CONTOUR EDUCATION

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VCE Mathematical Methods ¾
Polynomials Exam Skills [1.8]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 28
Supplementary Questions	Pg 29 – Pg 50

Section A: Compulsory Questions

Sub-Section [1.8.1]: Apply Transformations to Restrict the Number of Positive/Negative x-Intercept(s)

Question 1

Consider the following polynomials:

a. Given $f(x) = (x-4)(x+3)(x-6)$, determine the values of k such that $f(x+k)$ has no positive x-intercepts.

$k \geq 6$

b. Given $f(x) = (x-1)(x+2)(x-5)$, determine the values of k such that $f(x-k)$ has exactly one positive x-intercept.

$-5 < k \leq -1$

c. Given $f(x) = (x-2)(x-7)(x+1)$, determine the values of k such that $f(x-k)$ has exactly two positive x-intercepts.

$-2 < k \leq 1$

Question 2

Consider the following quadratic polynomials:

a. Given $f(x) = x^2 - 4x + 3$, factorise $f(x)$ and determine the values of k such that $f(x - k)$ has exactly one positive x-intercept.

$f(x) = (x-3)(x-1)$.
One positive x-intercept for $-3 < k \leq -1$

b. Given $f(x) = x^2 + 2x - 3$, factorise $f(x)$ and determine the values of k such that $f(x + k)$ has no positive x-intercepts.

$f(x) = (x-1)(x+3)$.
No positive x-intercepts for $k \geq 1$

c. Given $f(x) = x^2 - 5x + 6$, factorise $f(x)$ and determine the values of k such that $f(x - k)$ has exactly one negative x-intercept.

$f(x) = (x-3)(x-2)$.
One negative x-intercept for $-3 \leq k < -2$

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Question 3

Consider the following cubic polynomials:

a. Given $f(x) = x^3 - 4x^2 + x - 4$, factorise $f(x)$ and determine the values of k such that $f(x+k)$ has exactly one positive x-intercept.

$f(x) = (x - 4)(x^2 + 1)$
Exactly one positive x-intercept for $k < 4$

b. Given $f(x) = x^3 - 3x^2 - 4x + 12$, factorise $f(x)$ and determine the values of k such that $f(x - k)$ has one negative x-intercept.

$f(x) = (x - 3)(x + 2)(x - 2)$.
One negative x-intercept for $-2 \leq k < 2$

c. Given $f(x) = x^3 - 6x^2 + 9x$, factorise $f(x)$, and determine the values of k such that $f(x-k)$ has two positive x-intercepts.

$f(x) = x(x-3)^2$.
Two positive x-intercepts for $k > 0$

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Sub-Section [1.8.2]: Apply Discriminant to Solve Number of Solutions Questions

Question 4

For each of the following quadratic equations, determine the conditions on k for the equation to have the specified number of solutions.

a. $x^2 + x + 5k = 0$ has exactly two distinct real solutions.

$$\Delta > 0 \implies 1 - 20k > 0 \implies k < \frac{1}{20}$$

b. $x^2 - 4x + 4(k + 1) = 0$ has no real solutions.

$$\Delta < 0 \implies 16 - 16 - 16k < 0 \implies k > 0$$

c. $kx^2 - 3x + 2k = 0$ has exactly one real solution.

$$\Delta = 9 - 8k^2 = 0 \implies k = \pm\frac{3}{2\sqrt{2}}$$

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Question 5

For each of the following quadratic equations, determine the conditions on k for the equation to have the specified number of solutions.

a. $2x^2 + 4x + 2\log_3(k) = 0$ has exactly two distinct real solutions.

$$\Delta = 16 - 16\log_3(k) > 0 \implies \log_3(k) < 1 \implies 0 < k < 3$$

b. $\log_2(5)\, x^2 + 3x + \log_2(k) = 0$ has exactly one real solution.

$$\Delta = 9 - 4\log_2(5)\log_2(k) = 0 \Rightarrow k = 2^{\frac{9}{4\log_2(5)}}$$

c. $4k^2x^2 - 2kx + 1 = 0$ has no real solutions.

$$\Delta = 4k^2 - 16k^2 < 0 \implies k \in \mathbb{R}$$

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Question 6

For each of the following equations, determine the conditions on k for the equation to have the specified number of solutions.

a. $x^2 + kx + 3 = 0$ has two real solutions.

$\Delta = k^2 - 12 > 0 \implies k < -2\sqrt{3}$ or $k > 2\sqrt{3}$

b. $2x^2 - 4kx + k^2 + 3 = 0$ has no real solutions.

$\Delta = 16k^2 - 8(k^2 + 3) < 0 \implies -\sqrt{3} < k < \sqrt{3}$

c. $kx^3 + 4x^2 + 2kx = 0$ has three real solutions.

$$x(kx^2 + 4x + 2k)$$

So, we need $kx^2 + 4x + 2k = 0$ to have two solutions

$$\Delta = 16 - 8k^2 > 0 \implies -\sqrt{2} < k < \sqrt{2}$$

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Sub-Section [1.8.3]: Apply Shape/Graph to Solve Number of Solutions Questions

Question 7

The cubic function $f(x) = x^3 - 6x^2 + 9x + 2$ has turning points at (1,6) and (3,2). Determine the values of k for which the equation $f(x) = k$ has exactly two solutions.

From the shape of the graph $k = 2$ or $k = 6$

Question 8

Consider the quadratic function $g(x) = \frac{1}{2}x^2 - kx + 3$. Determine the values of k for which $g(x) = 2$ has exactly two solutions.

We find the values of k for which $\frac{1}{2}x^2 - kx + 1 = 0$ has two solutions.
$\Delta = k^2 - 2 > 0 \implies k < -\sqrt{2}$ or $k > \sqrt{2}$

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Question 9

The quartic function $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 2$ has turning points at $(-1, -7)$ and $(1,9)$ and $(3, -7)$.

Find the values of k for which the equation $f(x) = k$ has exactly two solutions.

From the shape of the graph $k > 9$ or $k = -7$

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Sub-Section [1.8.4]: Apply Odd and Even Functions

Question 10

For an odd function $f(x)$, it is known that $f(1) = 2$ and $f'(1) = 3$.

Find the values of $f(-1)$ and $f'(-1)$.

$f(-1) = -2$ and $f'(-1) = 3$.

Question 11

An odd function $f(x) = \frac{1}{2}x^3$, has a tangent line of $y = 6x - 8$ at the point $(2,4)$. Find the equation of the line tangent to $f(x)$ when $x = -2$.

Line with gradient 6 passing through $(-2, -4)$

$$y = 6x + 8$$

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Question 12

Let $f(x) = (x-3)(x-5)(x+1)(x+3)$. Find the value of k such that $f(x+k)$ is an even function.

Note that $f(x+1) = f(1-x)$, so $f(x)$ is symmetric about the line $x = 1$.
Therefore if f is translated 1 unit to the left it will be symmetric about $x = 0$ and therefore an even function.
$k = 1$.

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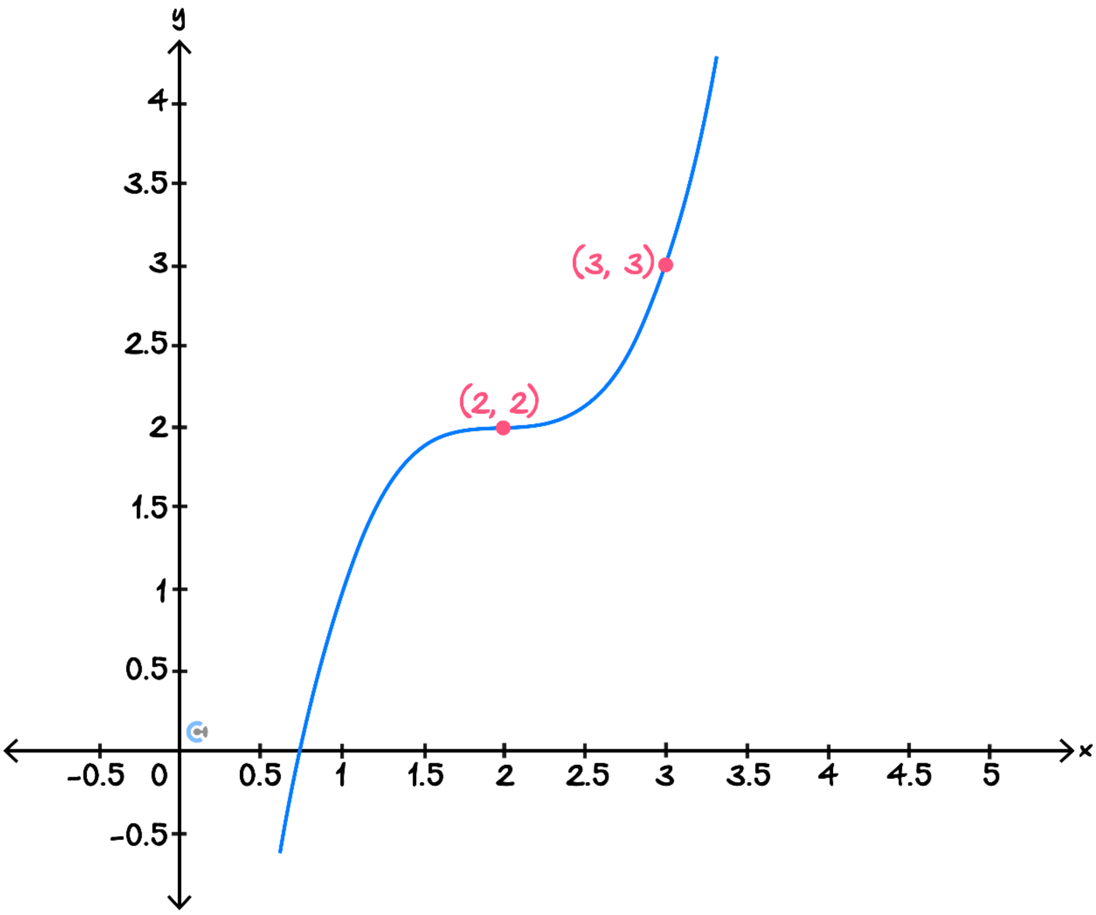
Sub-Section [1.8.5]: Identify Possible Rule(s) From a Graph

Question 13

Part of the graph of $y = f(x)$ is sketched below. The point $(2,2)$ is a stationary point of inflection. Determine the rule for $f(x)$.



$$(h, k) = (2,2)$$

So, rule of $f(x) = a(x-2)^3 + 2 \quad (1)$

Sub $x = 3$ and $y = 3$ into (1)

$$3 = a(3-2)^3 + 2$$

$$1 = a$$

Hence, $f(x) = (x-2)^2 + 2$

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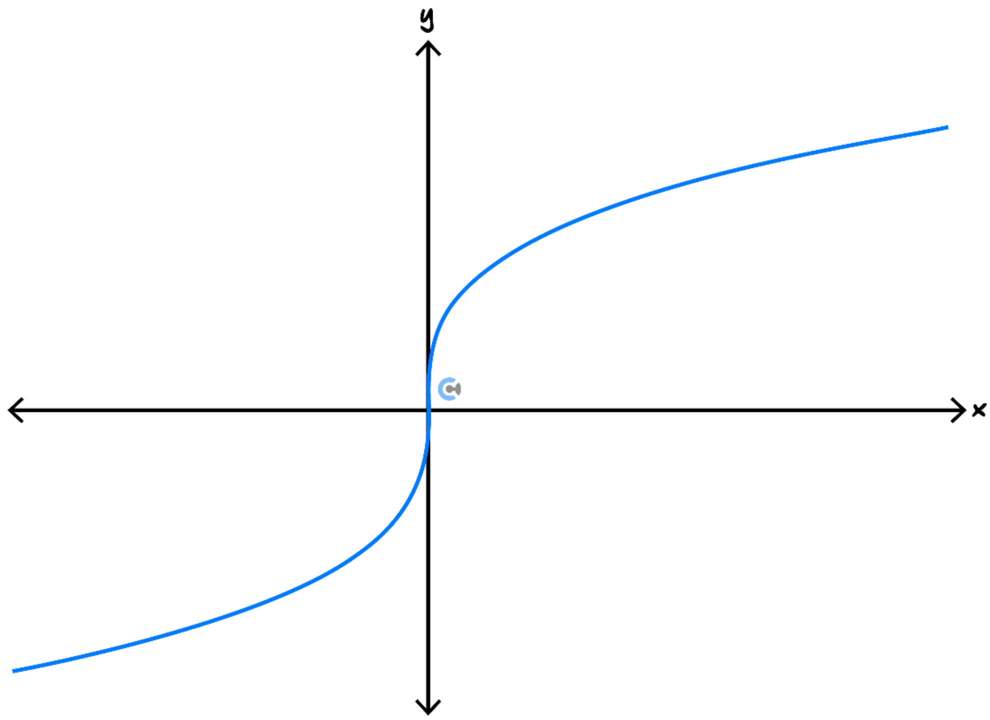
Question 14

Part of the graph of $y = x^{\frac{m}{n}}$, where m and n are positive integers, is shown below.



a. Is it true that $m > n$?

No. $n > m$

b. Determine whether m and n are odd or even.

Both are odd.

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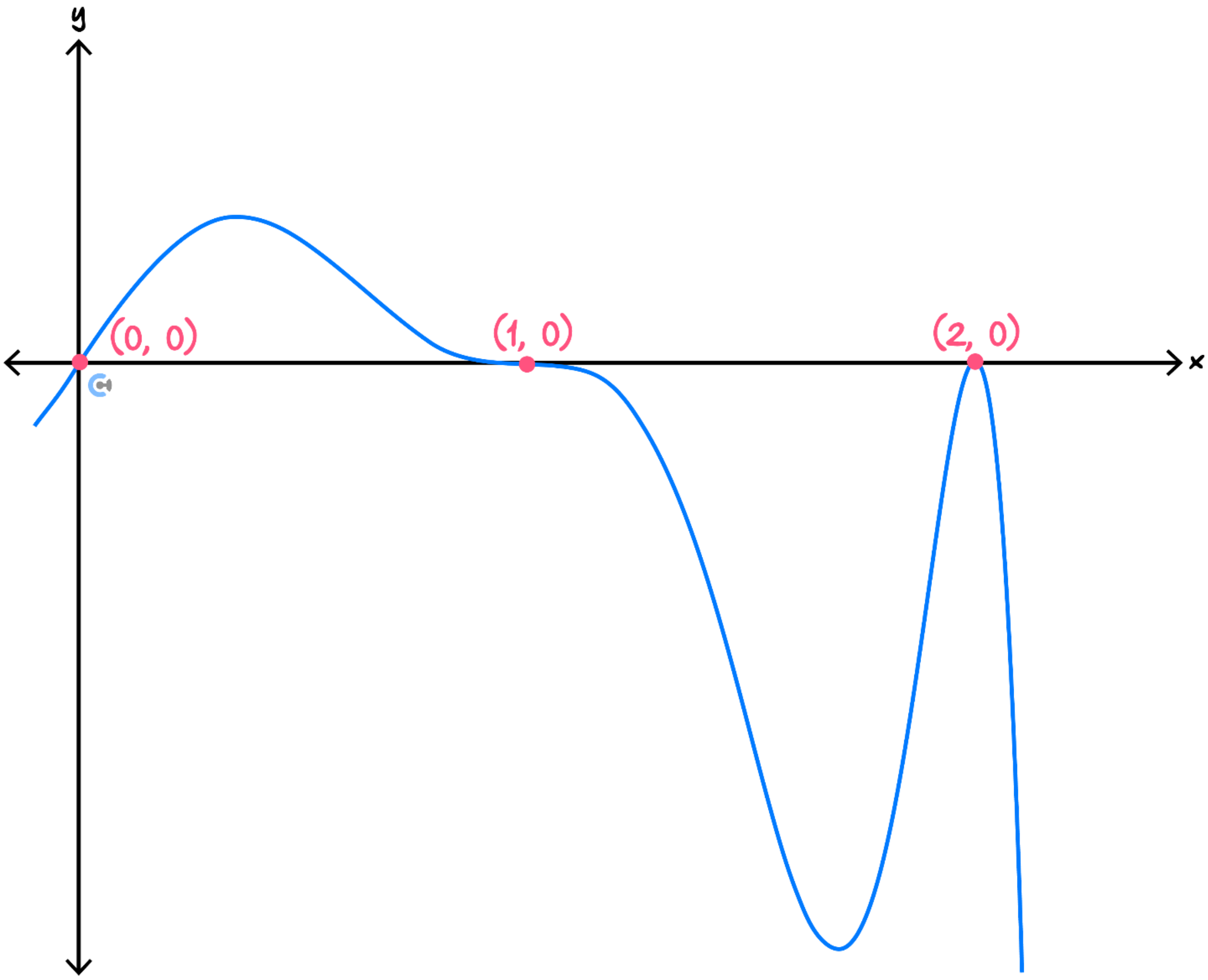
Question 15

Let $f(x)$ be an odd function. Part of the graph of $y = f(x)$ is shown below.

Determine a possible rule for $f(x)$.



Must have the factors $x(x-1)^3(x-2)^2$, then we use the shape and fact that $f(x)$ is odd to conclude that a possible rule is

$$f(x) = -x(x-1)^3(x-2)^2(x+1)^3(x+2)^2$$

Sub-Section: Exam 1 Questions

Question 16

Let $f: [-1,3] \to \mathbb{R}, f(x) = x^3 - 3x^2 + 4$.

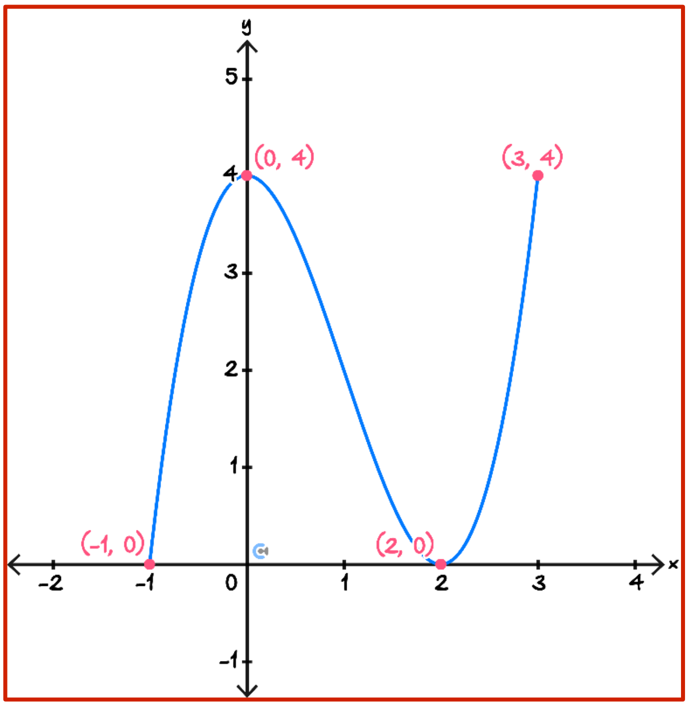
a. Show that $x - 2$ is a factor of $f(x)$.

$f(2) = 8 - 3 \times 4 + 4 = 0$
Therefore $x - 2$ is a factor.

b. Fully factorise $f(x)$.

$f(x) = (x-2)^2(x+1)$

c. It is known that the graph of $y = f(x)$ has a turning point on its y-intercept. Sketch the graph of $y = f(x)$, labelling all axes intercepts, turning points and end points.



d. Let $g: \mathbb{R} \to \mathbb{R}, g(x) = x^3 - 3x^2 + 4$.

Find the values of k such that $g(x - k) = 0$ has two positive solutions.

If the graph is translated by more than 1 unit to the right, therefore $k > 1$.

Question 17

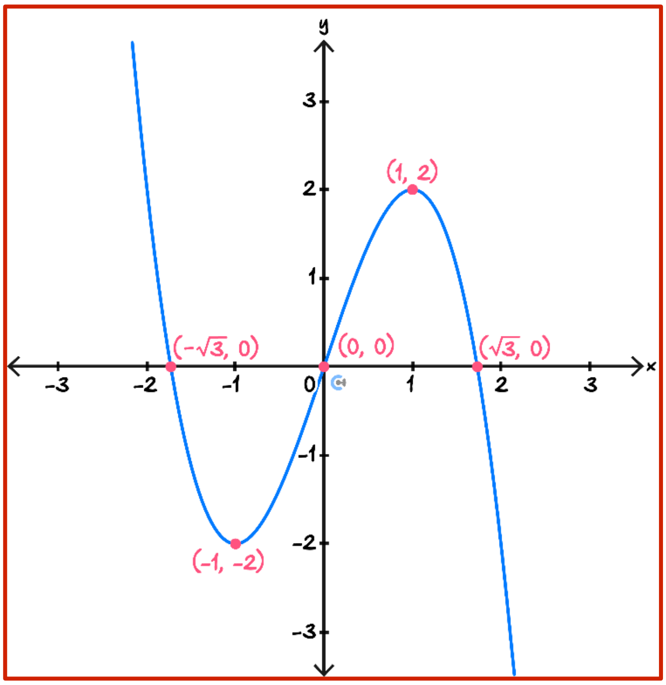
Let $f: \mathbb{R} \to \mathbb{R}, f(x) = 3x - x^3$.

It is known that the graph of $y = f(x)$ has a turning point when $x = 1$.

a. Show that f is an odd function.

$f(-x) = 3(-x) - (-x)^3 = -3x + x^3 = -(3x - x^3) = -f(x)$.

b. Sketch the graph of $y = f(x)$. Label all axes intercepts and turning points with coordinates.



Axes intercepts: $(-\sqrt{3}, 0), (0, 0), (\sqrt{3}, 0)$
Turning points: $(-1, -2)$ and $(1, 2)$

c. Consider the function $g: \mathbb{R} \to \mathbb{R}, g(x) = 3x - x^3 + k$, where k is a real constant.

i. Find the values of k for which $g(x)$ has exactly two x-axis intercepts.

$k = \pm 2$

ii. Find the values of k for which $g(x) = 1$ has exactly one solution.

$k < -1$ or $k > 3$

Question 18

Consider the function $f(x) = x^3 - ax^2 + bx + 8$, where a and b are integers.

It is known that $x - 2$ is a factor of $f(x)$ and that $f(x)$ has a remainder of -24 when divided by $x + 2$.

Find the values of a and b.

We know that $g(2) = 0$ and $g(-2) = -24$. This yields the equations

$$-4a + 2b + 16 = 0$$
$$-4a - 2b = -24$$

adding the two equations

$$\implies -8a + 16 = -24$$
$$a = 5$$
$$b = 2$$

Question 19

Consider $f: \mathbb{R} \to \mathbb{R}, f(x) = -x^3 + ax^2$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = ax$ where a is a positive real constant.

a. Find the coordinates of the x-intercepts of the graph of f in terms of a, where appropriate. (1 mark)

$(0, 0)$ and $(a, 0)$

b. Find the values of a for which the graphs of f and g have only one point of intersection.

Intersections occur when $f(x) = g(x)$

$$-x^3 + ax^2 = ax$$
$$x(x^2 - ax + a) = 0$$

There is always an intersection at $(0, 0)$.
For this to be the only intersection we require that $x^2 - ax + a$ has no real solutions.
Consider the determinant

$$\Delta = a^2 - 4a < 0 \implies 0 < a < 4.$$

One point of intersection if $0 < a < 4$.

The graphs of f and g have three points of intersection when $a > 4$. Let the x-coordinates of these three points of intersection be r, s and t where $r < s < t$.

c. Find the values of r, s and t, in terms of a, where appropriate.

We know that $r = 0$, now use the quadratic formula to solve

$$x^2 - ax + a = 0$$
$$x = \frac{a \pm \sqrt{a(a-4)}}{2}$$

so we have

$$r = 0, \quad s = \frac{a - \sqrt{a(a-4)}}{2} \quad \text{and} \quad t = \frac{a + \sqrt{a(a-4)}}{2}$$

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Sub-Section: Exam 2 Questions

Question 20

Let $p(x) = x^3 - 3ax^2 + 2x - 2$, where $a \in \mathbb{R}$. When p is divided by $x + 2$ the remainder is 10.

The value of a is:

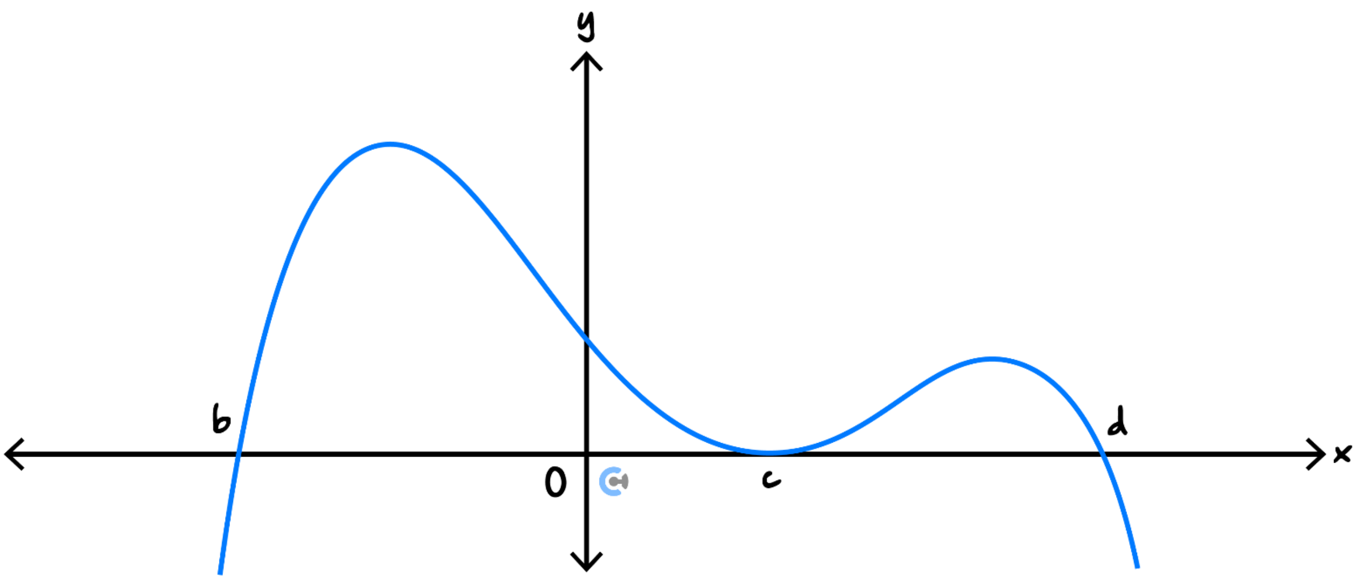
A. -2

B. -1

C. 1

D. 2

Question 21



The rule for a function with the graph above could be:

A. $y = -2(x + b)(x - c)^2(x - d)$

B. $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

D. $y = 2(x - b)(x - c)\,(x - d)$

Question 22

A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is:

A. $[0,4]$

B. $\{0,4\}$

C. $(0,4)$

D. $(-\infty, 4)$

Question 23

The equation $x^3 - 3x^2 - 9x + c = 0$ has only one solution for x when:

A. $-5 < c < 27$

B. $c \leq -5$

C. $c < -5$ or $c > 27$.

D. $c \leq -5$ or $c \geq 27$.

Question 24

A set of three numbers that could be the solutions of $x^3 + bx^2 - 22x + 40 = 0$, where $b \in \mathbb{R}$, is:

A. $\{-1,4,5\}$

B. $\{-2,2,4\}$

C. $\{-5,-4,2\}$

D. $\{-5,2,4\}$

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Question 25

Consider the quartic $f: \mathbb{R} \to \mathbb{R}, f(x) = 3x^4 - 4x^3 - 12x^2$.

a. Find the coordinates of the point M at which the minimum value of the function f occurs.

$(2, -32)$

b. State the values of $b \in \mathbb{R}$ for which the graph of $y = f(x) + b$ has no x-intercepts.

$b > 32$

A tangent line l is drawn to the graph of f when $x = \frac{1}{2}$ and has the equation $l(x) = -\frac{27}{2}x + \frac{55}{16}$.

c. Find the coordinates of all points where the line l intersects the graph of f.

$$\left(\frac{1}{2}, -\frac{53}{16}\right), \quad \left(\frac{1}{6}\left(1-\sqrt{166}\right), \frac{1}{16}\left(36\sqrt{166}+19\right)\right), \left(\frac{1}{6}\left(\sqrt{166}+1\right), \frac{1}{16}\left(19-36\sqrt{166}\right)\right)$$

Let $p: \mathbb{R} \to \mathbb{R}, p(x) = 3x^4 - 4x^3 - 12x^2 + 2a, a \in \mathbb{R}$.

d. Find the values of a for which:

i. $p(x) = 0$ has three solutions.

$f(x)$ has a local minimum at $(-1, -5)$. Note $p(x) = f(x) + 2a$.
$p(x) = 0$ has three solutions when $a = 0$ or $a = \frac{5}{2}$

ii. $p(x) = 0$ has two solutions.

$a < 0$ or $\frac{5}{2} < a < 16$

e. Find the value of k for which the function $g(x) = 3x^4 - (4 - k^2)x^3 - (12 + k)x^2 + (24 - 12k)x + 3k$ is an even function.

We just require that $4 - k^2 = 0$ and $24 - 12k = 0 \implies k = 2$

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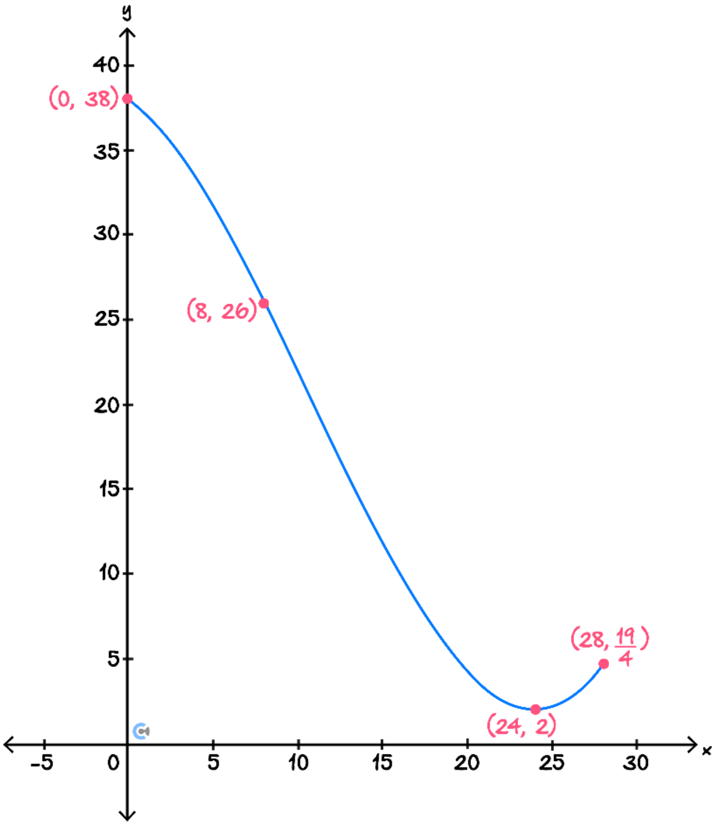
Question 26

James is designing a waterslide that launches you into the water. The waterslide's cross-section is modelled by a function:

$$f:[0,28] \to \mathbb{R}, f(x) = ax^3 + bx^2 + cx + d.$$

The graph of f is shown below.



a. Show that $a = \frac{1}{256}, b = -\frac{1}{8}, c = -\frac{3}{4}, d = 38$.

Solve the following system of equations:

$$f(0) = d = 38$$
$$f(8) = 512a + 64b + 8c + d = 26$$
$$f(24) = 13824a + 576b + 24c + d = 2$$
$$f(28) = 21952a + 784b + 28c + d = \frac{19}{4}$$

this yields

$$a = \frac{1}{256}, b = -\frac{1}{8}, c = -\frac{3}{4}, d = 38$$

b. $f(x)$ can be written as $f(x) = g(x)(x-8) + r$ where r is an integer.

Find $g(x)$ and r.

$$r = 26 \text{ and } g(x) = \frac{x^2}{256} - \frac{3x}{32} - \frac{3}{2}$$

c. The slide is supported by a support beam with equation $s(x) = 38 - ax$ where $a > 0$.

Find the values of a for which:

i. $f(x) = s(x)$ has three solutions.

We solve $f(x) = s(x)$ and get solutions

$$x = 0, \quad x = 16 - 8\sqrt{7-4a}, \quad , x = 16 + 8\sqrt{7-4a}$$

The solutions must be > 0 because of the restriction on f's domain.
So three solutions if $\frac{3}{4} < a < \frac{7}{4}$

ii. $f(x) = s(x)$ has one solution.

$$a > \frac{7}{4}$$

Let $h: \mathbb{R} \to \mathbb{R}, h(x) = f(x)$.

d. Describe a sequence of translations that map the graph of $h(x)$ onto the graph of an odd function.

Consider a translation a units right and b units up. The image of h under this transformation is

$$h(x-a)+b = \frac{x^3}{256} + \left(-\frac{3a}{256} - \frac{1}{8}\right)x^2 + \left(\frac{3a^2}{256} + \frac{a}{4} - \frac{3}{4}\right)x - \frac{a^3}{256} - \frac{a^2}{8} + \frac{3a}{4} + b + 38$$

We get an odd function if we set the coefficients of the constant and x^2 term equal to zero

$$-\frac{a^3}{256} - \frac{a^2}{8} + \frac{3a}{4} + b + 38 = 0$$
$$-\frac{3a}{256} - \frac{1}{8} = 0$$

this yields $a = -\frac{32}{3}$ and $b = -\frac{554}{27}$.
Therefore a sequence of translations to make $h(x)$ an odd function is

- A translation $\frac{32}{3}$ units left
- A translation $\frac{554}{27}$ units down.

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Section B: Supplementary Questions

Sub-Section [1.8.1]: Apply Transformations to Restrict the Number of Positive/Negative x-Intercept(s)

Question 27

Let $f(x) = (x-1)(x+4)(x-2)^2$. Find the values of k such that $f(x+k)$ has no positive x-intercepts.

We want $2 - k \leq 0$, so $k \geq 2$.

Question 28

Let $f(x) = x^3 - 2x^2 - 5x + 6$. Find the values of k such that $f(x+k)$ has exactly one negative x-intercept.

Note that $f(x) = (x-1)(x+3)(x+2)$.
For exactly one negative x-intercept, we need $-2 - k < 0$ and $1 - k \geq 0$.
Hence, $-2 < k \leq 1$.

Question 29

Let $f(x) = 2x^2 - 15x + 14$ and $g(x) = x^2 - 10x + 8$. Find the values of k such that $f(x+k)$ and $g(x+k)$ have exactly two intersections with negative x-coordinates.

Note that $f(x) = g(x)$ is equivalent to $f(x) - g(x) = x^2 - 5x + 6 = (x-2)(x-3) = 0$.

Therefore, $f(x+k) = g(x+k)$will have exactly two intersections with negative x-coordinates for $3 - k < 0$, therefore $k > 3$.

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Question 30

Let $f(x) = \frac{1}{2}x + 3$ and $g(x) = 2x^2 - 4x - 22$. Find the values of k such that $f\big(g(x + k)\big)$ has exactly one negative x-intercept.

Note that $f(g(x)) = x^2 - 2x - 8 = (x-4)(x+2)$. Therefore, $f(g(x+k))$ will have exactly one negative x-intercepts for $-2 < k \leq 4$.

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Sub-Section [1.8.2]: Apply Discriminant to Solve Number of Solutions Questions

Question 31

Find the values of k such that the equation $x^2 - 2^k x + 4$ has no solutions.

$(-2^k)^2 - 4 \cdot 1 \cdot 4 < 0 \implies 2^{2k} < 2^4 \implies 2k < 4 \implies k < 2.$

Question 32

Find the values of k such that the equation $x^2 - 2kx + 5k$ has exactly two solutions.

$(-2k)^2 - 4 \cdot 5k > 0 \implies 4k^2 - 20k > 0 \implies k^2 - 5k > 0 \implies k < 0 \text{ or } k > 5.$

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Question 33

Find the values of k such that the equation $(x^2 - kx + 4)(x^2 - 2\sqrt{3}x + k) = 0$ has exactly three solutions.

Either $x^2 - kx + 4$ gives two solutions (which requires $k < -4$ or $k > 4$) and $x^2 - 2\sqrt{3}x + k$ gives one solution (so that $k = 3$), or $x^2 - kx + 4$ gives one solution (so that $k = \pm 4$) and $x^2 - 2\sqrt{3}x + k$ gives two solutions (so that $k < 3$). Therefore, $k = -4$ is the only acceptable value.

Question 34

Let $f(x) = x^2 - 4x + 3$ and $g(x) = x^2 - 6x + k$. Find the values of k such that $f(g(x))$ has exactly four solutions.

The equation $f(g(x)) = 0$ gives $g(x) = x^2 - 6x + k = 1$ or $g(x) = x^2 - 6x + k = 3$.
These two equations in total will result in four solutions if both equations individually have two solutions.
The discriminant for each equation is positive whenever $(-6)^2 - 4(k - 1) > 0$
and also $(-6)^2 - 4(k - 3) > 0, i.e., k < 10$.

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Sub-Section [1.8.3]: Apply Shape/Graph to Solve Number of Solutions Questions

Question 35

Suppose $f(x) = x^2 - kx + 3$. Find the value of $k > 0$ so that $f(x) = k$ has exactly one solution.

We may solve $(-k)^2 - 4 \cdot 1 \cdot 3 = 0$ to find $k = \pm 2\sqrt{3}$. We reject $k = -2\sqrt{3}$ because $k > 0$ is specified in the question. Hence, $k = 2\sqrt{3}$.

Question 36

It is known that the quartic $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8.5$ has turning points at $(1, -0.5), (2, 0.5)$ and $(3, -0.5)$. Find the values of k such that $f(x) = k$ has exactly two solutions.

By inspection of the graph, $k > \frac{1}{2}$ or $k = -\frac{1}{2}$.

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Question 37

It is known that the quartic $f(x) = x^4 - 4x^3 - 8x^2 + 48x + 3$ has turning points at $(-2, -77)$, $(2, 51)$ and $(3, 48)$. Find the values of k such that $f(x) = k$ has exactly two solutions.

By inspection of the graph, $-77 < k < 48$ or $k > 51$.

Question 38

Let $f(x) = x^4 - 16x^3 + 46x^2 - 48x + 20$ and $g(x) = -x^4 + 2x^2 + 3$. It is known that the quartic $h(x) = 2x^4 - 16x^3 + 44x^2 - 48x + 17$ has turning points at $(1, -1)$, $(2, 1)$ and $(3, -1)$. Hence or otherwise, find the value of k such that $f(x) = g(x) + k$ has exactly three solutions.

First note that $f(x) = g(x) + k$ is equivalent to $f(x) - g(x) = k$, i.e. $2x^4 - 16x^3 + 44x^2 - 48x + 17 = k$. By inspection, we conclude $k = 1$

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Sub-Section [1.8.4]: Apply Odd And Even Functions

Question 39

Show that the function given by $f(x) = x^5 - 2x^2 + 1$ is neither even nor odd.

Observe that $f(1) = 0$ and $f(-1) = -2$. Notice that $f(1) \neq f(-1)$ therefore f is not even. Similarly, $f(1) \neq -f(-1)$, so f is not odd either.

Question 40

Let $f(x) = x^4 - (k^2 - 5k + 6)x^3 + k^3x^2 + 10$. Find the value(s) of k so that $f(x)$ is an even function.

Set k so that the coefficients of the odd power terms are zero. Hence $k^2 - 5k + 6 = 0$, i.e. $k = 2$ or $k = 3$.

Question 41

The tangent to the graph of $f(x) = x^2 - 4$ at the point $x = 2$ is given by $h(x) = 4x - 8$. Denote the tangent to $f(x)$ at $x = -2$ by $k(x)$. Find the rule for $k(x)$ by applying a reflection to $h(x)$.

Note that $f(x)$ is an even function, so it is symmetric about the y-axis. The tangent at $x = -2$ may obtained as a reflection of $h(x)$ across the y-axis. Hence, $k(x) = 4(-x) - 8 = -4x - 8$.

Question 42

The tangent to the graph of $f(x) = x^3 - 3x$ at the point $x = 2$ is given by $h(x) = 9x - 16$. Denote the tangent to $f(x)$ at $x = -2$ by $k(x)$. The rule for $k(x)$ can be obtained from the rule of $h(x)$ via the following sequence of transformations:

- A translation of a units in the positive direction of the x-axis.
- A translation of b units in the positive direction of the y-axis.

State the values of a and b and hence or otherwise, find the rule of $k(x)$.

We want to map the point $(2, 2)$ to $(-2, -2)$, so we need $a = -4, b = -4$ and $k(x) = 9(x + 4) - 16 - 4 = 9x + 16$.

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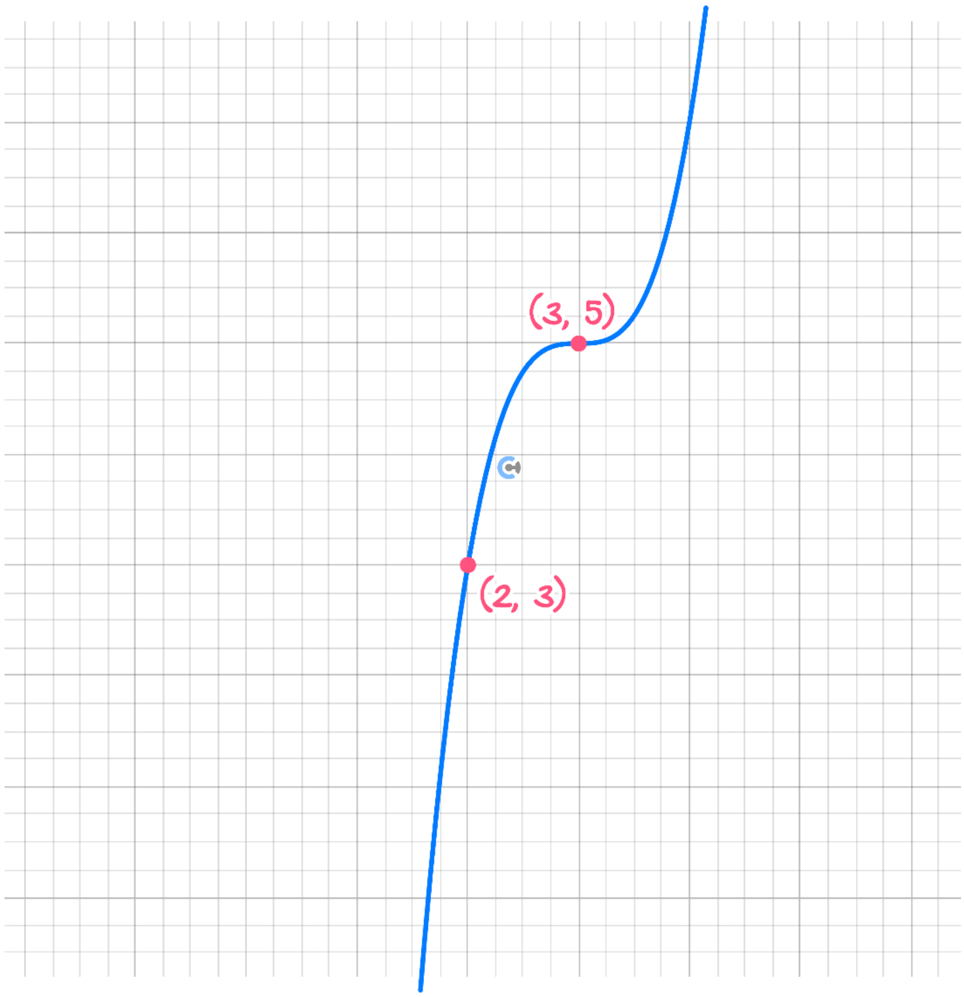
Sub-Section [1.8.5]: Identify Possible Rule(s) From a Graph

Question 43

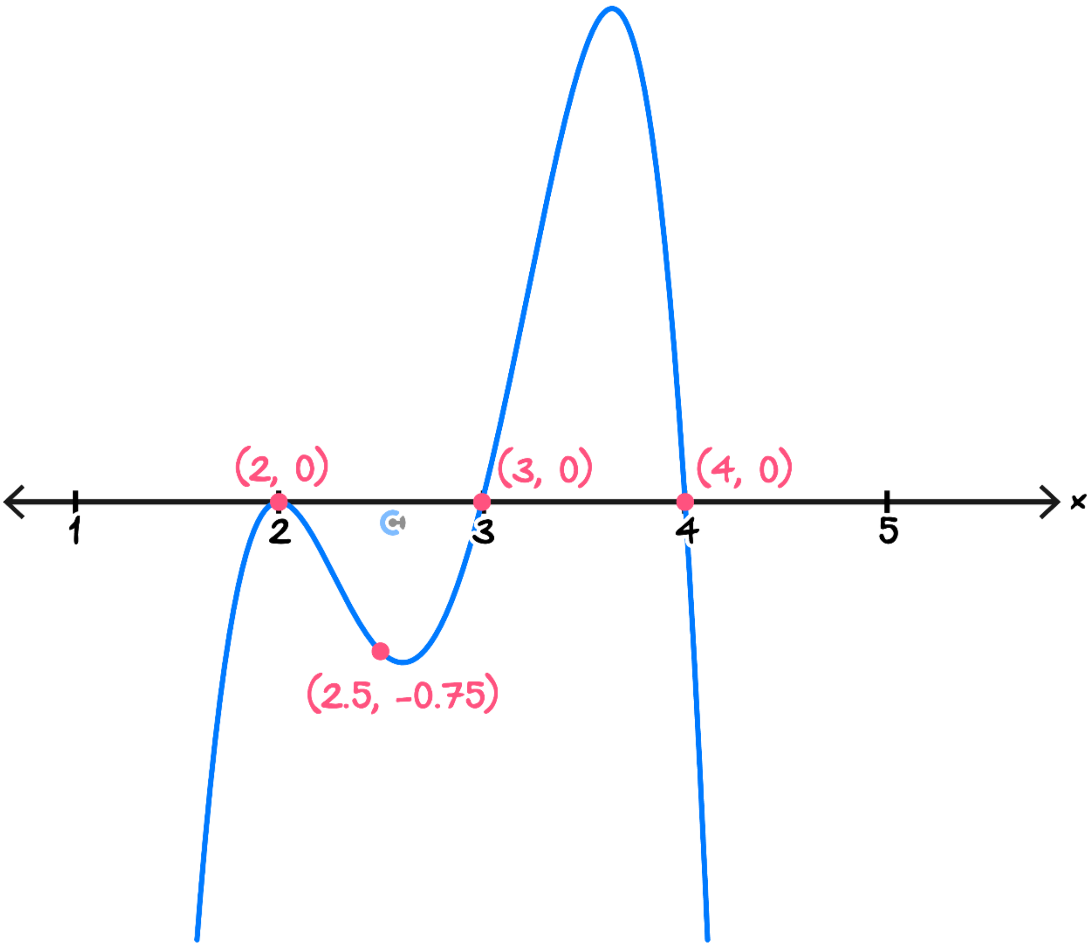
Part of the graph of $f(x)$ is plotted below. The point $(3,5)$ is a stationary point of inflection. Find a possible rule for the function.



The function could be $f(x) = 2(x - 3)^3 + 5$.

Question 44

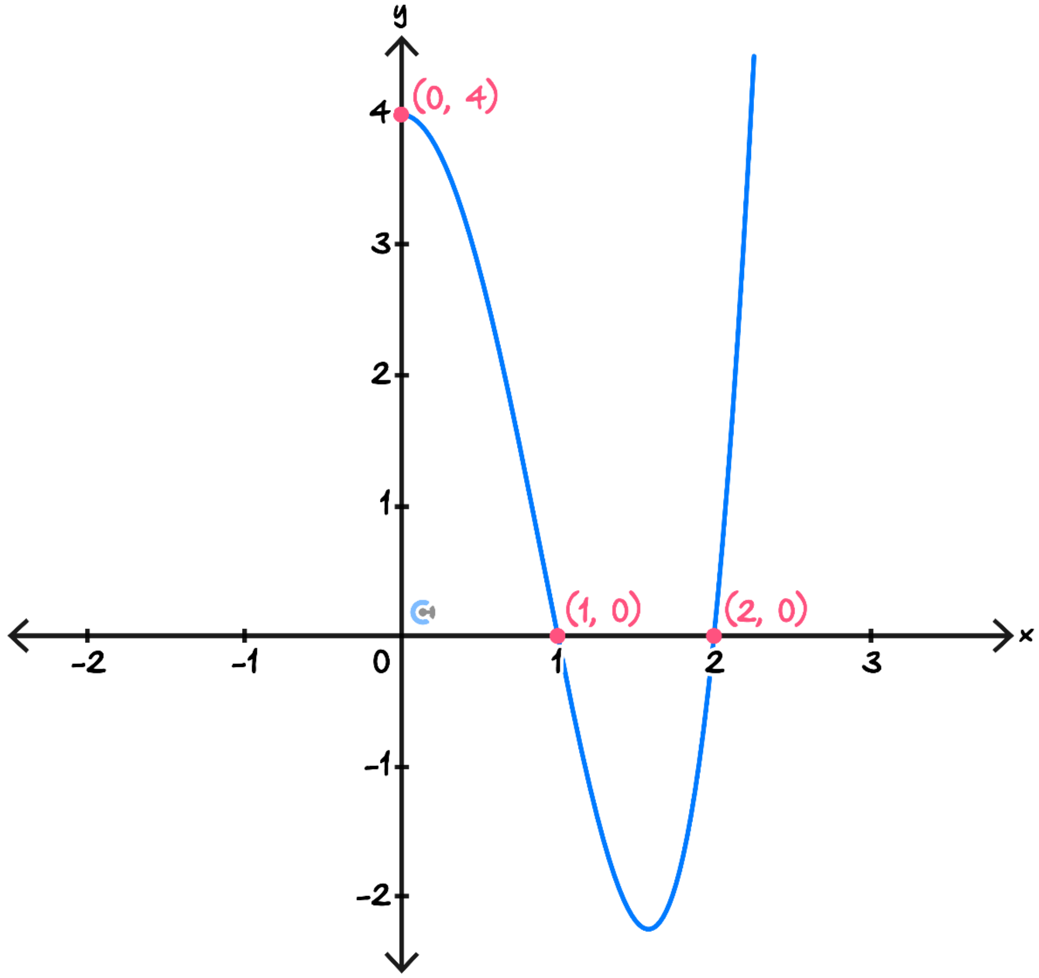
Part of the graph of $f(x)$ is plotted below. Find a possible rule for the function.



The rule for this function could be $f(x) = -4(x-2)^2(x-3)(x-4)$.

Question 45

Part of the graph $f(x)$ is plotted below. Find a possible rule for the function if the function is known to be even.



The rule could be given by $f(x) = (x-1)(x-2)(x+1)(x+2)$.

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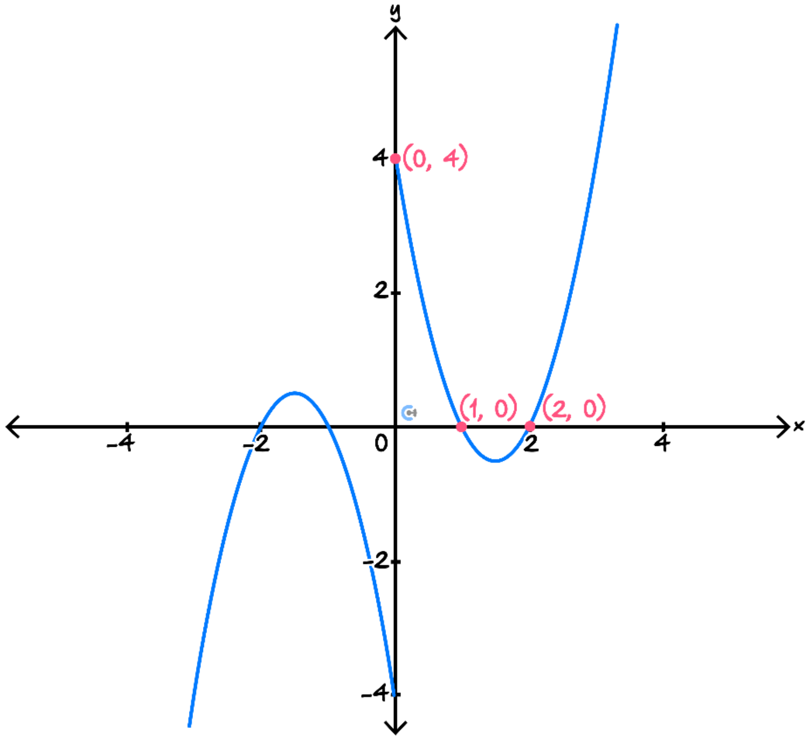
Question 46

Part of the graph $f(x)$ is plotted below.



Find a possible rule for the function if the function is known to be odd. Write your answer in the form.

$$f(x) = \begin{cases} f_1(x), & x < 0 \\ f_2(x), & x > 0 \end{cases}$$

The function could have the rule

$$f(x) = \begin{cases} 2(x-1)(x-2), & x > 0 \\ -2(-x-1)(-x-2), & x < 0 \end{cases}.$$

Note that the rule for $x < 0$ can be obtained by applying a reflection in the y-axis, followed by a reflection in the x-axis to the rule for $x > 0$.

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Sub-Section: Exam 1 Questions

Question 47

Find the value(s) of k so that the equation $(x^2 - kx + 16)(x^2 - 2\sqrt{7}x + k) = 0$ has:

a. Exactly one solution.

Either $x^2 - kx + 16$ has no solutions and $x^2 - 2\sqrt{7}x + k$ has one solution, or $x^2 - kx + 16$ has one solution and $x^2 - 2\sqrt{7}x + k$ has no solutions. Hence $k = 7$ or $k = 8$.

b. Exactly four solutions.

We need both $x^2 - kx + 16$ and $x^2 - 2\sqrt{7}x + k$ to have two solutions. Therefore we need $-8 < k$ or $k > 8$ and also $k < 7$. Hence, $k < -8$.

Question 48

Suppose that $f(x) = x^2 - 7x + 6$ and $g(x) = x^2 - kx + 1$. Find the values of k so that the equation $f(g(x))$ has:

a. Exactly two solutions.

The equation $f(g(x)) = 0$ gives $g(x) = 1$ or $g(x) = 6$. Therefore, there will be two solutions if

- $x^2 - kx + 1 = 1$ gives two solutions and $x^2 - kx + 1 = 6$ gives no solutions: This never happens because $x^2 - kx + 1 = 6$ will always give two solutions.
- $x^2 - kx + 1 = 1$ gives no solutions and $x^2 - kx + 1 = 6$ gives two solutions: Note that this never happens because $x^2 - kx$ always leads to a solution (the discriminant is never negative).
- $x^2 - kx + 1 = 1$ and $x^2 - kx + 1 = 6$ gives a single solution each. Note that this never happens because $x^2 - kx - 5 = 0$ always gives two solutions.

Hence, no such value of k exists.

b. Exactly four solutions.

Similar to above, now both $g(x) = 1$ and $g(x) = 6$ need to give two solutions each. Therefore, $k \neq 0$.

Question 49

Suppose that $f(x)$ is an odd function such that $f(x) = (x-2)^2$ for $x > 0$.

a. Write down a possible rule for $f(x)$ in the form:

$$f(x) = \begin{cases} f_1(x), & x < 0 \\ f_2(x), & x > 0 \end{cases}$$

$$f(x) = \begin{cases} (x-2)^2, & x > 0 \\ -(-x-2)^2, & x < 0 \end{cases}.$$

We can apply reflections across the x-axis and y-axis to obtain the rule for $x < 0$. Note that for $x < 0$, one can simplify the rule further to obtain $-(x+2)^2$.

b. It is known that the tangent to $f(x)$ at the point $x = 3$ is given by the rule $h(x) = 2x - 5$. By applying an appropriate sequence of transformations to $h(x)$, find the rule for the tangent at the point $x = -3$.

By sketching out the function, we can notice that the tangent at $x = -3$ can be obtained through translating the tangent at $x = 3$ so that $(3, 1)$ is mapped to $(-3, -1)$. Therefore, we should translate 6 units to the left and 2 units down. Thus, $k(x) = 2(x+6) - 5 - 2 = 2x + 5$. Alternatively, one could also apply reflections across both the x- and y-axes.

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Question 50

Consider a quartic of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. It is known that the quartic satisfies the following conditions:

- $f(1) = 0$.
- $f(2) = 0$.
- $f(0) = 4$.
- Also, $f(x)$ is even.

a. Find the values of a, b, c, d and e.

We require $b = 0$ and $d = 0$ since $f(x)$ is even. Furthermore, $f(1) = 0$ tells us that $a + c + e = 0$, $f(2) = 0$ tells us that $16a + 4c + e = 0$ and $f(0) = 4$ tells us that $e = 4$. Therefore, $a + c = -4$ and $16a + 4c = -4$. Solving this system of equations, we conclude $a = 1$ and $c = -5$.

b. Verify that $f(x)$ can be factorised to $(x - 1)(x + 1)(x - 2)(x + 2)$.

We have found previously $f(x) = x^4 - 5x^2 + 4$. Now, we can factorise $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$.

c. Find the values of k so that $f(x + k)$ has exactly two positive x-intercepts.

Setting $1 - k > 0$ and $-1 - k \leq 0$ gives $-1 \leq k < 1$.

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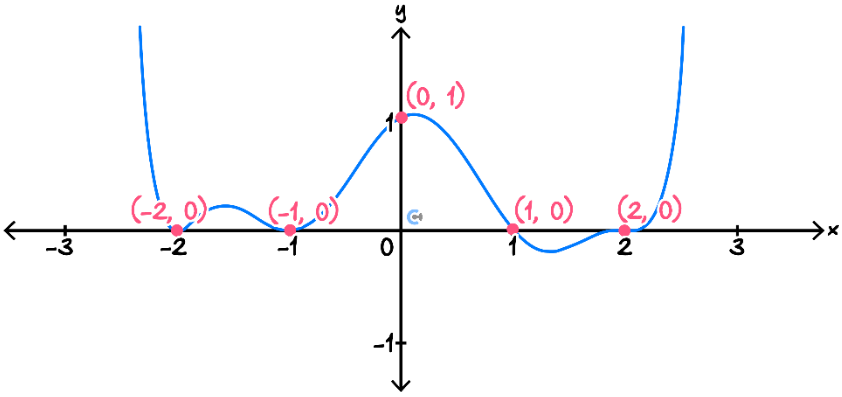
Sub-Section: Exam 2 Questions

Question 51

The minimum degree of the following polynomial is:



A. 2

B. 4

C. 6

D. 8

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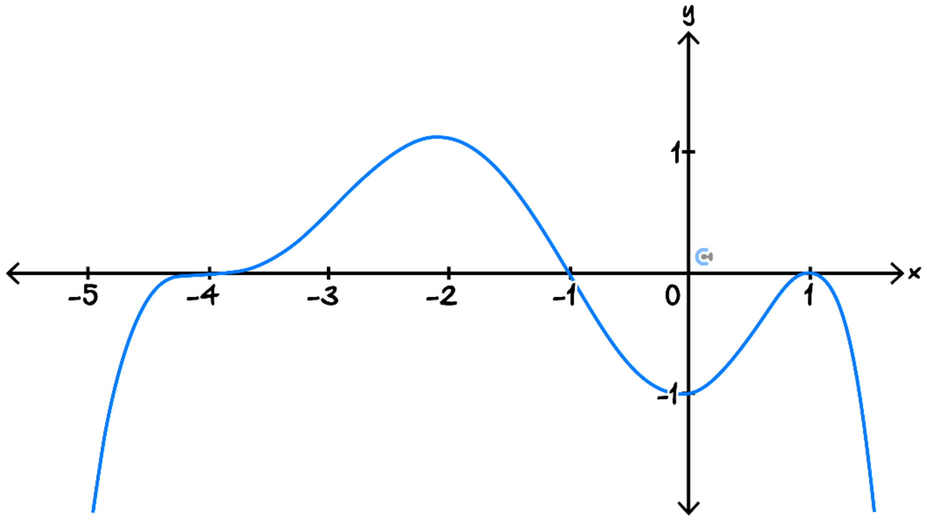
Question 52

A possible rule for the following function given below is:



A. $a(x-1)^3(x+4)^2(x+1)$ where $a<0$.

B. $a(x-1)^3(x+4)^2(x+1)^3$ where $a>0$.

C. $a(x-1)^2(x+4)^3(x+1)$ where $a<0$.

D. $a(x-1)\ (x+4)^3(x+1)$ where $a>0$.

Question 53

Let $f(x)=x^3-(k^2-5k+6)x^2-(k^3+5k)x$. If $f(x)$ is odd, then k must equal:

A. 1 or 3.

B. 1 or 2.

C. 2 or 3.

D. 2 or 6.

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Question 54

Let $g(x) = (x - 1)^2(x - 5)^2 - 4$. There will be exactly four solutions to the equation given by $g(x) = k$ whenever:

A. $-16 < k < 8$

B. $-4 < k < 12$

C. $-4 < k < 0$

D. $-4 < k < 16$

Question 55

Let $h(x) = x^4 - 10x^2 + 9$. The function $h(x + k)$ will have exactly three negative x-intercepts whenever:

A. $1 < k \leq 3$

B. $1 \leq k \leq 3$

C. $-3 < k \leq 1$

D. $-3 \leq k \leq 1$

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Question 56

Consider a cubic of the form $f(x) = ax^3 + bx^2 + cx + d$. Suppose that $f(x)$ satisfies the following conditions:

- $f(0) = 4$.
- $f(1) = 0$.
- $f(-2) = 0$.
- $f(4) = 0$.

a. Calculate the values of a, b, c and d.

Since $f(1) = 0, f(-2) = 0, and\ f(4) = 0$

$\Rightarrow f(x) = n(x-1)(x+2)(x-4) \quad (1)$

Now, at $x = 0, f(x) = 4$

So, sub $x = 0$ and $f(x) = 4$ into (1)

$$4 = n(0-1)(0+2)(0-4)$$

$$\Rightarrow n = \frac{1}{2}$$

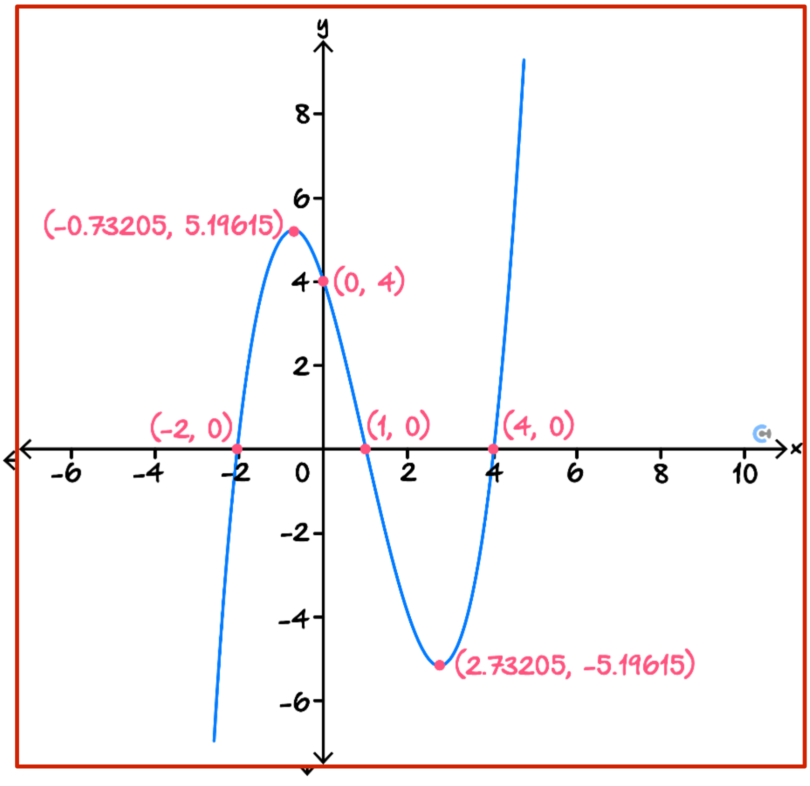
Sub $n = \frac{1}{2}$ into $(1)$and expand.

$$f(x) = \frac{1}{2}(x-1)(x+2)(x-4)$$

$$f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 - 3x + 4$$

Thus, $a = \frac{1}{2}, b = -\frac{3}{2}, c = -3$, and $d = 4$

b. Sketch the graph of the function $y = f(x)$, labelling all turning points and intercepts.



c. Find the value(s) of k such that $f(x) - k = 0$ has exactly:

i. 2 solutions.

$k = 5.19615$ or $k = -5.19615$

ii. 3 solutions.

$-5.19615 < k < 5.19615$

d. Let $g(x) = x^2 - kx + 5$. State the values of k such that $f(g(x)) = 0$ gives the maximum number of solutions possible.

The equation $f(g(x)) = 0$ is solved whenever $g(x) = 1, or \ g(x) = -2, or \ g(x) = 4$. These three equations give exactly two solutions each if $k < -2\sqrt{7}$ or $k > 2\sqrt{7}$.
Therefore, the maximum number of solutions is six, and we have six solutions whenever $k < -2\sqrt{7}$ or $k > 2\sqrt{7}$.

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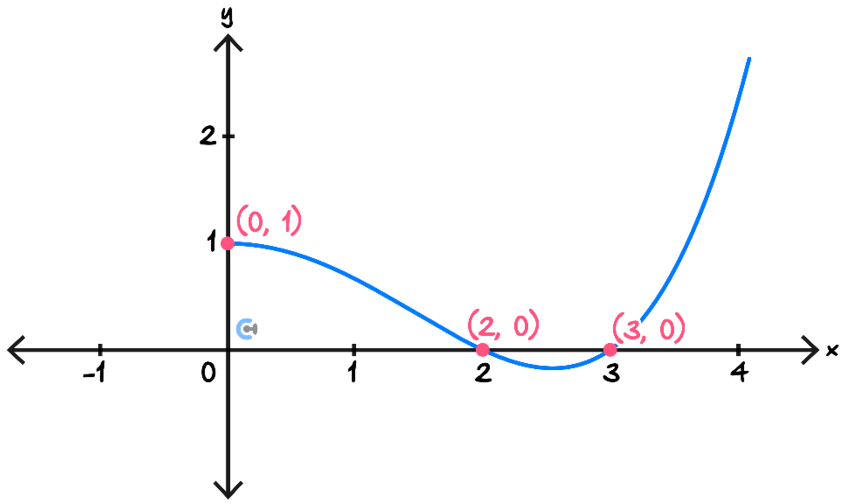
Question 57

The part of the graph of $f(x)$ is shown below. Furthermore, it is known that the function $f(x)$ is a quartic and also even.



a. State the rule for $f(x)$.

$$f(x) = \frac{1}{36}(x-2)(x+2)(x+3)(x-3)$$

b. The tangent to the graph of $f(x)$ at $x = 3$ is given by $y = \frac{5}{6}x - \frac{5}{2}$.

i. Describe a sequence of transformation(s) that can be applied to $h(x)$ to obtain the tangent to the graph of $f(x)$ at $x = -3$.

Reflection across the y-axis.

ii. Hence, write down the rule for the tangent to the graph of $f(x)$ at $x = -3$.

$$y = -\frac{5}{6}x - \frac{5}{2}$$

c. State the values of k so that $f(x-k)$ has exactly:

i. 3 positive x-intercepts.

$2 < k \leq 3$

ii. 3 negative x-intercepts.

$-3 \leq k < -2$

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