the function is not defined whenever the denominator vanishes, i.e. at $x = 1$ and $x = 3$. Thus, the function is not differentiable at $x = 1$ and $x = 3$.

c. $f(x) = \begin{cases} 2x, & x < 0 \\ 2x+1, & x \geq 0 \end{cases}$

By inspection, we can see a discontinuity at $x = 0$. If a function is not continuous at a given point, it is also not differentiable there either.

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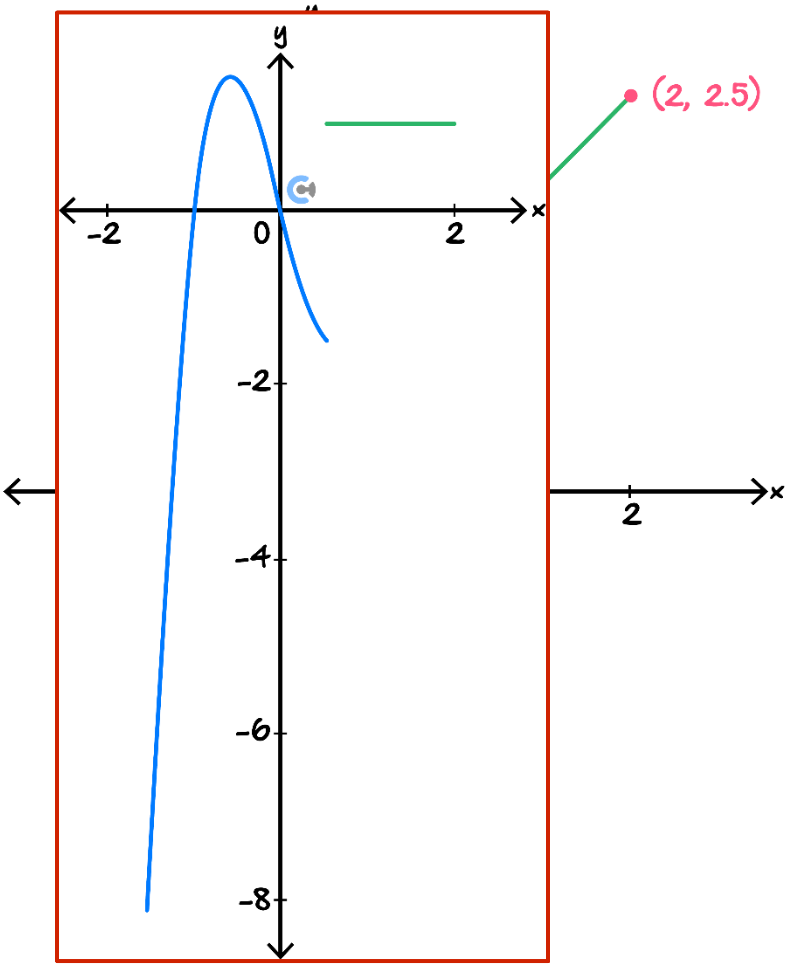
Question 18

Consider the following function.



a. Sketch the corresponding derivative function on the same set of axes above.

b. Furthermore, state the domain of the derivative function.

The function will be differentiable everywhere except the endpoints and the discontinuity, i.e. $\text{dom } f' = (-1.5, 2)\backslash\{0.5\}$.

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Question 19

Consider the following function $f(x)$ with the rule:

$$f(x) = \begin{cases} 2x^2 - 6x + 5, & x < 2 \\ ax + b, & x \geq 2 \end{cases}$$

Find the value of a and b such that $f(x)$ is differentiable at $x = 2$.

Since the two rules must join at $x = 2$, imposing $f(2^+) = f(2^-)$ gives $2a + b = 1$. Also, $f'(2^+) = f'(2^-)$ gives $a = 2$. By solving the resulting system of equations, we conclude that $a = 2$ and $b = -3$.

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Question 20

Consider the following function $f(x)$ with rule:

$$f(x) = \begin{cases} x^3 - 3x + 5, & x < -1 \\ g(x), & -1 \le x < 1 \\ x^2 - 5x + 2, & x \ge 1 \end{cases}$$

The goal for this question is to find a suitable rule $g(x)$ making $f(x)$ differentiable for all $x \in \mathbb{R}$.

a. State the four equations that $g(x)$ and $g'(x)$ must satisfy at $x = 1$ and $x = -1$.

$g(1) = -2, g'(1) = -3, g(-1) = 7$ and $g'(-1) = 0$.

b. A natural choice would be to let $g(x)$ be a polynomial. As there are four equations that need to be satisfied, explain why it is suitable to set $g(x)$ to be a cubic polynomial.

A cubic polynomial $y = ax^3 + bx^2 + cx + d$ has four parameters, which should possibly give a unique solution for a, b, c and d based on the system of four equations obtained in the previous part.

c. Hence, find a suitable rule for $g(x) = ax^3 + bx^2 + cx + d$ assuming $g(x)$ is a polynomial. It may be necessary to use a CAS to solve the system of equations obtained in the working.

By using the first question, we see that a, b, c and d must satisfy

$$\begin{aligned} a + b + c + d &= -2 \\ 3a + 2b + c &= -3 \\ -a + b - c + d &= 7 \\ 3a - 2b + c &= 0 \end{aligned}$$

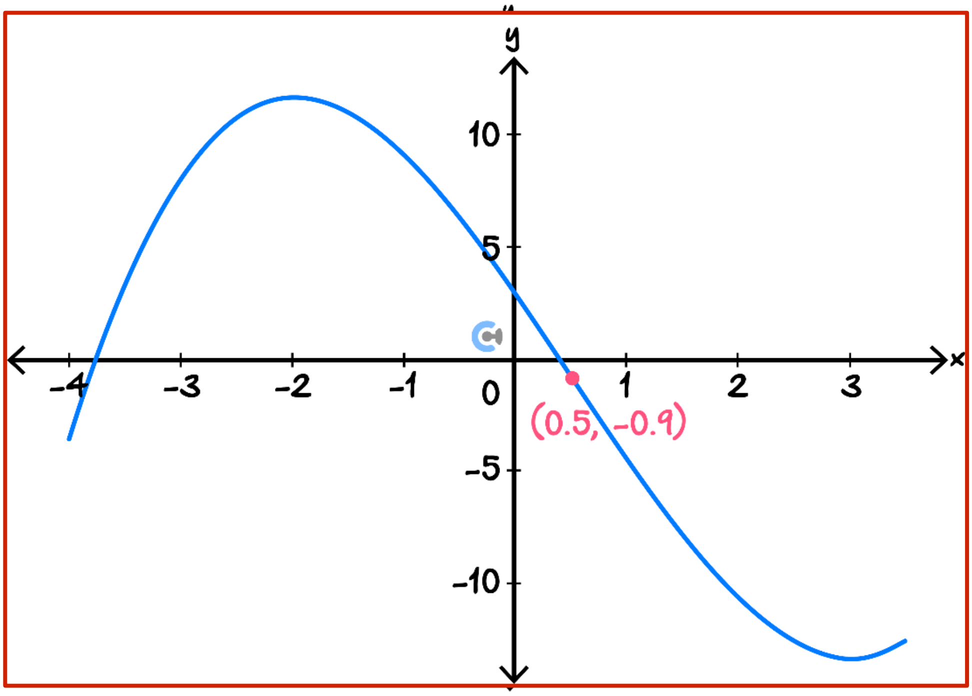
Hence, $a = \frac{3}{2}, b = -\frac{3}{4}, c = -6$ and $d = \frac{13}{4}$. Furthermore, $g(x) = \frac{3}{2}x^3 - \frac{3}{4}x^2 - 6x + \frac{13}{4}$ is a possible rule for the polynomial.

Note: This is a common technique called cubic splines used for interpolation.

Sub-Section [2.2.3]: Identify Concavity and Find Inflection Points

Question 21

Consider the following graph for $f(x)$.



a. Circle the point of inflection on the above graph.

b. State the values of x such that the function is concave up.

Notice how the gradient of the function increases after $x = 1/2$ (i.e. it is becoming less negative, and after the turning point becomes more positive). So, the answer is $x > 1/2$.

c. State the values of x such that the function is concave down.

Similar to above, $x < 1/2$.

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Question 22

Consider a function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 - 2x^3 - 36x^2 + 5x + 1$.

a. Calculate the second derivative of the function $f(x)$.

$f''(x) = 12x^2 - 12x - 72 = 12(x^2 - x - 6) = 12(x-3)(x+2)$.
Note: You did not need to factorise the final answer, but it is helpful for the following question where you need to solve for the points of inflection.

b. Find the points of inflection of the function $f(x)$.

The points of inflection occur where $f''(x) = 0$, i.e. when $12(x-3)(x+2) = 0$. Therefore, the points of inflection occur when $x = -2$ or $x = 3$. Note also that $f''(2.9) < 0$, $f''(3.1) > 0$, $f''(-2.1) > 0$ and $f''(-1.9) < 0$, so $f''(x)$ does indeed switch signs around $x = -2$ and $x = 3$.

c. Find the values of x where the function is concave up.

We solve for the values of x such that $f''(x) > 0$. Thus, the function is concave up whenever $x < -2$ or $x > 3$
Note: Remember to exclude the endpoints!

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Question 23

Suppose that a function $f(x)$ is double differentiable for all $x \in (0,2)$, and satisfies the following properties:

- $f''(1) = 0$
- $f'(0) = 1$
- $f'(0.5) = 0$
- $f'(0.75) = -0.71$
- $f'(1) = -1$
- $f'(1.25) = -0.71$
- $f'(1.5) = 0$

Find the values of x such that the function is concave up.

Notice that the point of inflection occurs at $x = 1$ and that the gradient starts increasing after $x > 1$ (e.g. it becomes less negative). Thus, the values of x such that the function is concave up is $x \in (1, 2)$.

Question 24

Find a rule of a polynomial $g(x)$ so that $g(0) = 12,\ g(1) = 9,\ g(2) = 0$, and so that there is a point of inflection when $x = 2$.

Since there are four equations, we should use a polynomial that involves four parameters – i.e. a cubic. Hence, assume $g(x) = ax^3 + bx^2 + cx + d$. Based on the conditions, we obtain four equations

$$\begin{aligned} d &= 1 \\ a + b + c + d &= 9 \\ 8a + 4b + 2c + d &= 0 \\ 12a + 2b &= 0 \end{aligned}$$

The last equation comes from substituting $x = 2$ into $g''(x) = 6a + 2b$. Solving these equations gives us the following solution $a = 1, b = -6, c = 2$ and $d = 12$. Therefore, a possible rule for the cubic could be $x^3 - 6x^2 + 2x + 12$.

Section C: [2.3] – Differentiation Exam Skills (Checkpoints)

Sub-Section [2.3.1]: Find General Derivatives with Functional Notation

Question 25

If f is a differentiable function, find $\frac{dy}{dx}$ for the following:

a. $y = f(x)\tan(x)$

$$\frac{dy}{dx} = f'(x)\tan(x) + \frac{f(x)}{\cos^2(x)}$$

b. $y = \sqrt{f(x)}$

$$\frac{dy}{dx} = \frac{f'(x)}{2\sqrt{f(x)}}$$

Question 26

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = f(e^x) \cdot g(x)$

$$\frac{dy}{dx} = e^x f'(e^x)g(x) + g'(x)f(e^x)$$

b. $y = f(g(\cos(3x))$

$$\frac{dy}{dx} = -3\sin(3x)g'(\cos(3x))f'(g(\cos(3x)))$$

Question 27

If f and g are differentiable functions, find $\frac{dy}{dx}$ for the following:

a. $y = \sqrt{f(3x^2) + g(2x + f(x))}$

$$\frac{dy}{dx} = \frac{6xf'(3x^2) + (2 + f'(x))g'(2x + f(x))}{2\sqrt{f(3x^2) + g(2x + f(x))}}$$

b. $y = \frac{e^{f(x^2)}}{g(f(x^2)) + f(x^2)}$

$$\frac{dy}{dx} = 2xf'(x^2)\frac{e^{f(x^2)}(g(f(x^2)) + f(x^2)) - (g'(f(x^2)) + 1)e^{f(x^2)}}{(g(f(x^2)) + f(x^2))^2}$$

Question 28

If f and g are differentiable increasing functions, with $g'(x)$ also being one-to-one, what is the maximum amount of stationary points that $y = f(x) + 3x + g(-f(x) - 3x)$ has?

We know that $\frac{dy}{dx} = f'(x) + 3 - (f'(x) + 3)g'(-f(x) - 3x) = (f'(x) + 3)(1 - g'(-f(x) - 3))$.
If we solve it to be 0, since $f'(x) \geq 0$, we must have $(1 - g'(-f(x) - 3)) = 0$.
Since $g'(x)$ is one to one, and so is $-f(x) - 3$, we will only have one stationary point.

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Sub-Section [2.3.2]: Apply Differentiability to Join Two Functions Smoothly

Question 29

A hybrid function is defined as:

$$f(x) = \begin{cases} e^{2x} - 2, & x < 0 \\ ax + b, & x \geq 0 \end{cases}$$

Find the values of a and b such that $f(x)$ is smooth and continuous at $x = 0$.

Continuous: $b = -1$ Smooth: $a = 2$ So $a = 2$ and $b = -1$

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Question 30

A hybrid function is defined as:

$$f(x) = \begin{cases} \log_e(ax), & x < 1 \\ bx^2, & x \geq 1 \end{cases}$$

Where $a > 0$. Find the values of a and b such that $f(x)$ is both continuous and differentiable at $x = 1$.

Continuous: $\log_e(a) = b$
Smooth: $2b = 1$
Solving simultaneously yields $b = \frac{1}{2}$ and $a = \sqrt{e}$.

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Question 31

A hybrid function, $f : \mathbb{R} \to \mathbb{R}$, is defined as:

$$f(x) = \begin{cases} 2x + 4 & x < -2 \\ ax^3 + bx^2 + cx + d & -2 \le x \le 2 \\ x^2 - 6x + 10 & x > 2 \end{cases}$$

Find the values of a, b, c and d such that $f(x)$ is both continuous and smooth over its entire domain.

Let $g(x) = ax^3 + bx^2 + cx + d$.
For continuity we require $g(-2) = 0$ and $g(2) = 2$. This yields the equations,

$$-8a + 4b - 2c + d = 0 \quad \text{and} \quad 8a + 4b + 2c + d = 2$$

For smoothness we require $g'(-2) = 2$ and $g'[2] = -2$. This yields the equations,

$$12a - 4b + c = 2 \quad \text{and} \quad 12a + 4b + c = -2$$

Solving simultaneously yields $a = -\frac{1}{16}, b = -\frac{1}{2}, c = \frac{3}{4}$ and $d = 3$.

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Question 32 Tech-Active.

a. A hybrid function $f : \mathbb{R} \to \mathbb{R}$, is defined as:

$$f(x) = \begin{cases} \sin(x) + 3 & x < 0 \\ g_1(x) & 0 \leq x < 1 \\ g_2(x) & 1 \leq x < 2 \\ g_3(x) & 2 \leq x < 3 \\ \log_e\left(\frac{e^2x^3}{27}\right) & x \geq 3 \end{cases}$$

Where g_1, g_2 and g_3 are cubic polynomials. Find g_1, g_2, g_3 if both f and f' are smooth on $\mathbb{R}$.

We solve the following equations simultaneously,

$$g_1(0) = 3, g_1'(0) = 1, g_1''(0) = 0$$
$$g_2(1) = g_1(1), g_2'(1) = g_1'(1), g_2''(1) = g_1''(1)$$
$$g_3(2) = g_2(2), g_3'(2) = g_2'(2), g_3''(2) = g_2''(2)$$
$$2 = g_3(3), 1 = g_3'(3), \frac{1}{3} = g''(3)$$

Thus,

$$g_1(x) = -\frac{37}{54}x^3 + x + 3$$
$$g_2(x) = \frac{151}{108}x^3 - \frac{25}{4}x^2 + \frac{29}{4}x + \frac{11}{12}$$
$$g_3(x) = -\frac{83}{108}x^3 + \frac{27}{4}x^2 - \frac{75}{4}x + \frac{73}{4}$$

b. A different hybrid function, $h : \mathbb{R} \to \mathbb{R}$, is defined as:

$$h(x) = \begin{cases} \sin(x) + 3 & x < 0 \\ g_4(x) & 0 \leq x < 3 \\ \log_e\left(\frac{e^2x^3}{27}\right) & x \geq 3 \end{cases}$$

Where g_4 is a polynomial. If both h and h' are smooth on $\mathbb{R}$, what is the minimum degree of $g_4(x)$?

Degree 5, we will have 6 equations, and thus require 6 unknowns.

Section D: [2.4] - Applications of Differentiation (Checkpoints)

Sub-Section [2.4.1]: Find Tangents and Normals

Question 33

Find the equation of the normal to the graph of $f(x) = \cos(5x)$ at the point $x = \frac{\pi}{4}$.

We first calculate the derivative $f'(x) = -5\sin(5x)$. Thus, the gradient of the tangent is $f'\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$ Hence, the normal has gradient equal to $-\frac{2}{5\sqrt{2}}$. Furthermore, $f\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. Therefore, the equation of the normal is

$$\begin{aligned} y + \frac{\sqrt{2}}{2} &= -\frac{2}{5\sqrt{2}}\left(x - \frac{\pi}{4}\right) \\ \implies y &= -\frac{2}{5\sqrt{2}}x + \frac{\pi}{10\sqrt{2}} - \frac{\sqrt{2}}{2}. \\ \implies y &= -\frac{\sqrt{2}}{5}x + \frac{\sqrt{2}\cdot\pi}{20} - \frac{\sqrt{2}}{2} \end{aligned}$$

Question 34

Find the equation of the normal to the graph of $f(x) = x^2 - 3x - 1$ which has a gradient of $-\frac{1}{5}$.

Firstly, we calculate the derivative $f'(x) = 2x - 3$. If the gradient of the normal is $-\frac{1}{5}$, then the gradient of the tangent at the same point will be equal to 5. Therefore, we solve $2x - 3 = 5$, which gives $x = 4$. Furthermore, $f(4) = 3$. Therefore, the equation of the normal is

$$\begin{aligned} y - 3 &= -\frac{1}{5}(x - 4) \\ \implies y &= -\frac{1}{5}x + \frac{19}{5}. \end{aligned}$$

Question 35

Find the equation of the normal to the graph of $f: (2, \infty) \to \mathbb{R},\ f(x) = x^2 - 2x$ at the point $x = a$. Hence by using a CAS, obtain the equation of the normal that passes through the point $(-1,4)$.

The derivative is $f'(x) = 2x - 2$. The gradient of the tangent is given by $2a - 2$ and furthermore, the function passes through the point $(a, a^2 - 2a)$. Thus, the equation of the normal is

$$\begin{aligned} y - (a^2 - 2a) &= -\frac{1}{2a-2}(x-a) \\ \implies y - (a^2 - 2a) &= -\frac{1}{2a-2}x + \frac{a}{2a-2} \\ \implies y &= -\frac{1}{2a-2}x + \frac{a}{2a-2} + a^2 - 2a \end{aligned}$$

Furthermore, the normal is known to contain the point $(-1, 4)$. Therefore,

$$4 = \frac{1}{2a-2} + \frac{a}{2a-2} + a^2 - 2a$$

Using a CAS, we solve to find that $a = 3$. We reject the other roots as they are not within the specified domain. Therefore, the equation of the normal is

$$y = -\frac{1}{4}x + \frac{15}{4}$$

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Question 36

Consider the function given by $f(x) = e^{x^2} - \cos(x)$.

a. Find the equation of the tangent to the graph of $f(x)$ at the point $x = 1$.

We first calculate $f'(x) = 2xe^{x^2} + \sin(x)$ and thus $f'(1) = 2e + \sin(1)$. Furthermore, $f(1) = e - \cos(1)$. Therefore, we can conclude that the equation of the tangent is given by the rule

$$\begin{aligned} y - e + \cos(1) &= (2e + \sin(1))(x - 1) \\ \implies y &= (2e + \sin(1))x - e - \sin(1) - \cos(1) \end{aligned}$$

b. Without needing to do any further differentiation / solving, find the equation of the normal that passes through the point $x = -1$.

As $f(x)$ is actually an even funtion, we may simply relect the tangent from the above part to obtain the tangent at $x = -1$, i.e. with $(x, y) \to (-x, y)$, we deduce $y = -(2e + \sin(1))x - e - \sin(1) - \cos(1)$.

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Sub-Section [2.4.2]: Find Minimum and Maximum

Question 37

Find the maximum and minimum values of the function $f(x) = x^3 - 3x^2 - 24x + 15$ with domain $x \in [0,5]$.

The global extrema occur either where $f'(x) = 0$ or at an endpoint. Firstly, we consider $f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x-4)(x+2)$. Thus, $f'(x) = 0$ for $x = -2$ or $x = 4$. However, we only consider $x = 4$ as $x = -2$ is not in the domain. Hence, the possible values of x where the global extrema occur are $x = 0, 4$ and 5. Indeed, the values of $f(x)$ at these points are $f(0) = 15$, $f(4) = -65$ and $f(5) = -55$. Thus, the global maximum is 15 and the global minimum is -65

Question 38

Find the maximum area of a rectangle with a perimeter equal to 18 m.

Let x denote the width of the rectangle, then the height of the rectangle must be $9 - x$. Hence, we see the rectangle has area given by $A(x) = x(9-x)$ with $x \in (0,9)$. Solving $A'(x) = 0$ gives us the value $x = 9/2$. As a result, the maximum area of the rectangle is $81/4\ \text{m}^2$.

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Question 39

Find the maximum rate of change of the function $f(x) = -x^3 + 6x^2 + 10x - 5$.

The rate of change of $f(x)$ is given by $f'(x) = -3x^2 + 12x + 10$. To find the minimum rate of change, we should solve for where $f''(x) = 0$. Indeed, $f''(x) = -6x + 12$ and $f''(x) = 0$ if $x = 2$. By inspection, $f'(x)$ is a quadratic with negative leading coefficient, so $x = 2$ corresponds to a maximum. The maximum rate of change is $f'(2) = 22$.

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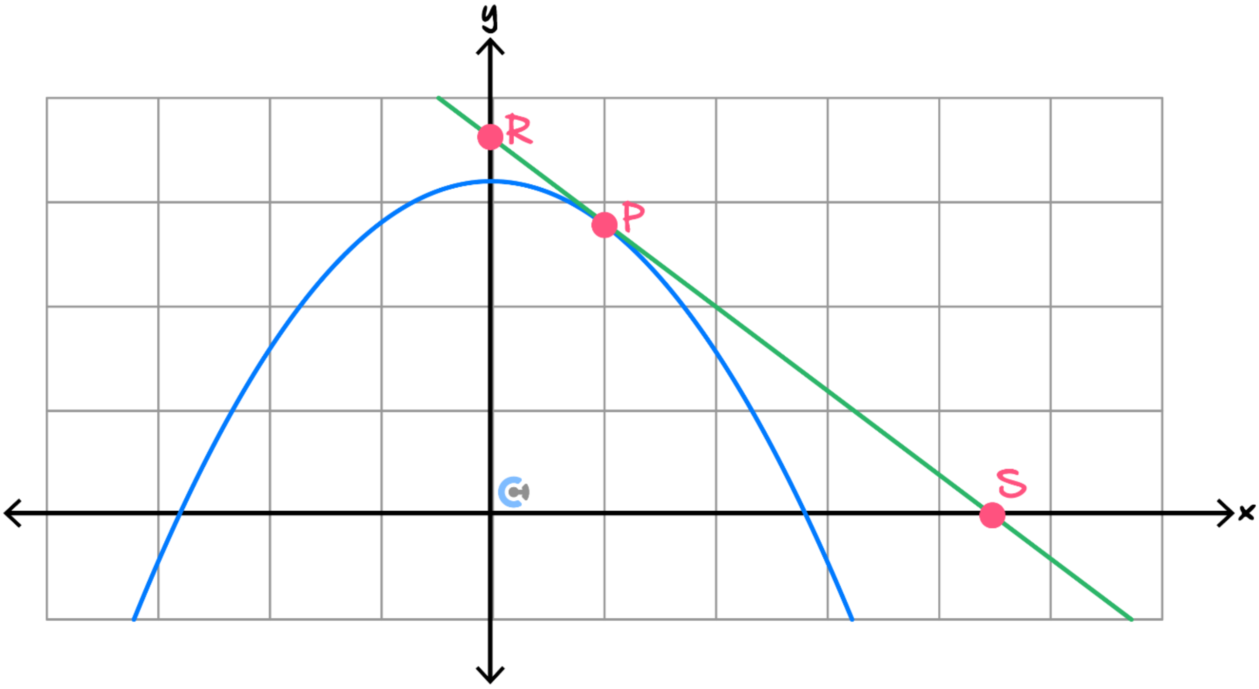
Question 40 Tech-Active.

The diagram below shows the graph of the function $f(x) = 16 - 2x^2$.



The graph of the tangent to the curve at the point $P(p. f(p))$, where $p \in \left[\frac{1}{2}, \frac{5}{2}\right]$ is also shown.

Determine the equation of the tangent line in terms of p.

$$f'(p) = -4p$$

$$y - f(p) = -4p(x - p)$$
$$y - (16 - 2p^2) = -4p(x - p)$$
$$y = -4px + 2p^2 + 16$$

Which is the required equation of tangent.

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Sub-Section [2.4.3]: Apply Newton's Method to Find the Approximation of a Root and its Limitations

Question 41

Approximate the root of the equation $x^3 - 2x^2 + 5x - 6$ using Newton's method with an initial value of $x_0 = 1.2$ and a tolerance level of 0.01. Leave your answer correct to two decimal places.

Begin the algorithm with $x_0 = 1.2 \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.4549$. Note that $|x_1 - x_0| > 0.01$, so we must perform another iteration of Newton's method. Thus, $x_2 = 1.4549 - \frac{f(1.4549)}{f'(1.4549)} = 1.4331$. Again, $|x_2 - x_1| > 0.01$, so we must perform another iteration. Now, $x_3 = 1.4331 - \frac{f(1.4331)}{f'(1.4331)} = 1.4329$. Now, $|x_3 - x_2| < 0.01$, thus we may end Newton's method and approximate the root to be $x = 1.4329$.

Hence, the approximated root is: 1.43

Question 42

Approximate a solution of the equation $e^x = \cos(2x-1)$ using Newton's method with an initial value of $x_0 = -2$. Use only one iteration for your approximation.

We want to approximate a solution to $e^x = \cos(2x-1)$. We can equivalently approximate a root of $f(x) = e^x - \cos(2x-1)$. Thus, by noting that $f'(x) = e^x + 2\sin(2x-1)$, we obtain that $x_1 = (-2) - \frac{e^{-2} - \cos(2\cdot -2-1)}{e^{-2} + 2\sin(2\cdot -2-1)} = -1.93$.

For interest, the root is actually approximately $x = -1.92909$, which you can check on a CAS.

Question 43

Consider the function $f(x) = \sin(x) - e^{2x}$. Explain why it would be unsuitable to choose an initial value that solves the equation $\cos(x) - 2e^{2x}$.

If x_0 solves $\cos(x_0) - 2e^{2x_0} = 0$, then $f'(x_0)$ is undefined, so x_1 would be undefined as well. *Notice also that the tangent to the graph would become parallel to the x-axis, and therefore would not have a (unique) x-intercept, meaning we would not know the value of the next iteration.*

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Question 44

An issue that can arise when using Newton's method is that the derivative may not be easy to calculate.

a. Explain why Newton's method is impractical for approximating the roots of $f(x) = \sin^{-1}\left(x^2 - \frac{\pi}{2}\right)$ within the context of VCE Mathematical Methods Units 3 and 4.

We are not taught how to differentiate $f(x) = \sin^{-1}(x)$ in Methods 3/4. Hence, we would not be able to find the rule for $f'(x)$ needed in the formula.

b. Nevertheless, we can try to use a similar method known as the secant method with an initial guess of $x_0 = 1.5$. Approximate the tangent to $x_0 = 1.5$ by finding the equation of secant (i.e. the straight line) passing through the points $(1.5, 0.7467)$ and $(1.51, 0.7885)$. Your answer should be given to two decimal places.

Firstly, we calculate the gradient is $m = \frac{0.7885-0.7467}{1.51-1.5} = 4.18$. Therefore, $y - 0.7647 = 4.18(x - 1.5)$ gives $y = 4.18x - 5.5233$. The final answer is $y = 4.18x$ -5.51

c. Hence, obtain an approximation for a root of $f(x) = \sin^{-1}\left(x^2 - \frac{\pi}{2}\right)$ based on the line obtained above.

Note that this method approximates the root which is $x = \sqrt{\pi/2} \approx 1.2533$.

We should calculate the x-intercept of the above secant, which is $x = 1.32$.

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Section E: [2.5] - Applications of Differentiation Exam Skills (Checkpoints)

Sub-Section [2.5.1]: Advanced Tangents and Normal Questions

Question 45

a. Find the equation of the tangent to the function $f(x) = 2x^2 + 2x - 3$ that is parallel to the line $y = 3 - 2x$.

$f'(x) = 4x + 2$
$m = -2$
Hence,
$4x + 2 = -2$
$4x = -4$
$x = -1$
$f(-1) = 2(-1)^2 + 2(-1) - 3$
$f(-1) = -3$
Tangent at $(-1, -3)$
$y - (-3) = -2(x + 1)$
$y = -2x - 2 - 3$
$y = -2x - 5$

b. Find the equation of the tangent to the function $f(x) = 3x^2 - 13x + 8$, that is perpendicular to the line $y + x = 3$.

$f'(x) = 6x - 13$
Slope of the tangent $= -1$
Hence, $6x - 13 = -1$
$6x = 12$
$x = 2$
$f(2) = 3(2)^2 - 13(2) + 8 = -6$
Equation of tangent at $(2, -6)$
$y - (-6) = -1(x - 2)$
$y = -x - 4$

c. Find the equation of a tangent to the function $f(x) = e^x$, which makes an angle of $45°$ with the positive x-axis.

Gradient of the tangent $= 1 (\tan 45° = 1)$

$$f'(x) = 1$$
$$e^x = 1$$
$$x = 0$$
$$f(0) = 1$$

Tangent at $(1, 0) =$

$$y - 0 = 1(x - 1)$$
$$y = x - 1$$

Question 46

a. Find the equation of the tangent to the function $f(x) = \frac{2}{x}$ that is parallel to the tangent of f when $x = -1$.

$$f'(x) = -\frac{2}{x^2}$$
$$f'(-1) = -2$$
$$f'(x) = -2$$
$$-\frac{2}{x^2} = -2$$
$$x = \pm 1$$

So, we find the equation of the tangents at $x = 1$

Tangent at $(1, 2)$

$$y - 2 = -2(x - 1)$$
$$y = -2x + 4$$

b. Find the equation of the tangent to the function $f(x) = \frac{1}{3}x^3 - 3x$, that is parallel to the tangent of f when $x = 3$.

$$f'(x) = x^2 - 3$$
$$f'(3) = 3^2 - 3$$
$$f'(3) = 6$$
Now we solve
$$x^2 - 3 = 6$$
$$x = \pm 3$$
We need to find tangent at $x = -3$
$$f(-3) = \frac{1}{3}(-3)^3 - 3(-3) = -9 + 9 = 0$$
Equations of tangent
$$y = 6(x - 3)$$
$$y = 6x - 18$$

c. Find the equation of the tangents to the curve $y = x^2 + 2x + 2$ that passes through the point $(-1, 0)$.

Let the slope of the tangent be m.
Equation of tangent: $y = m(x + 1)$
Equation touches the curve at one point.
$$x^2 + 2x + 2 = m(x + 1)$$
$$x^2 + (2 - m)x + (2 - m) = 0$$
For the quadratic to have one solution the discriminant must be 0.
$$(2 - m)^2 - 4(2 - m) = 0$$
$$m = \pm 2$$
Hence, equation of tangents are
$y = 2x + 2$ and $y = -2x - 2$.

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Question 47

a. A tangent to $f(x) = \frac{x^3}{3} - x^2 - 2x + 4$ makes an angle of 45° with the positive x-axis and passes through the point $(3, b)$, where $b < 0$. Find the value of b.

$$f(x) = \frac{x^3}{3} - x^2 - 2x + 4$$
$$f'(x) = x^2 - 2x - 2$$
Slope is 1 as $\tan 45° = 1$
Now $x^2 - 2x - 2 = 1$
$$x^2 - 2x - 3 = 0$$
$$-2x - 3 = 0$$
$$x = -3, 1$$
$$f(-3) = -8$$
Tangent at $x(-3, -8)$
$$y + 8 = x + 3$$
$y = x - 5$.. (tangent 1)

$$f(1) = \frac{4}{3}$$
Tangent at $\left(1, \frac{4}{3}\right)$
$$y - \frac{4}{3} = x - 1$$
$y = x + \frac{1}{3}$.. (tangent 2)
Now at $x = -3$, $y = -2$ for tangent 1
At $x = -3, y = \frac{4}{3}$ for tangent 2
Since $b < 0$, hence $b = -2$.

b. A tangent to the function $f(x) = \cos(2x) + 2$, makes an angle of 120°, with the positive x-axis and passes through the point $\left(0, \frac{\pi}{6} + \frac{5}{2\sqrt{3}}\right)$.

Determine the point on f that this tangent is drawn to.

Tangent to the curve has slope $-\sqrt{3}$
$$f'(x) = -\sqrt{3}$$
$$-2\sin 2x = -\sqrt{3}$$
$$\sin 2x = \frac{\sqrt{3}}{2}$$
$x = \frac{\pi}{6}$ is an immediate solution.
We find the tangent line at $x = \frac{\pi}{6}$
$$f\left(\frac{\pi}{6}\right) = \cos 2\left(\frac{\pi}{6}\right) + 2 = \frac{5}{2}$$
Equation of tangent:
$$y - \frac{5}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$$
$$y = -\sqrt{3}x + \frac{\sqrt{3}\pi}{6} + \frac{5}{2}$$
At $x = \frac{\pi}{6}, y = \frac{3}{2}$

c. Find the equation of a tangent to the function $f(x) = x^2 - 3x + 3$ that makes an angle of $15°$ with the line $\sqrt{3}y = x$. The slope of the tangent is greater than the slope of the line.

Slope of the line $\sqrt{3}y = x$ is $\frac{1}{\sqrt{3}}$.

Angle $= 30°$

So, tangent will be $(30 + 15) = 45°$

Slope of tangent $= \tan 45° = 1$

Now $f'(x) = 2x - 3$

$2x - 3 = 1$

$x = 2$

$f(2) = (2)^2 - 3(2) + 3 = 1$

Tangent through $(2, 1)$

$y - 1 = 1(x - 2)$

$y = x - 1$

Question 48 Tech-Active.

a. Find the equation of the tangent to $f(x) = x^2 + 2x - 5$ when $x = 1$.

$f(x) = x^2 + 2x - 5$

$f(1) = 1 + 2 - 5 = -2$

$f'(x) = 2x + 2$

$f'(1) = 2(1) + 2 = 4$

Tangent equation through $(1, -2)$

$y - (-2) = 4(x - 1)$

$y = 4x - 6$

b. Find the equation of the normal to $y = 2x^3 - 3x + 1$ when $x = -2$.

$f(x) = 2x^3 - 3x + 1$

$f(-2) = 2(-2)^3 - 3(-2) + 1 = -9$

$f'(x) = 6x^2 - 3$

$f'(-2) = 6(-2)^2 - 3 = 21$

Slope of normal $= -\frac{1}{21}$

Equation of normal through $(-2, -9)$

$$y + 9 = -\frac{1}{21}(x + 2)$$

$$y = -\frac{1}{21}x - \frac{2}{21} - 9$$

$$y = -\frac{1}{21}x - \frac{191}{21}$$

c. Find the equation of the normal to the function $f(x) = x^2 - 2x - 1$ that passes through the point $(4, -2)$.

$f'(x) = 2x - 2$

The normal passes through $(4, -2)$.

Slope of the normal at $x = a$.

$\text{Slope} = -\frac{1}{2a-2}$

Tangent equation:

$$y + 2 = -\frac{1}{2a+2}(x - 4)$$

Solving with $(a, f(a))$

$$(a^2 - 2a - 1) + 2 = -\frac{1}{2a-2}(a - 4)$$

$a = 2$ (only real root)

$f(2) = -1$

Equation of normal through $(2, -1)$

$\text{Slope} = -\frac{1}{2(2)-2} = -\frac{1}{2}$

$$y + 1 = -\frac{1}{2}(x - 2)$$

$2y + x = 0$

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Sub-Section [2.5.2]: Advanced Maximum / Minimum Questions

Question 49

a. The sum of three positive numbers is 26. The second number is 3 times as large as the first. If the sum of the squares of these numbers is minimum, find the numbers.

$$x + y + z = 26$$
$$y = 3x$$
$$x + 3x + z = 26$$
$$z = 26 - 4x$$
We want to minimise
$$f(x) = x^2 + y^2 + z^2$$
$$f(x) = x^2 + (3x)^2 + (26 - 4x)^2$$
$$f(x) = 26x^2 - 208x + 676$$
$$f'(x) = 52x - 208 = 0$$
$$x = 4$$
Hence, numbers are $4, 12$ and 10.

b. Find the maximum area of a field that can be enclosed by 40 metres of fencing.

Let one side be x.
Other side $= 20 - x$
Area $A(x) = x(20 - x) = 20x - x^2$
$$A'(x) = 20 - 2x$$
$$A'(x) = 0$$
$$x = 10$$
Maximum area $= 10 \times 10 = 100$ square metres.

c. Find the minimum distance from the origin to a point on the line $y = x - 3$.

Let the point with minimum distance is $(a, a-4)$

Distance from $(0,0)$

$$D = \sqrt{a^2 + (a-4)^2} = \sqrt{2a^2 - 8a + 16}$$

So we minimise $2a^2 - 8a + 16$

$$d'(a) = 4a - 8$$

$$d'(a) = 0; \ a = 2$$

Minimum distance $= \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$ units.

Question 50

a. Water is being poured into a container. The volume of the container at time t seconds, in mL, is given by:

$$V(t) = 20t^2 - \frac{1}{3}t^3, 0 \leq t \leq 430$$

At what time is the rate of increase in volume the greatest, and what is this rate of increase?

$$V(t) = 20t^2 - \frac{1}{3}t^3$$

$$V'(t) = 40t - t^2$$

$$V"(t) = 40 - 2t$$

$V"(t) = 0; t = 10$ seconds

$V'(10) = 40(10) - 10^2 = 300\ mL$ per second

This is the greatest rate of increase when $t = 10$ seconds.

b. Part of a roller coaster track can be described by the rule $y = 8\sin\left(\frac{\pi x}{20}\right) + 5, x \in [0, 40]$.

State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

$$y' = \frac{2\pi}{5}\cos\frac{\pi x}{20}$$
$$y' = 0$$
$$\frac{2\pi}{5}\cos\frac{\pi x}{20} = 0$$
$\cos\frac{\pi x}{20} = 1$ for maximum value.

Solving, we get $x = 0, 20, 40..$

$y = 5$ at $x = 0$

Hence, the point is $(0, 5)$.

c. The population of foxes $P(t)$ on an island at time t in years is given by $P(t) = 20te^{-\frac{t}{2}}$. Find the maximum rate of increase in the population and the time at which this occurs.

$$N'(t) = 20e^{-\frac{t}{2}} - 10te^{-\frac{t}{2}}$$
$$N''(t) = -10e^{-\frac{t}{2}} - 10e^{-\frac{t}{2}} + 5te^{-\frac{t}{2}}$$
$$N''(t) = -20e^{-\frac{t}{2}} + 5te^{-\frac{t}{2}}$$
Now for $N''(t) = 0$ we have,
$$-20e^{-\frac{t}{2}} + 5te^{-\frac{t}{2}} = 0$$
$$5e^{-\frac{t}{2}}(-4 + t) = 0$$
$t = 4$ years
$$N'(4) = 20e^{-\frac{4}{2}} - 10(4)e^{-\frac{4}{2}} = -20e^{-2}$$
Hence, $-20e^{-2}$ has the maximum rate of decrease when $t = 4$ years.

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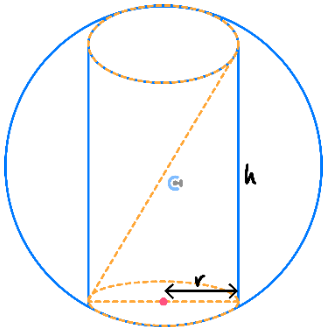
Question 51

A cylinder fits inside a sphere of radius $3\sqrt{3}\ cm$.



a. If the radius of the cylinder is r and the height of the cylinder is h, show that $r = \frac{1}{2}\sqrt{108 - h^2}$.

In the right triangle we use Pythagoras Theorem.
We know radius $= 3\sqrt{3}$
Diameter $= 6\sqrt{3}$

$$(2r)^2 + h^2 = \left(6\sqrt{3}\right)^2$$
$$4r^2 = 108 - h^2$$
$$r = \frac{1}{2}\sqrt{108 - h^2}$$

b. The volume of a cylinder is given by $V = \pi r^2 h$. Find the maximum volume of the cylinder.

$$V = \pi r^2 h$$
$$V = \pi\left(\frac{108-h^2}{4}\right)h$$
$$V = 27\pi h - \frac{h^3\pi}{4}$$
$$V' = 27\pi - \frac{1}{2}h^3\pi$$

Now for stationary point we have $V' = 0$.

$$27\pi - \frac{3}{4}h^2\pi = 0$$
$$h^2 = 36$$
$$h = 6\ cm$$

Hence, max volume when $h = 6$,

$$\text{Max volume} = \pi\left(\frac{108-h^2}{4}\right)h$$
$$V = \pi\left(\frac{72}{4}\right)6 = 108\pi \text{ cubic } cm.$$

Space for Personal Notes

Section F: [2.6] - Families of Functions & Its Exam Skills (Checkpoints)

Sub-Section [2.6.1]: Applying Family of Functions

Question 52

Consider the following family of functions $f(x) = e^{ax} - 1, a > 0$.

a. Identify the "surname" (common aspect(s) of the family) and the "first name" (unique aspect(s) of the family).

Surname: All will pass through (0,0), all have a horizontal asymptote at $y = -1$.
First Name: Each graph has a different dilation from the y-axis.

b. Hence, state what happens to the graph of f in terms of a transformation when the value of a increases.

As the value of a increases, the graph of f is dilated less from the y-axis, that is f moves closer to the y-axis as the value of a increases.

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Question 53

Consider the following family of functions $f(x) = (x-2)^2 + k, k \in \mathbb{R}$.

a. Show that the graph of f always has a stationary point at $x = 2$ and find the nature of this stationary point.

$f'(x) = 2(x-2) = 0$, solve for x:

$x - 2 = 0 \quad \therefore x = 2 \quad \therefore$ S.P at $x = 2$

x	1	2	3
$f'(x)$	-2	0	2
	\	—	/

$\therefore$ local minimum at $x = 2$

b. Hence, identify the "surname(s)" and "first name(s)" of the family.

Surname: Local minimum at $x = 2$, same shape.
First name: Differing vertical translations $\therefore$ different y and x-intercepts.

Question 54

Consider the family of functions $f(x) = \sin\left(kx + \frac{\pi}{2}\right), k \in \mathbb{R}$.

a. Identify the effects of k on the graph.

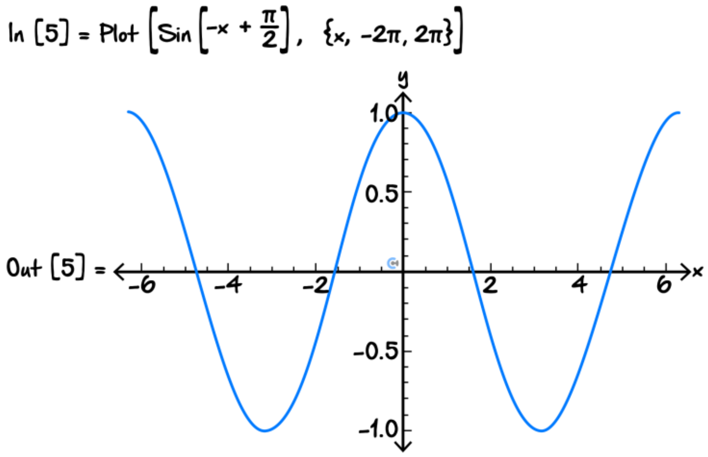
Increasing k will decrease the period of $f(x)$, and vice versa (decreasing k will increase the period of $f(x)$.

b. Hence, identify the "surname" and "first name(s)" of the family.

Surname: y-intercept at $(0,1)$ since $f(0) = \sin\left(\frac{\pi}{2}\right) = 1$ for all $k \in \mathbb{R}$, same general shape.
First name: Differing periods $\therefore$ differing dilations from the y-axis.

c. Express $f(x)$ without using sin when $k = \pm 1$.

Hint: List out the transformations and sketch the resulting graph if you get stuck!



$f(x) = \cos(x)$. Translating the graph of $\sin(x)$ $\frac{\pi}{2}$ units to the left results in $\cos(x)$, which is an even function ($f(-x) = f(x)$) therefore a reflection in the y-axis when $k = -1$ results in the same graph.

Space f

Question 55 Tech-Active.

Consider the family of functions $f(x) = e^{ax} - ax + 1, a \in \mathbb{R}^+$.

a. State one transformation that maps the graph of $g(x) = e^x - x + 1$ onto the graph of $f(x)$.

Dilation by a factor of $\frac{1}{a}$ from the y-axis.

b. Identify a "surname" of the family.

y-intercept at $(0,2)$.

c. Describe what happens to the shape of $f(x)$ as a increases.

f dilates closer to the y-axis, which results in the graph curving upwards for positive values of x, and the graph approaches a linear shape for negative values of x.

d. By plotting $h(x) = f(x) - e^{ax}$ on the same axes as $f(x)$, state the equation of the asymptote of $f(x)$.

$y = -ax + 1$

Space for Personal Notes

Sub-Section [2.6.2]: Finding Unknowns for a Certain Number of Intersections

Question 56

Find the value of a where $a \in \mathbb{R}$ such that the graph of $f(x) = e^x + a$ intersects the line $y = x$ exactly once.

$f(x) = x$ ① $f'(x) = 1$ ②

$\Rightarrow e^x + a = x$ $e^x = 1$ $\therefore x = 0$

$\Rightarrow 1 + a = 0$, $\therefore a = -1$

Question 57

Consider the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = e^x + a$ where $a \in \mathbb{R}$ and $g(x) = x - 3$.

a. Find the inverse function of $f(x)$.

$f(y) = x$ where $y = f^{-1}(x)$, solve for y:

$e^y + a = x \Rightarrow e^y = x - a \therefore y = \log_e(x - a)$

b. Find the value of a such that the graph of $f^{-1}(x)$ intersects with $g(x)$ exactly once.

$\log_e(x-a) = x-3$ ① $\frac{dy}{dx} = \frac{1}{x-a} = g'(x) = 1$ ②

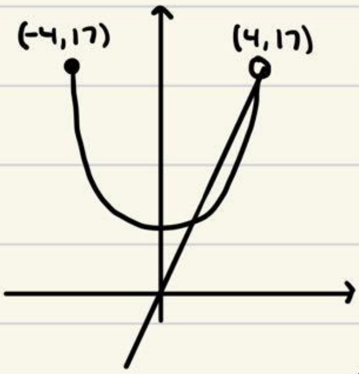
from ②: $x - a = 1$ $\therefore$ $x = a+1$, sub into ①:

$\log_e(a+1-a) = a+1-3$

$\Rightarrow \log_e(1) = a-2 \Rightarrow 0 = a-2$ $\therefore a = 2$

Question 58

Consider the functions $f: [-4,4) \to \mathbb{R}, f(x) = x^2 + 1$ and $g(x) = mx, m \in \mathbb{R}$. Find the value(s) of m where f and g intersect exactly once.



Case 1: $f(x) = g(x)$, $f'(x) = g'(x)$

$x^2 + 1 = mx$ $2x = m$

$\Rightarrow x^2 + 1 = 2x^2$

$\Rightarrow x^2 - 1 = 0, x = \pm 1$ $\therefore m = \pm 2$

Case 2: $g(4) \geq 17 \Rightarrow 4m \geq 17$ $\therefore m \geq \frac{17}{4}$

Case 3: $g(-4) > 17 \Rightarrow -4m > 17$ $\therefore m < \frac{17}{4}$

$\therefore m \geq \frac{17}{4}$ or $m < -\frac{17}{4}$ or $m = 2$ or $m = -2$

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Question 59 Tech-Active.

Consider the function $f : [k, \infty) \to \mathbb{R}, f(x) = (x - k)^2 + 4$. Find the value(s) of k such that $f(x)$ and its inverse intersect exactly once.

```
In[28]:= Solve[(y - k)^2 + 4 == x, y]
Out[28]= {{y → k - √(-4 + x)}, {y → k + √(-4 + x)}}

(*domain of f is x≥k ∴ range of f^-1 must be y≥k so take +ve solution*)

In[36]:= Solve[k + √(-4 + x) == x, x]
   Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[36]= {{x → 1/2 (1 + 2 k - √(-15 + 4 k))}, {x → 1/2 (1 + 2 k + √(-15 + 4 k))}}

(*one solution must be when these points are equal*)

In[39]:= Solve[4 k - 15 == 0, k]
Out[39]= {{k → 15/4}}

In[40]:= (*there must also be only one intersection when the left solution is outside of the domain of f*)
         Reduce[1/2 (1 + 2 k - √(-15 + 4 k)) < k, k]
Out[40]= k > 4
```

$k \in (4, \infty) \cup \left\{\frac{15}{4}\right\}$ Use a slider to check your answers.

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Sub-Section [2.6.3]: Finding Unknowns for Maximums and Minimums

NOTE: This entire section can be done tech-active.

Question 60

For what value of $k \in \mathbb{R}$, will the function, $f(x) = 2 + e^{kx}$ have a minimum on the x-axis?

```
In[41]:= Solve[{2 x + E^(k x) == 0, D[2 x + E^(k x), x] == 0}, {k, x}]
Out[41]= {{k -> -2/e, x -> -e/2}}
```

$$k = -\frac{2}{e}$$

Question 61

For what value(s) of $k \in \mathbb{R}$ will the function $f(x) = (x - k)^2 \log_e(x)$ have a minimum at $x = 4$?

```
In[42]:= f[x_] := (x - k)^2 Log[x]
In[45]:= Solve[f'[4] == 0, k]
Out[45]= {{k -> 4}, {k -> 4 (1 + 2 Log[4])}}
```

Inserting these values into a slider shows a minimum when $k = 4$ but a maximum for the other solution of k ∴ we must reject the other solution, leaving just $k = 4$. ALWAYS PLOT AFTER SOLVING JUST TO CHECK.

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Question 62

Find the value(s) of $k \in \mathbb{R}$ such that the minimum of the function $f: [-4, 10] \to \mathbb{R}, f(x) = 3\left(\frac{x}{2} - 3k\right)^2 - 6$ occurs at $x = -4$.

```
In[72]:= Solve[3 (-2 - 3 k) == 0, k]
Out[72]= {{k → -2/3}}

(*Slider shows that minimum occurs for k≤2/3*)
```

$$\therefore k \leq -\frac{2}{3}$$

Space for Personal Notes

Section G: [2.7] - Pseudocode & Its Exam Skills (Checkpoints)

Question 63

The following pseudocode segments each print a value. Write down the value that is printed.

a. $x \leftarrow 8$
if $x > 5$
  $x \leftarrow x - 3$
end if
print(x)

5

b. $y \leftarrow 4$
if $y < 3$
  $y \leftarrow y + 6$
else
  $y \leftarrow y - 1$
end if
print (y)

3

c. $z \leftarrow 2$
while $z < 6$
  $z \leftarrow z + 2$
end while
print (z)

6

Space for Personal Notes

Question 64

The following pseudocode segments each print a value. Write down the value that is printed.

a. $a \leftarrow 2$
for $i \leftarrow 1$ **to** 4
  if a is even
    $a \leftarrow a + 3$
  else
    $a \leftarrow a + 2$
  end if
end for
print(a)

11

b. $b \leftarrow 10$
while $b > 4$
  if b is even
    $b \leftarrow b - 2$
  else
    $b \leftarrow b - 3$
  end if
end while
print(b)

4

c. $c \leftarrow 1$
for $j \leftarrow 1$ **to** 5
  if j is even
    $c \leftarrow c + 2$
  else
    $c \leftarrow c + 4$
  end if
end for
print(c)

17

Space for Personal Notes

Question 65

The following pseudocode segments each print values. Write down the values that are printed.

a.

```
a ← 2
b ← 3
for i from 1 to 2
    for j from 1 to 3
        if a is even
            b ← b + j
            a ← a + 1
        else
            a ← a + 2
        end if
    end for
end for
print(a, b)
```

$a = 13, b = 4$

b.

```
x ← 5
y ← 12
while y > 6
    for j from 1 to 3
        if x is even
          y ← y − 2
        else
          x ← x + 1
        end if
     end for
end while
print(x, y)
```

$x = 6, y = 2$

c.

```
p ← 1
q ← 10
for j from 1 to 4
    for k from 1 to 2
        if q is even
            p ← p + k
            q ← q − 2
        else
            q ← q − 1
        end if
    end for
end for
print(p, q)
```

$p = 13, q = -6$

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Sub-Section [2.7.1]: Evaluate and Understand the Pseudocode for Different Implementations of Newton's Method

Question 66

An implementation of Newton's method is shown below.

define newton($f(x), x_0, n$):
$df(x)$ ← the derivative of $f(x)$
for i **from** 1 **to** n **do**
  if $df(x_0) = 0$ **then**
    return "Error: Division by zero"
  else
    $x_0 \leftarrow x_0 - \frac{f(x_0)}{df(x_0)}$
  end if
end while
return x_0

Consider calling the function newton $(x^2 - 5, 2, 5)$.

a. How many iterations are performed?

5

b. What is the final value of x_0. Give your answer correct to four decimal places.

2.2361

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.	2.25	0.25
1	2.25	2.23611	0.0138889
2	2.23611	2.23607	0.0000431332
3	2.23607	2.23607	4.16014×10^{-10}
4	2.23607	2.23607	0.

Space for Personal Notes

Question 67

An implementation of Newton's method is shown below.

define newton($f(x), x_0, n, tol$):
$df(x)$ ← the derivative of $f(x)$
$i \leftarrow 0$
$x_n \leftarrow x_0$
while $i < n$ **do**
  if $df(x_n) = 0$ **then**
    return "Error: Division by zero"
  $x_{n+1} \leftarrow x_n - \frac{f(x_n)}{df(x_n)}$
  if $-tol < x_{n+1} - x_n < tol$ **then**
    return x_{n+1}
  $x_n \leftarrow x_{n+1}$
  $i \leftarrow i + 1$
end while
return x_n

Consider calling the function newton $(x^3 - x^2 + 5, 2, 30, 0.0001)$.

a. Find the final return value. Give your answer correct to four decimal places.

-1.4334

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.	0.875	1.125
1	0.875	-8.09286	8.96786
2	-8.09286	-5.31609	2.77676
3	-5.31609	-3.49774	1.81836
4	-3.49774	-2.35292	1.14481
5	-2.35292	-1.71662	0.636306
6	-1.71662	-1.47176	0.244855
7	-1.47176	-1.43426	0.0374961
8	-1.43426	-1.43343	0.000836399
9	-1.43343	-1.43343	4.10702×10^{-7}

b. State the value of i when the algorithm terminates.

$i = 9$

Question 68

An implementation of Newton's method is shown below.

```
define newton(f(x), x_0, n, tol):
df(x) ← the derivative of f(x)
i ← 0
x_n ← x_0
while i < n do
    if df(x_n) = 0 then
      return "Error: Division by zero"
    x_{n+1} ← x_n − f(x_n)/df(x_n)
    if −tol < x_{n+1} − x_n < tol then
       return x_{n+1}
    x_n ← x_{n+1}
    i ← i + 1
end while
return x_n
```

Consider calling the function.

a. Find the final return value of newton $(x^3 - 2x^2 + 4, 1, 2, 0.001)$.

2.875

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	1.	4.	3.
1	4.	2.875	1.125
2	2.875	2.03026	0.844741

b. The function newton $\left(x(x+7)(x-7), \frac{7}{\sqrt{5}}, 5001, 0.0001\right)$ is called. Find the final return value.

Oscillating sequence. We return $x_{5001} = -\frac{7}{\sqrt{5}}$

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