CONTOUREDUCTION

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VCE Mathematical Methods ¾
Polynomials Exam Skills [1.8]
Workbook

Outline:

Learning Objectives:

- ☐ **MM34 [1.8.1]** - Apply transformations to restrict the number of positive/negative x-intercept(s).
- ☐ **MM34 [1.8.2]** - Apply discriminant to solve number of solutions questions.
- ☐ **MM34 [1.8.3]** - Apply shape/graph to solve number of solutions questions.
- ☐ **MM34 [1.8.4]** - Apply odd and even functions (MHS Investigation 2023 S).
- ☐ **MM34 [1.8.5]** - Identify possible rule(s) from a graph.

Section A: Recap

If you were here last week skip to Section B Warm-Up Test.

Roots of Polynomial Functions

Definition

$$\textit{Roots} = x\textit{-intercept}$$

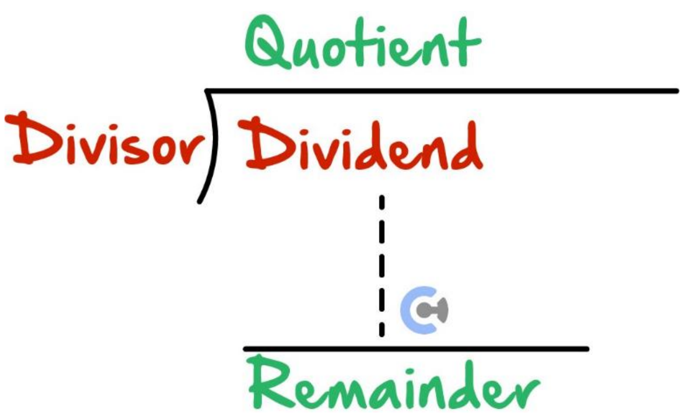
Question 1

Find the roots of the following polynomials:

$$(x-2)^2(x+4)^4$$

Polynomial Long Division

Definition



$$\frac{\textit{dividend}}{\textit{divisor}} = \textit{quotient} + \frac{\textit{remainder}}{\textit{divisor}}$$

TIPS

- Always focus on the highest degree term first.
- Always remember to fill any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Question 2

Simplify the following using polynomial long division:

$$\frac{x^4 - 5x^3 + 5x^2 - 10x + 6}{x^2 + 2}$$

<u>Remainder Theorem</u>

- **Definition**:
  - Find the remainder of the long division without the need for long division.

$$\textit{When } P(x) \textit{ is divided by } (x - \alpha), \textit{ the remainder is } P(\alpha)$$

- **Steps:**
  1. Find x-values which makes the divisor equal to 0.
  2. Substitute it into the dividend function.

Question 3

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where $f(x) = x^3 - x^2 + 4x - 2$ and $g(x) = 3x + 6$.

<u>Factor Theorem</u>

Definition

- For every x-intercept, there is a factor.

$$\boldsymbol{If\ P(\alpha) = 0, then\ (x - \alpha)\ is\ a\ factor\ of\ P(x)}$$

Question 4

If $(x - \alpha)$ is a factor of $4x^5 - 11\alpha x^3 + x^4$, what must be the value of α, where $\alpha \neq 0$?

Definition

Factorising Cubic Polynomials

- **Steps:**

1. Find a single root by trial and error.
   - (Factor theorem: Substitute into the function and see if we get zero.)
2. Use long division to find the quadratic factor.
3. Factorise the remaining factor.

Question 5

Find all the roots of $f(x) = x^3 + 13x^2 + 20x - 100$.

Definition

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$potential\ root = \pm\frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$$

- If the roots are rational numbers, the roots can only be $\pm\frac{factors\ of\ constant\ term\ a_0}{factors\ of\ leading\ coefficient\ a_n}$.

Question 6

Find all the roots of $f(x) = 2x^3 - 3x^2 - 23x + 12$.

Sum and Difference of Cubes

Definition

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Question 7

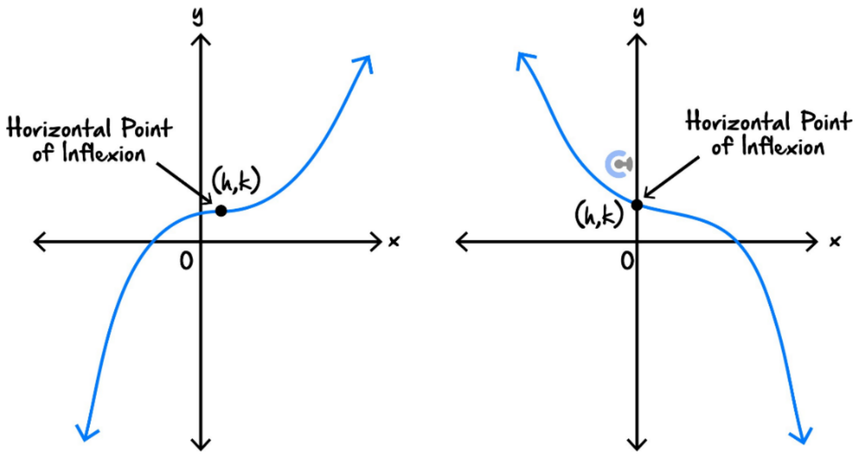
Factorise the following polynomials as much as possible:

$$27x^3 + 216$$

Definition

Graphs of $a(x-h)^n+k$, Where n is Odd and Positive

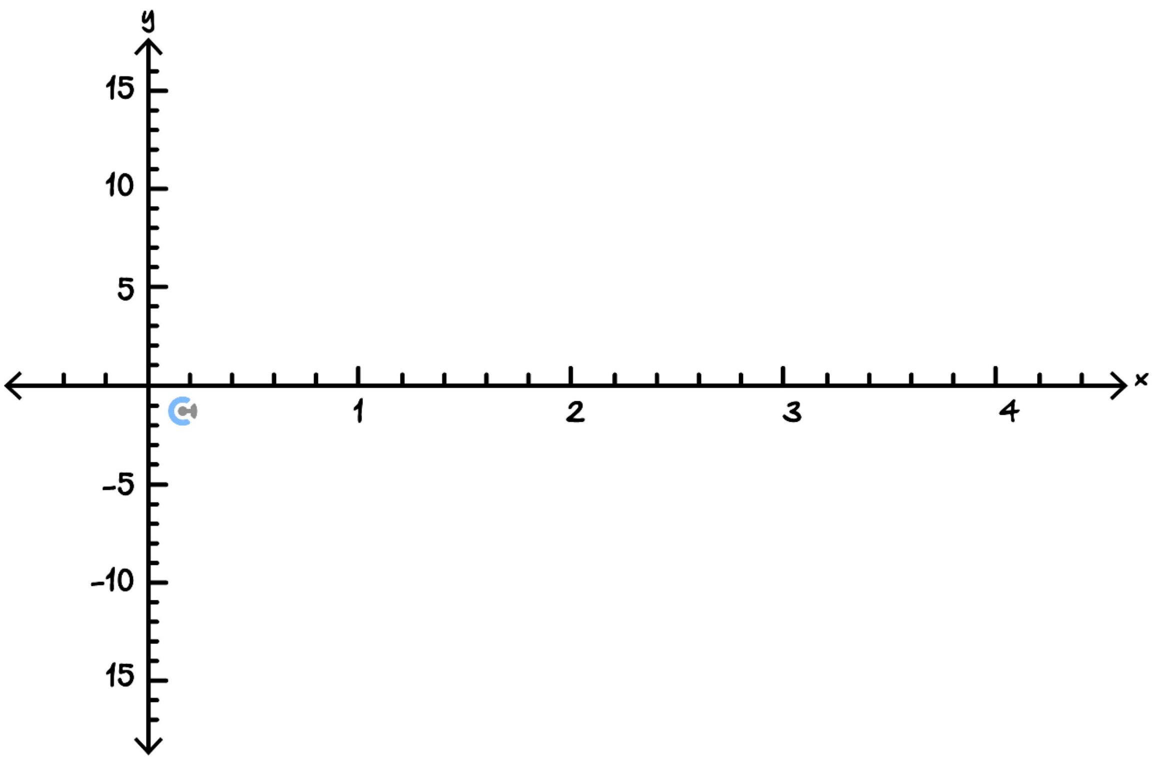
- All graphs look like a "cubic".



- The point (h,k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Question 8

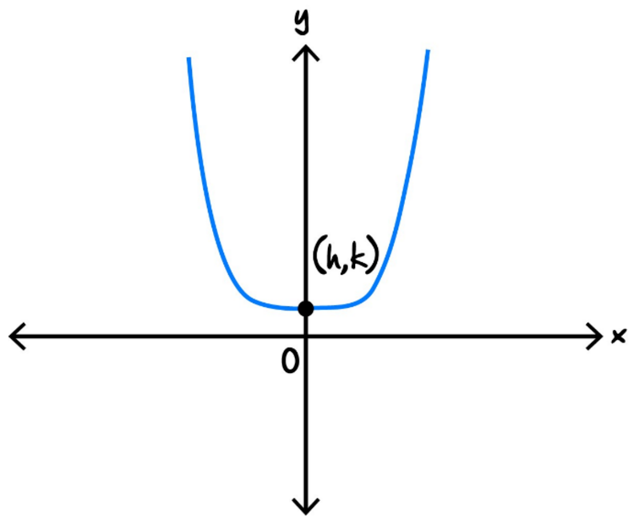
Sketch the graph of $y=(x-2)^3-1$ on the axes below.

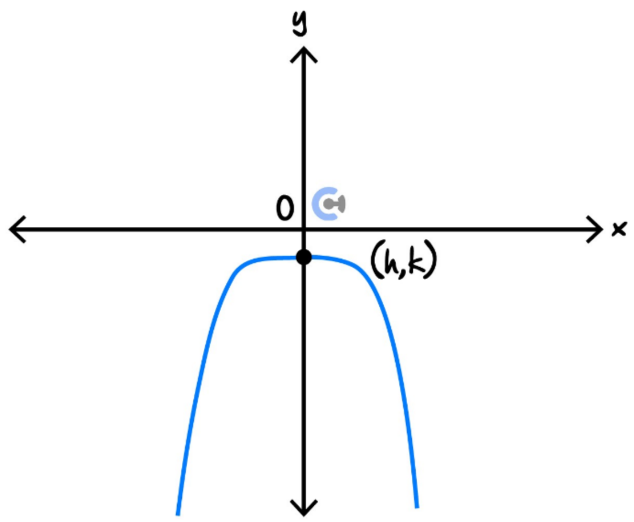


Definition

Graphs of $a(x-h)^n+k$, where n is Even and Positive

- All graphs look like a "quadratic".

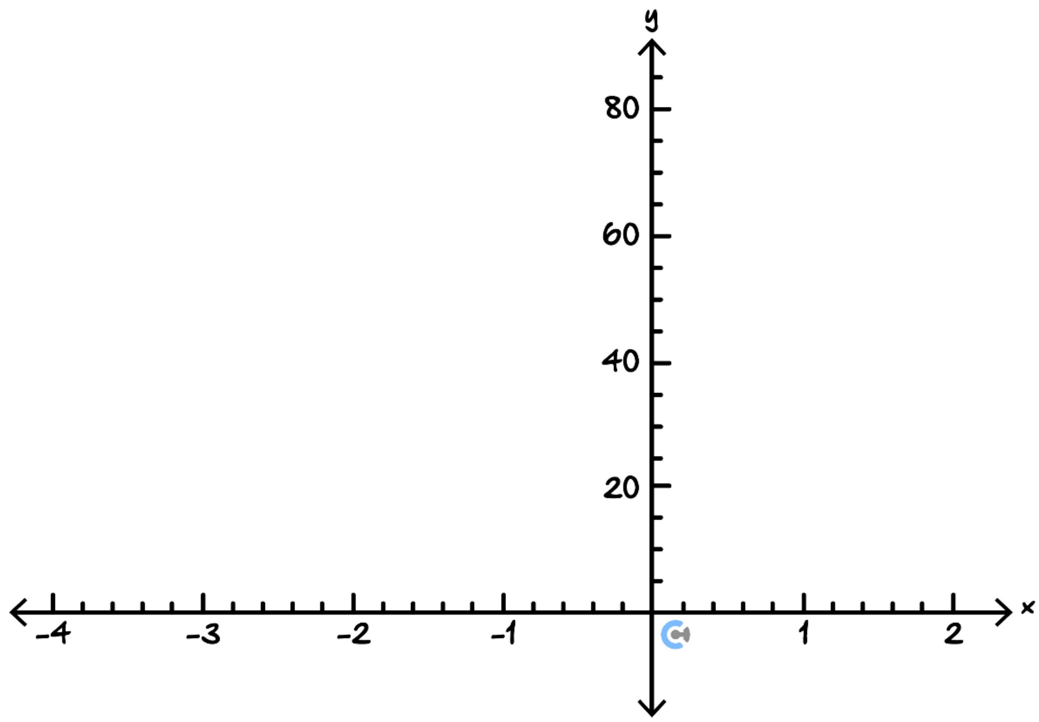




- The point (h,k) gives us the turning point.

Question 9

Sketch the graph of $y=(x+1)^4+2$ on the axes below.



Definition

Graphs of Factorised Polynomials

- **Steps:**

  1. Plot x-intercepts.

  2. Determine whether the polynomial is positive or negative.

  3. Use the repeated factors to deduce the shape:

     - Non-repeated: Only x-intercept.

     - Even Repeated: x-intercept and a turning point.

     - Odd Repeated: x-intercept and a stationary point of inflection.

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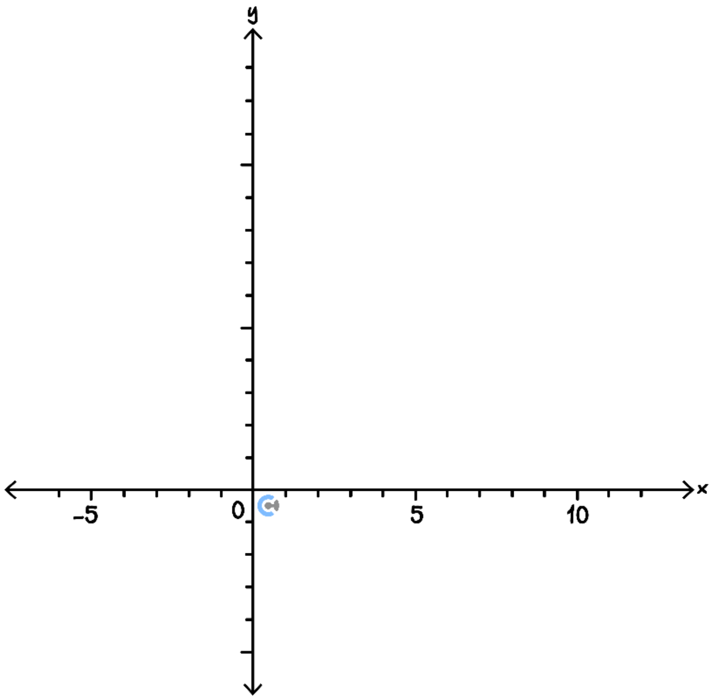
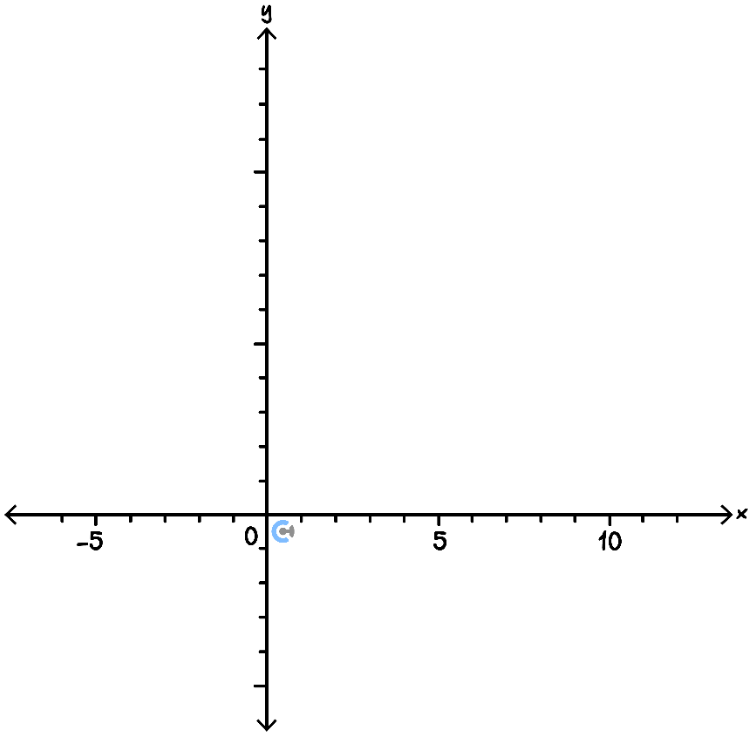
Question 10

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (1 + x)(6 - x)^2$

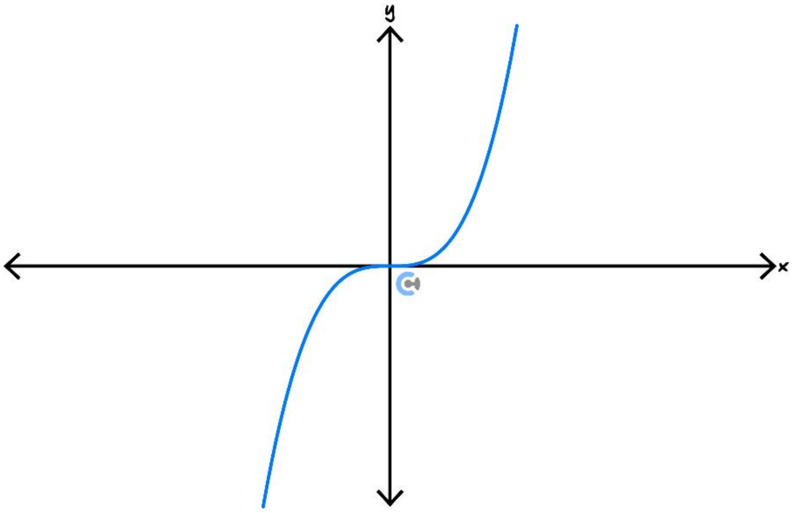


b. $y = (x + 8)^3(x - 3)(x - 7)$



Definition

Odd Functions



- E.g.: $x^3, x^5, x^7 - x^3$... see the pattern?

$$\boldsymbol{f(-x) = -f(x)}$$

- Property: Reflecting around the ______________ is the same as reflecting around the _____________.

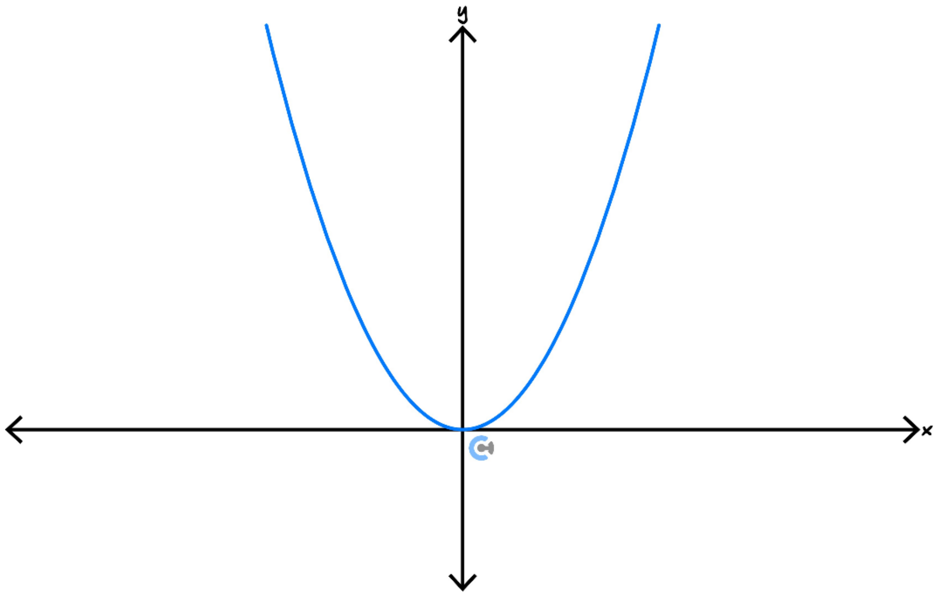
Question 11

Show that the following function is an odd function.

$$h(x) = -x^3 + 5x$$

Definition

Even Functions



- E.g.: $x^2, x^4, -x^{10}, x^4 - 4$... see the pattern?

$$f(-x) = f(x)$$

- Property: It is symmetrical around the ______________.

Question 12

Identify whether the function $f(x) = 2x^2 - 4x$ is an even function, odd function or neither.

Definition

Power Functions

$$y = x^{\frac{n}{m}}$$

- m: Dictates the number of **tails**.
  - **Odd m = Two tails.**
  - **Even m = One tail.**
- n: Dictates the **range**.
  - Odd n: Range could be all real.
  - Even n: Range must be non-negative.
- $\frac{n}{m}(Power)$:
  - Power > 1: Looks like a polynomial function.
  - Power < 1: Looks like a root function.

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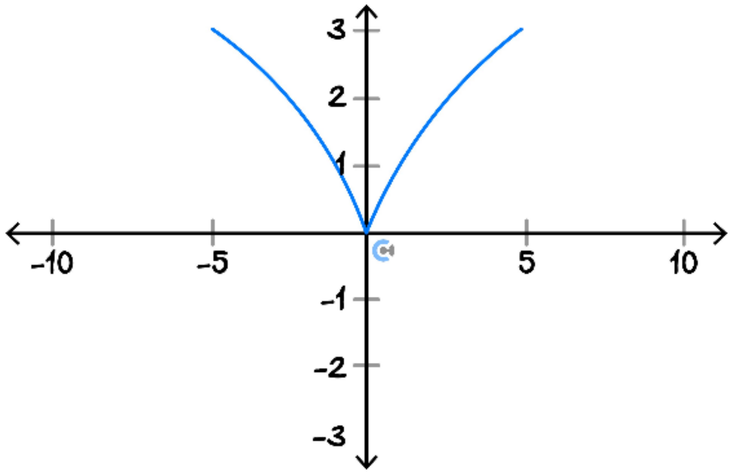
Question 13 (1 mark)

Find a possible rule for the given function out of the 4 options below.



A. $x^{\frac{2}{3}}$

B. $x^{\frac{3}{2}}$

C. $x^{\frac{1}{5}}$

D. $-x^{\frac{2}{7}}$

Space for Personal Notes

Section B: Warm-Up Test

INSTRUCTION: 15 Marks. 15 Minutes Writing.

Question 14 (2 marks)

Consider the function $f(x) = x^3 - 8x$. Show that $f(x)$ is an odd function.

Question 15 (2 marks)

Write $f(x) = \frac{3x^3+12x^2+4x-8}{x+3}$ in the form $f(x) = Q(x) + \frac{B}{x+3}$, where $Q(x)$ is a quadratic and $B \in R$.

Question 16 (3 marks)

Consider the function $f(x) = x^3 + ax^2 + bx + 6$. If $x + 2$ is a factor of $f(x)$ and the remainder of $f(x) \div (x + 1)$ is given by 8, find the values of a and b.

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Question 17 (3 marks)

Solve the following equation for x.

$$2x^3 - 3x^2 - 23x + 16 = 4$$

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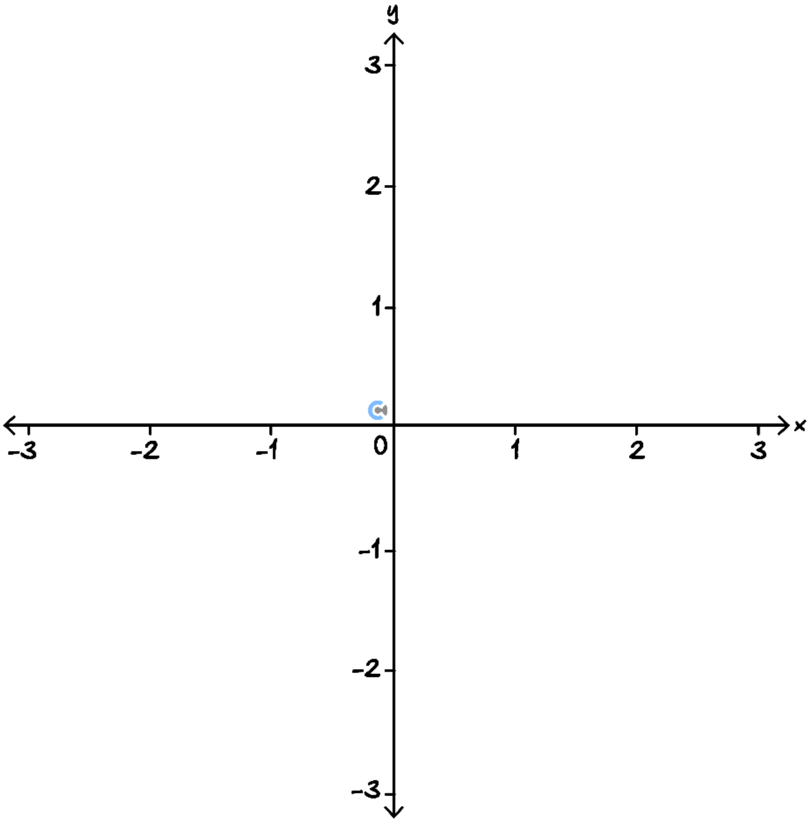
Question 18 (3 marks)

Sketch the graph of the following function on the axes below. Label all axes intercepts with their coordinates.

$$y = (x-1)^2(x-2)(x+1)$$



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Question 19 (2 marks)

It is known that $f(x)$ is an even function where $f(-2) = 3$ and $f'(-2) = 4$.

Let $g(x) = f(x) + 3$.

Find the values of $g(2)$ and $g'(2)$.

Space for Personal Notes

Section C: Polynomial Exam Skills

Sub-Section: Apply Transformations to Restrict the Number of Positive/Negative x-intercept(s)

Definition: Steps for Transforming a Graph to Have n Positive x-intercepts

1. Find all x-intercepts in terms of n.
2. List them out in ascending order.
3. Restrict each x-intercepts to achieve the number of positive/negative solutions we want.

NOTE: If you're looking for negative x-intercepts, count intercepts from the left and translate to the left of the origin.

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Question 20 Walkthrough.

Transform a cubic to have 2 positive x intercepts.

Suppose $f(x) = (x-1)(x-3)(x+1)$. Let $g(x) = f(x-k)$. For what values of k does $g(x)$ have two positive x-intercepts?

Question 21

Suppose $f(x) = (x-2)(x-5)(x+3)$. Let $g(x) = f(x+k)$. For what values of k does $g(x)$ have two negative x-intercepts?

Question 22 Extension.

Suppose $f(x) = x^4 - 3x^2 - 4$. Let $g(x) = f(x-k)$.

For what values of k does $g(x)$ have at least two positive x-intercepts?

Sub-Section: Number of Solutions

What if the function is factorisable?

Question 23 Walkthrough.

Consider $f(x) = x^3 - kx^2 + 5x$.

Find the value(s) of k such that $f(x) = 0$ has 2 solutions.

NOTE: This can only be done if there is no constant term.

Your Turn!

Question 24

Consider $f(x) = 2x^3 + 2kx^2 + 30x$.

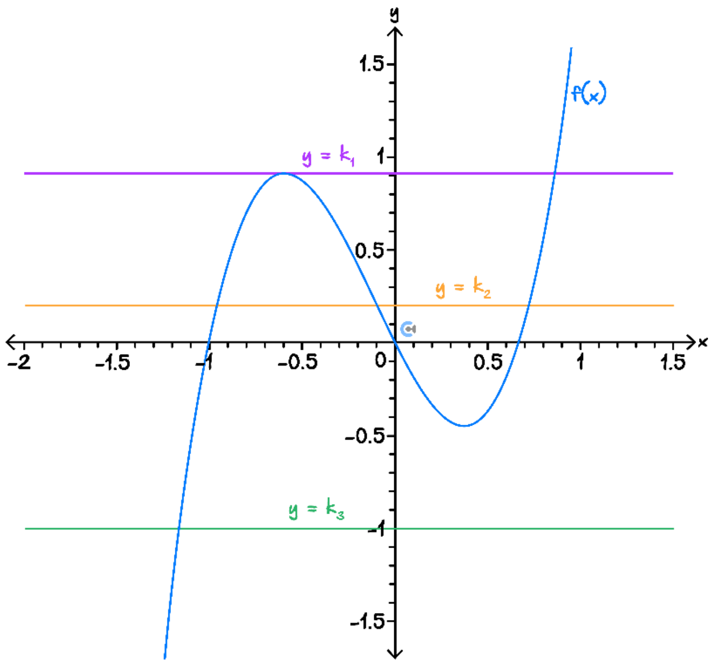
Find the value(s) of k such that $f(x) = 0$ has 1 solution.

Sub-Section [1.8.3]: Apply Shape/Graph to Solve Number of Solutions Questions.

What if you cannot factorise the function?

Exploration: How to Solve Number of Solutions to $f(x) = k$

- Consider the graphs $y = f(x)$ and $y = k$. The solution to $f(x) = k$ is just the intersection of these two graphs!



- Look at the graph above. Which of these lines has one solution to $f(x) = k_n$?

$$y = k_3$$

- What about two solutions?

$$y = k_1$$

- Three solutions?

$$y = k_2$$

Definition: Steps for Solving Number of Solutions to $f(x) = k$

1. Find turning points.
2. Draw a horizontal line between each pair of turning points.
3. Draw a horizontal line outside all turning points.
4. Draw a horizontal line on each turning point.
5. Count the number of intersections on each line.

Question 25 Walkthrough. Tech-Active.

Consider $f(x) = 2x^3 - 15x^2 + 36x + 5$.

Find the value(s) of k such that $f(x) + k = 0$ has 2 solutions.

NOTE: Always visualise!

Your Turn!

Question 26 Tech-Active.

Consider $f(x) = 2x^3 - 3x^2 - 72x + 10$.

Find the value(s) of k such that $f(x) + k = 0$ has 1 solution.

In Summary

<u>Number of Solutions</u>

- If the polynomial can be factorised:

Discriminant!

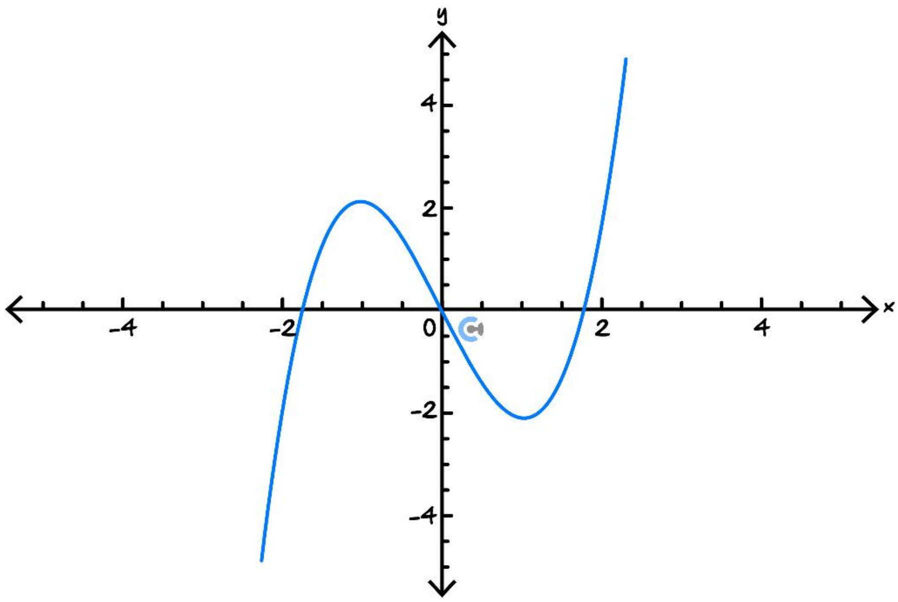
- If the polynomial cannot be factorised:

Visualise and use Turning Points!

Sub-Section: Calculus and Odd and Even Function Properties

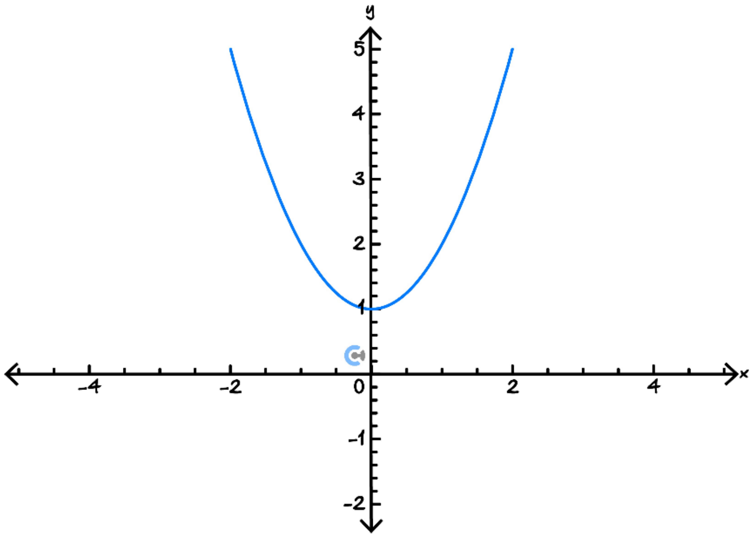
Implication of Odd and Even Functions (Calculus)

- The properties of odd and even functions can become crucial for later topics.
- **Odd Functions**



- Gradients at $x = a$ and $x = -a$ are ______________________________.
- Integral Property: $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0$.

Even Functions



- Gradients at $x = a$ and $x = -a$ are ______________________________.
- Integral Property: $\int_{-a}^{a} f(x)dx = 2\int_{-a}^{0} f(x)dx = 2\int_{0}^{a} f(x)dx$.

Question 27

For a given odd function it is known that $f(2) = 5, f'(2) = -3$.

Find the values for $f(-2)$ and $f'(-2)$.

Sub-Section: Identify Possible Rule(s) From a Graph

Definition: Shape of Power Functions

$$y = x^{\frac{n}{m}}$$

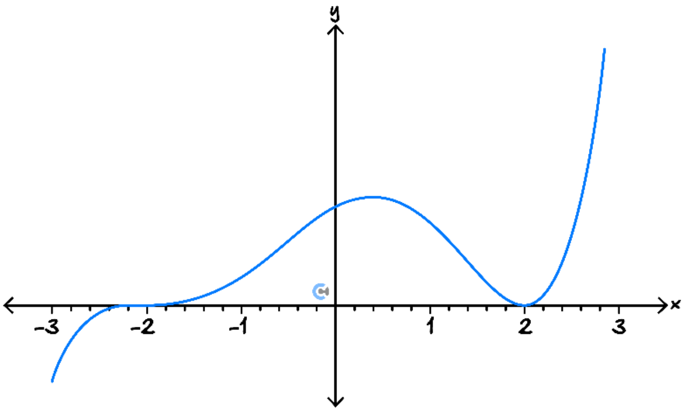
- m: Dictates the number of **tails**.
  - Odd m = Two tails.
  - Even m = One tail.
- n: Dictates the **range**.
  - Odd n: Range could be all real.
  - Even n: Range must be non-negative.
- $\frac{n}{m}$ $(Power)$:
  - Power > 1: Looks like a polynomial function.
  - Power < 1: Looks like a root function.

Definition: Finding Possible Rules of a Polynomial

- **Steps:**
  1. Find all the intercepts.
  2. Write out the intercept form e.g. $(x-1)^a(x-4)^b$.
  3. Figure out the possible type of power (even or odd).
  4. Turning point $=$ even, point of inflection $=$ odd, passes straight through $= 1$.
  5. Figure out the sign of the biggest power.
  6. Find the matching rule through elimination.

Question 28 (1 mark) **Walkthrough.**

A possible rule for the following graph is:



A. $y = (x-2)^2(x+2)^4$

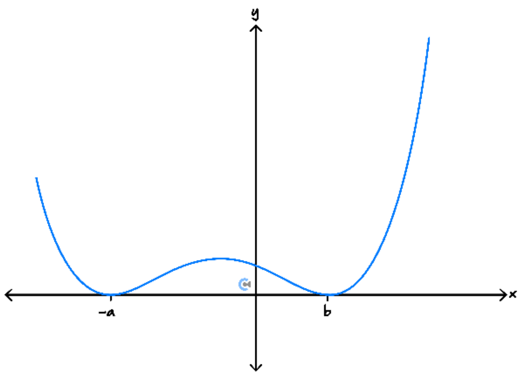
B. $y = (x-2)^2(x+2)^3$

C. $y = (x-2)^2(x+2)$

D. $y = (x-2)^3(x+2)$

Question 29 (1 mark)

A possible rule for the graph below, where $a, b \in R^+$ is:



A. $y = (x-a)^2(x+b)^2$

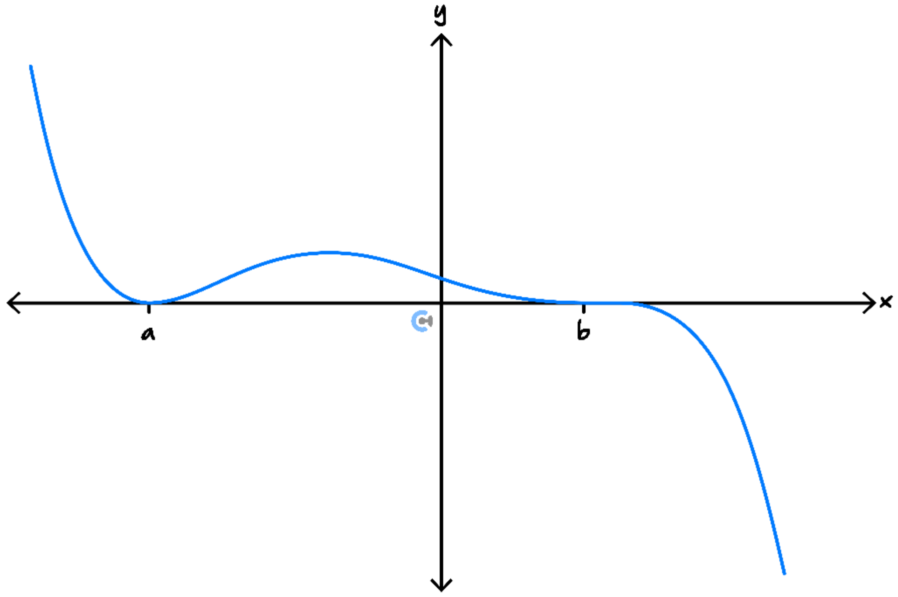
B. $y = x(x-a)(x-b)$

C. $y = (x+a)^2(x-b)^2$

D. $y = (x+a)^3(x-b)^2$

Question 30 (1 mark) **Extension.**

A possible rule for the graph below, where $a < 0$ and $b > 0$ is:



A. $y = (x - a)^2(x + b)^3$

B. $y = -(x - a)^2(x - b)^3$

C. $y = (x + a)^2(x - b)^3$

D. $y = -(x + a)^2(x - b)^3$

Space for Personal Notes

Section D: Exam 1 (14 Marks)

Question 31 (3 marks)

Find the values of k such that the equation $(x^2 - kx + 9)(x^2 - 4x + k) = 0$ has exactly one solution.

Question 32 (3 marks)

Consider the function $f(x) = 2x - x^3$. It is known that the graph of f has a tangent line with equation $y = -10x - 16$ when $x = -2$. Without using calculus, find the equation of the tangent to the graph of $g(x) = f(x) + 4$ when $x = 2$.

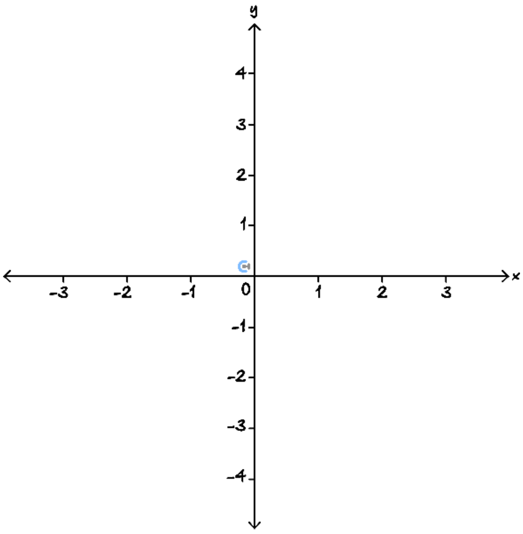
Question 33 (8 marks)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$.

a. Show that $x - 1$ is a factor of $f(x)$. (1 mark)

b. Fully factorise $f(x)$. (2 marks)

c. Sketch the graph of $y = f(x)$, labelling all axial intercepts with coordinates. Note that turning points occur at approximately $(-0.37, -4.84)$ and $(1.36, 0.35)$. (3 marks)



d. Find the values of k such that $f(x-k)=0$ has two positive solutions. (2 marks)

Space for Personal Notes

Section E: Tech-Active Exam Skills

Calculator Commands: Turning Point

- ALWAYS sketch the graph to find approximate bounds for where the turning point you want is located.
- To find a local maximum we maximise the function over a specific domain.
- To find a local minimum we minimise the function over a specific domain.
- **TI and Casio:** Use fmin(expression, variable, lower (optional), upper (optional)) or fmax(expression, variable, lower (optional), upper (optional)).
- **TI:** Menu→ $4 \rightarrow \frac{7}{8}$.

Define $f(x)=x^3-4\cdot x$ — *Done*

$\text{fMin}(f(x),x,0,2)$ — $x=\frac{2\cdot\sqrt{3}}{3}$

$f\left(\frac{2\cdot\sqrt{3}}{3}\right)$ — $\frac{-16\cdot\sqrt{3}}{9}$

- **Casio:** Action→Calculation→ $\frac{f_{min}}{f_{max}}$.

fmin(x^3−4x, x, 0, 2)

$\left\{\text{MinValue}=\frac{-16\cdot\sqrt{3}}{9}, x=\frac{2\cdot\sqrt{3}}{3}\right\}$

.

NOTE: TI only gives the x value for the $\frac{min}{max}$ so we then need to sub it back into our function. Casio gives us both!

Calculator Commands

- **Mathematica:** Minimize[ ] and Maximize[ ] commands.
- Minimize[$f[x], x$] will minimize $f[x]$ over its whole domain.
- To restrict the domain we must use Minimize[$\{f[x], a \leq x \leq b\}, x$].

```
In[34]:= Minimize[{x^3 - 4 x, 0 < x < 2}, x]
```

Out[34]= $\left\{-\frac{16}{3\sqrt{3}}, \left\{x \to \frac{2}{\sqrt{3}}\right\}\right\}$

- **TI UDF:** We can use the analyse function.

- **Analyse a Function**

- **Overview:**

This program will find for a given function:

- Coordinates of endpoints.
- The maximal domain.
- The equations of straight-line asymptotes.
- The rule of the derivative.
- Inflection points and their nature.
- Stationary points and their nature.

There are two analyse programs:

- Analyse which analyses a function over the domain R or the maximal domain.
- Analysed which analyses over a domain with specified start and end points.

Both are found in the methods_func library. You can switch between the two on the calculator page by adding/removing the 'd' to reference the appropriate program.

$$\text{analysed}\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

- Start Point: $\begin{bmatrix} -5 & \frac{262}{77} \end{bmatrix}$
- End Point: $\begin{bmatrix} 5 & \frac{-316}{529} \end{bmatrix}$
- Maximal Domain:
  $x\neq-1.68469$ and
  $x\neq-0.629579$ and
  $x\neq0.314273$ and
  $-5\leq x\leq5$
- Asymptotes: (4)
  $x=-1.68469$ (Vertical)
  $x=-0.629579$ (Vertical)
  $x=0.314273$ (Vertical)
  $y=\frac{4}{3}-\frac{x}{3}$ (Oblique)
- x –Intercepts: (4)
  $[-1.3772 \quad 0]$, $[-0.273891 \quad 0]$,
  $[1 \quad 0]$, $[2.65109 \quad 0]$
- Vertical Intercept: $[0 \quad 1]$
- Derivative:
  $$\frac{-(3\cdot x^6+12\cdot x^5-26\cdot x^3-24\cdot x^2-6\cdot x-4)}{(3\cdot x^3+6\cdot x^2+x-1)^2}$$
- Inflection Points: (2)
  $[-1.11377 \quad 1.48672]$ (Increasing)
  $[-0.11198 \quad 0.604642]$ (Increasing)
- Stationary Points: (2)
  $[-3.45719 \quad 3.17894]$ (Local min.)
  $[1.6173 \quad 0.124612]$ (Local max.)

Done

Input:

$analyse(< function >, < variable >)$

$analysed(< function >, < variable >,$
$< lower\ bound >, < upper\ bound >)$

Other notes:

- It is recommended to use the analysed program when working with trigonometric functions.
- Be careful when using functions with parameters since some parts of the programs may not be able to give a solution. :/
- If at least one of the bounds is “?”, the asymptote finder will be disabled and the program will analyse over the maximal domain.

Question 34 Walkthrough.

Find the turning points of $f(x) = x^3 - 3x - 1$. Hence, find the values of k for which the equation $f(x) = k$ has $1, 2,$ and 3 solutions.

Question 35

Find the turning points of $f(x) = x^3 + 3x^2 - x - 3$. Hence, find the values of k for which the equation $f(x) = k$ has $1, 2,$ and 3 solutions.

Calculator Commands: Using Sliders/Manipulate on CAS

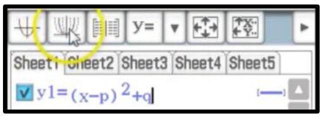
Mathematica

```
Manipulate[Plot[function, {x, xmin, xmax}],
    {unknown, lowerbound, upperbound}]
```

NOTE: The function **must** be typed out instead of using its saved name.

TI-Nspire

Casio Classpad



Question 36 (1 mark) **Walkthrough.**

Consider the function $f(x) = 2x^3 - 15x^2 + 24x + 4$.

The equation $f(x) = k$ has 1 solution for:

A. $1 < k < 4$

B. $-12 < k < 15$

C. $k < -12$ or $k > 15$

D. $k \geq 15$

Question 37 (1 mark)

Consider the function $f(x) = 3x^4 - 40x^3 + 186x^2 - 360x + 250$.

The equation $f(x) = k$ has 4 solutions for:

A. $-25 < k < 7$

B. $2 < k < 7$

C. $-25 < k < 2$

D. $-25 \leq k \leq 2$

Section F: Exam 2 (12 Marks)

Question 38 (1 mark)

Let $p(x) = x^3 - 5ax^2 + 2ax - 5$, where $a \in \mathbb{R}$. When p is divided by $x + 2$ the remainder is 11. The value of a is:

A. -2

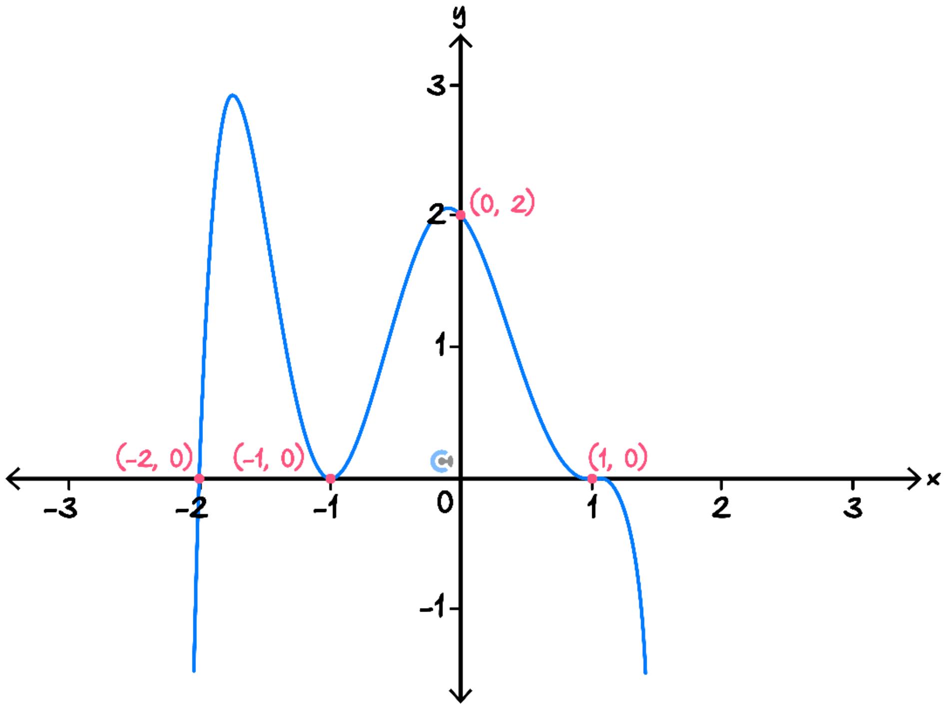
B. -1

C. 1

D. 2

Question 39 (1 mark)

The rule for a function with the graph below could be:



A. $y = -(x-1)^2\,(x+2)(x\ +\ 1)^3$

B. $y\ =\ (x-1)^3\,(x+2)(x+1)^2$

C. $y\ =\ -(x-1)^3(x+2)(x+1)^2$

D. $y\ =\ (x-1)^2\,(x+2)(x+1)^3$

Question 40 (1 mark)

Let $f(x) = x^5 - (k^2 - 5k + 6)x^4 + x^3 - (k^2 - 7k + 10)x^2 - (k^2 - 3)x$
If $f(x)$ is odd, then k must equal:

A. 2 or 3

B. 5 only

C. 2 only

D. 2 or 5

Question 41 (1 mark)

Let $g(x) = (x - 1)^2(x - 4)^2 - 10$. There will be exactly four solutions to the equation given by $g(x) = k$ whenever:

A. $-10 < k < 5$

B. $-10 < k < -5$

C. $5 < k < 10$

D. $5 \leq k \leq 10$

Question 42 (1 mark)

Let $h(x) = x^4 - 5x^2 + 4$. The function $h(x + k)$ will have exactly three negative x-intercepts whenever:

A. $1 \leq k \leq 2$

B. $1 < k \leq 2$

C. $-2 < k \leq 1$

D. $-2 \leq k \leq 1$

Space for Personal Notes

Question 43 (7 marks)

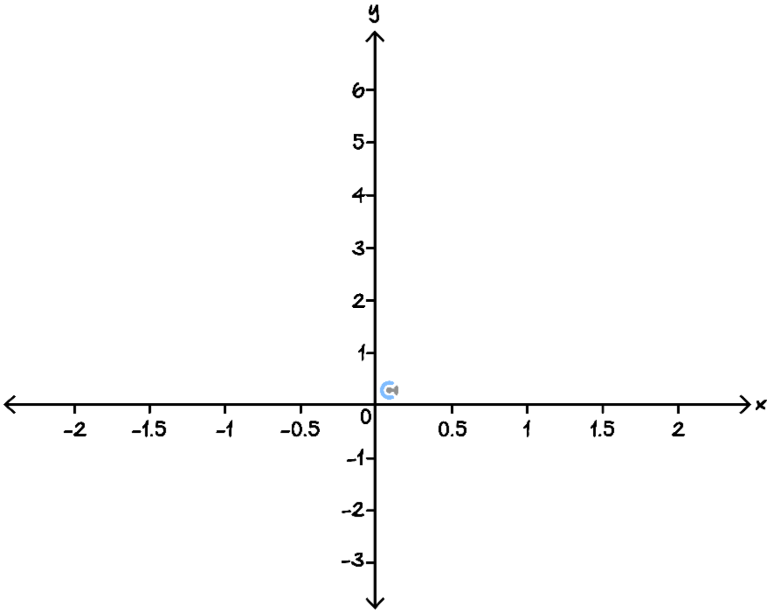
Consider a quartic of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. It is known that f satisfies the following conditions:

- $f(0) = 5$
- $f(1) = -1$
- $f(2) = 5$
- $f(x)$ is even function.

a. Show that $f(x) = 2x^4 - 8x^2 + 5$. (3 marks)

b. Solve the equation $f(x) = 0$. (1 mark)

c. Sketch the graph of $y = f(x)$ on the axes below. Label all turning points with coordinates.



d. Find the values of k, where $k \in \mathbb{R}$, for which:

i. $f(x) = k$ has no solutions. (1 mark)

ii. $f(x) = k$ has exactly four solutions. (1 mark)

iii. $f(x) = k$ has exactly two solutions. (1 mark)

Contour Checklist

Learning Objective: [1.8.1] – Apply Transformations to Restrict the Number of Positive/Negative x-intercept(s)

Key Takeaways

- [ ] To solve these questions, figure out how to translate the relevant intercept to the _____________.

Learning Objective: [1.8.2] – Apply Discriminant to Solve Number of Solutions Questions

Key Takeaways

- [ ] There are no real solutions for a quadratic when Δ _______0.
- [ ] There is one real solution for a quadratic when Δ__________ 0.
- [ ] There are two unique real solutions for a quadratic when Δ______________ 0.

Learning Objective: [1.8.3] – Apply Shape/Graph to Solve Number of Solutions Questions

Key Takeaways

- [ ] To find the number of solutions for $f(x) = k$, draw a _________________________ line at ________________ $= k$ and count the intersections.

Learning Objective: [1.8.4] - Apply Odd and Even Functions (MHS Investigation 2023 S)

Key Takeaways

- [ ] For an odd function, ______________________________.
- [ ] For an even function, ____________________________.

Learning Objective: [1.8.5] - Identify Possible Rule(s) From a Graph

Key Takeaways

- [ ] A turning point x-intercept has a(n) ______________________ power on its factor.
- [ ] A stationary point of inflection x intercept has a(n) __________________ power on its factor.
- [ ] If the x-intercept passes straight through, the power of the factor is ___________________.