Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Polynomials [1.7]
Workbook

Outline:

Learning Objectives:

- ☐ **MM34 [1.7.1]** – Apply Factor Theorem and Remainder Theorem to identify the roots, remainders and find unknown of a function.
- ☐ **MM34 [1.7.2]** – Find factored form of polynomials.
- ☐ **MM34 [1.7.3]** – Graph factored and unfactored polynomials.
- ☐ **MM34 [1.7.4]** – Identify odd, even functions and correct power functions.

Section A: Algebra of Polynomial Functions

Sub-Section: Roots of a Polynomial

Roots of Polynomial Functions

$$\text{Roots} = x\text{-intercept}$$

Discussion: Can there be more roots than the degree?

Question 1

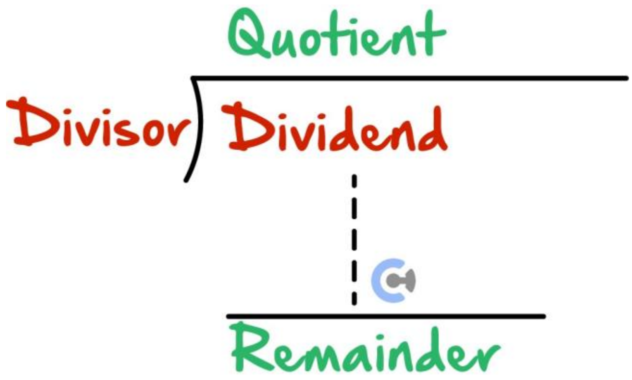
Find the roots of the following polynomials:

$$(x-2)^2(x+4)^4$$

Space for Personal Notes

Sub-Section: Long Division

Polynomial Long Division



$$\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$$

Question 2 Walkthrough.

Simplify the following using polynomial long division:

$$\frac{3x^3 + 10x + 20}{2x + 4}$$

TIPS:

- Always focus on the highest degree term first.
- Always remember to fill any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Question 3

Simplify the following using polynomial long division:

$$\frac{x^4 - 5x^3 + 5x^2 - 10x + 6}{x^2 + 2}$$

Space for Personal Notes

Sub-Section: Remainder Theorem

How can we find the remainder without long division?

Exploration: Derivation of the remainder theorem.

- Consider $\frac{f(x)}{g(x)}$.

$$\frac{f(x)}{g(x)} = q(x) + \frac{R}{g(x)}, \text{where } R = \text{Remainder}$$

- Let's multiply everything by $g(x)$.

$$f(x) =$$

- Remember, we are trying to find the remainder R before we do long division.
  - What functions do we already have before long division?

$$f(x) = q(x) \cdot g(x) + R$$

- How can we get $f(x)$ to equal to the remainder R?
  - We can substitute a value of x such that the ________________ is equal to 0.

$$f(\alpha) =$$

$$f(\alpha) =$$

Space for Personal Notes

Definition

Remainder Theorem

- **Definition**:
  - Find the remainder of long division without the need of long division.

$$\text{when } P(x) \text{ is divided by } (x-\alpha), \text{the remainder is } P(\alpha)$$

- **Steps:**

1. Find x-values which makes the divisor equal to 0.
2. Substitute it into the dividend function.

Question 4 Walkthrough.

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where $f(x) = x^3 - x^2 + x + 1$ and $g(x) = x - 1$.

Your turn!

Active Recall: Remainder Theorem

1. Find x-values which makes the ______________ equal to 0.
2. Substitute it into the ________________ function.

Question 5

Find the remainder of the division, $\frac{f(x)}{g(x)}$, where $f(x) = x^3 - x^2 + 4x - 2$ and $g(x) = 3x + 6$.

Question 6 Extension.

For the polynomial $f(x) = -2x^3 + x^2 + (10 - 3\alpha)x + 9$, we get a remainder of 23 when $f(x)$ is divided by $g(x) = x + 2$. Find the value of α.

Sub-Section: Factor Theorem

Factor Theorem

- For every x-intercept, there is a factor.

$$\boldsymbol{if\ P(\alpha) = 0, then\ (x - \alpha)\ is\ a\ factor\ of\ P(x)}$$

How does the factor theorem work?

Exploration: Derivation of factor theorem.

- Consider $\frac{f(x)}{g(x)}$ and $g(\alpha) = 0$.

$$\boldsymbol{Remainder =}$$

  - What happens if $f(\alpha) = 0$?

$$\boldsymbol{Remainder = \_\_\_\_\_}$$

  - Hence, what can we say about $g(x)$?

$$\boldsymbol{f(x)\ is\ divisible\ by\ \_\_\_\_\_\_\_\_\_\_}$$

$$\boldsymbol{hence, g(x)\ is\ a\ \_\_\_\_\_\_\_\_\_\_\_\_\_\ of\ f(x)}$$

Space for Personal Notes

Question 7 Walkthrough.

Determine if $x+1$ is a factor of $P(x) = 2x^3 + x^2 - 4x - 3$.

Your turn!

Question 8

If $(x-\alpha)$ is a factor of $4x^5 - 11\alpha x^3 + x^4$, what must be the value of α, where $\alpha \neq 0$?

Sub-Section: Factorising Polynomials

Factorising Cubic Polynomials

Definition

- **Steps:**

1. Find a single root by trial and error.
   - (Factor Theorem: Substitute into the function and see if we get zero.)
2. Use long division to find the quadratic factor.
3. Factorise the remaining factor.

Question 9 Walkthrough.

Find all the roots of $f(x) = x^3 + 4x^2 + x - 6$.

Space for Personal Notes

Your turn!

Active Recall: Factorising Cubic Polynomials

- **Steps:**
  1. Find a single root by trial and error.
     - (Factor Theorem: Substitute into the function and see if we get ________.)
  2. Use long division to find the ____________________ factor.
  3. Factorise the remaining factor.

Question 10

Find all the roots of $f(x) = x^3 + 13x^2 + 20x - 100$.

Question 11 Extension.

Find all the roots of $f(x) = x^3 - 2x^2 - 29x - 42$.

Sub-Section: Rational Root Theorem

That was quite tedious! Is there a better way of finding the first root?

Rational Root Theorem

- Rational Root Theorem **narrows down** the possible roots.

$$\text{potential root} = \pm \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$$

- If the roots are rational numbers, the roots can only be $\pm \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$.

Question 12 Walkthrough.

Find all the roots of $f(x) = x^3 + \frac{3}{2}x^2 - 25x + 12$.

NOTE: You should always factorise the fraction factors out first. In above example, the $\frac{1}{2}$!

Question 13

Find all the roots of $f(x) = 2x^3 - 3x^2 - 23x + 12$.

Question 14 Extension.

Find all the roots of $f(x) = 2x^3 + \frac{7}{2}x^2 - 49x + 12$.

Sub-Section: Sum and Difference of Cubes

Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Question 15

Factorise the following polynomials as much as possible:

$$27x^3 + 216$$

Discussion: What is the discriminant of $a^2 \pm ab + b^2$ factor equal to?

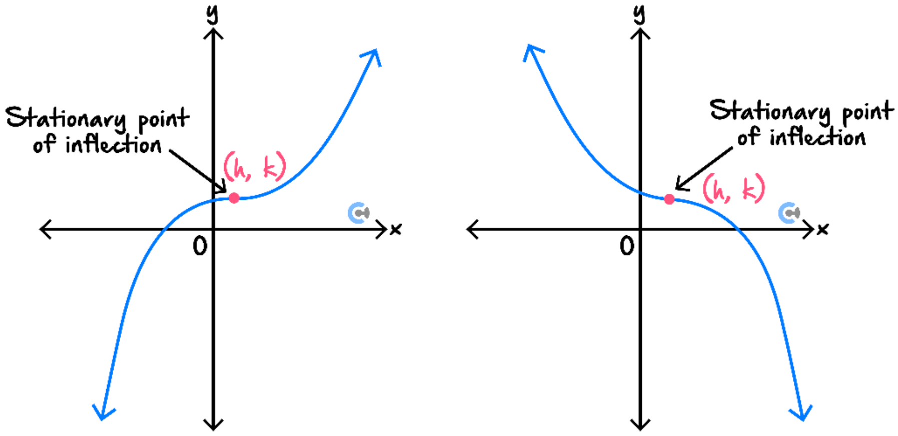
Section B: Graphs of a Polynomial

Sub-Section: Graphing Polynomials in the Form of $a(x-h)^n + k$

Let's look at odd powers first.

Graphs of $a(x-h)^n + k$, Where n is Odd and Positive

- All graphs look like a "cubic".

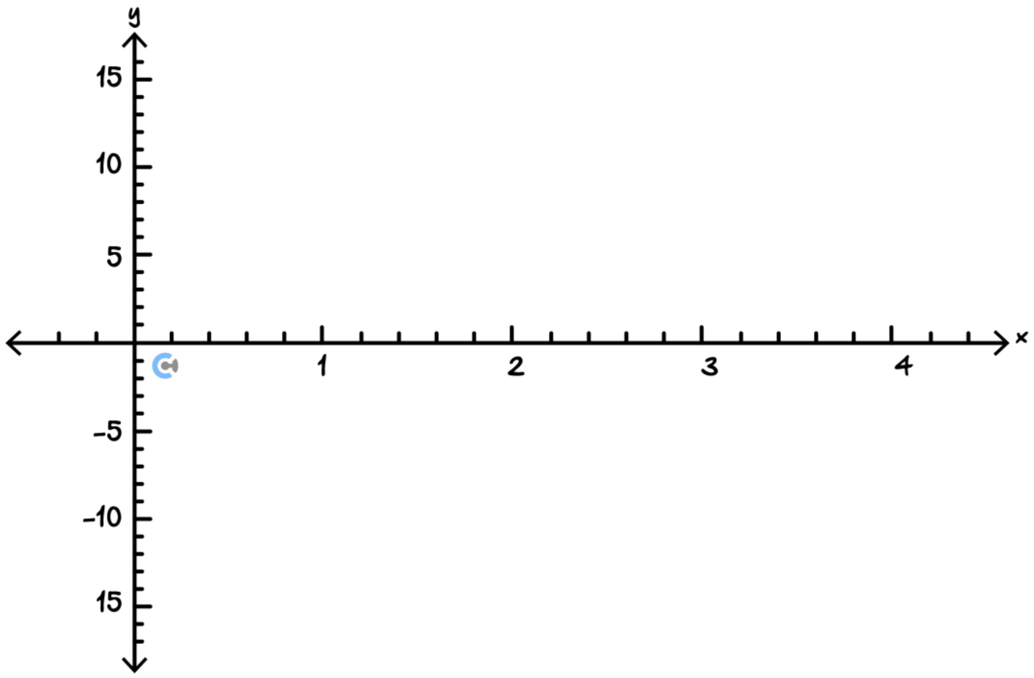


- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Space for Personal Notes

Question 16

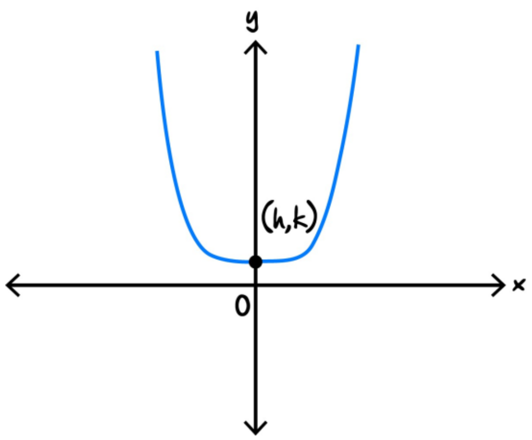
Sketch the graph of $y = (x-2)^3 - 1$ on the axes below.

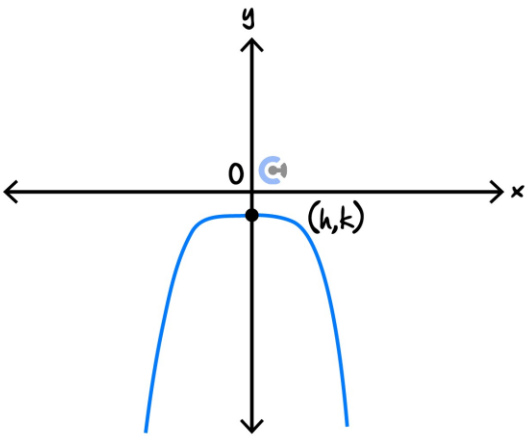


What about even powers?

Graphs of $a(x-h)^n + k$, where n is Even and Positive

- All graphs look like a "quadratic".

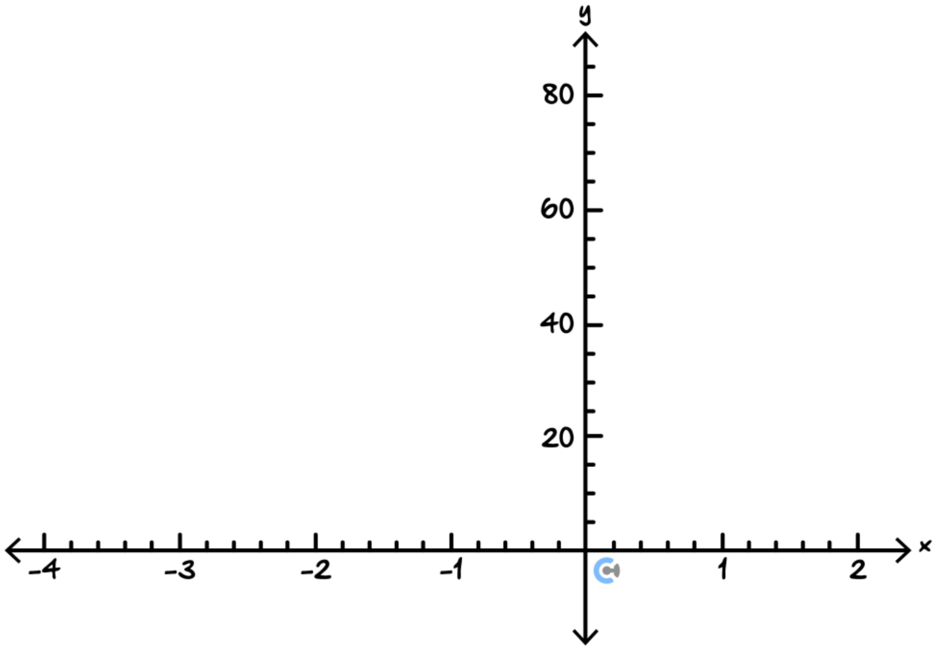




- The point (h, k) gives us the turning point.

Question 17

Sketch the graph of $y = (x + 1)^4 + 2$ on the axes below.



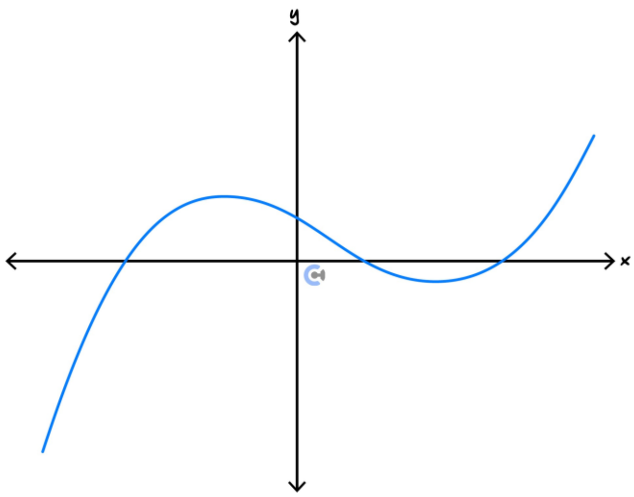
Space for Personal Notes

Sub-Section: Graphing Factorised Polynomials

What about the factorised polynomial?

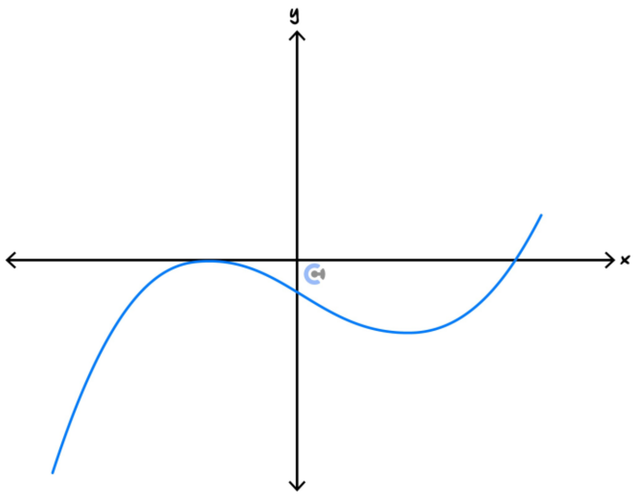
Exploration: Graphs of Factorised Polynomials

All ______________ linear factors correspond to ______________________ of the graph



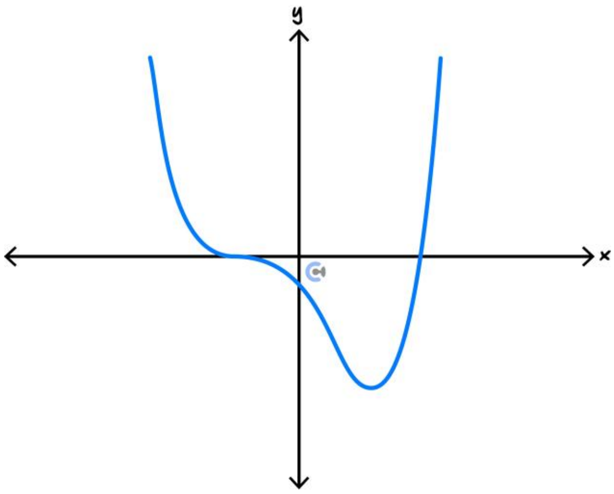
- E.g., $f(x) = (x-a)(x-b)(x-c)$ results in x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

All ______________ linear factors correspond to __________________________________ of the graph



- E.g., $f(x) = (x-a)^2(x-b)$ will have an x-intercept $(a, 0)$ which is also a local minimum/maximum.

All _______________ linear factors correspond to ______________________________________ of the graph.



- E.g., $f(x) = (x-a)^3(x-b)$ has an x-intercept $(a, 0)$ which is also a stationary point of inflection.

Definition

<u>Graphs of Factorised Polynomials</u>

- **Steps:**

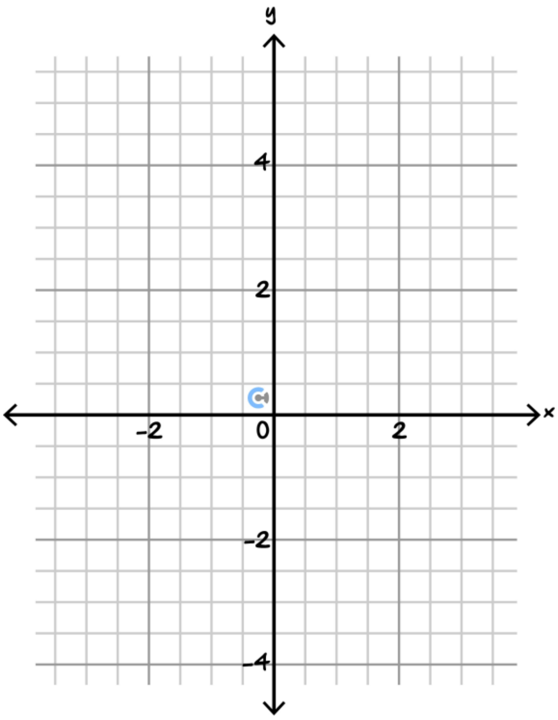
1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape:
   - Non - Repeated: Only x-intercept.
   - Even Repeated: x-intercept and a turning point.
   - Odd Repeated: x-intercept and a stationary point of inflection.

Space for Personal Notes

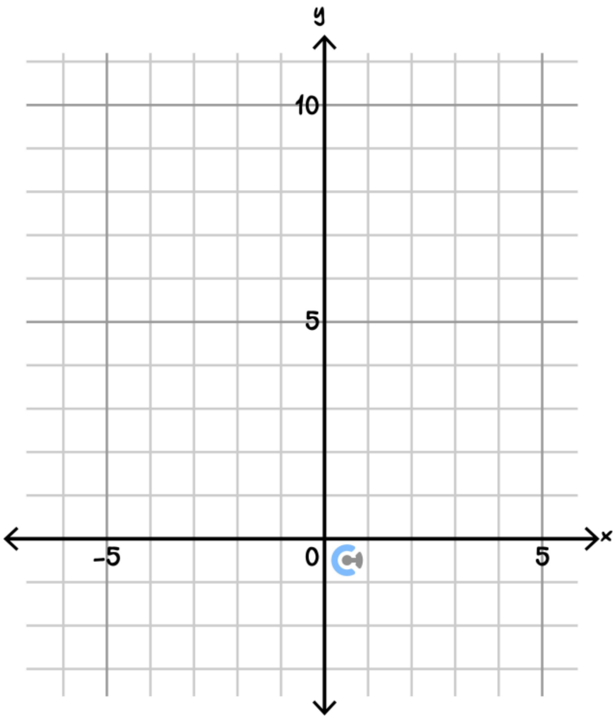
Question 18 Walkthrough.

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (x-1)^2(x+1)$



b. $y = \left(x + \frac{5}{2}\right)^3 (3 - x)$



Your turn!

Active Recall: Steps of sketching factorised polynomials

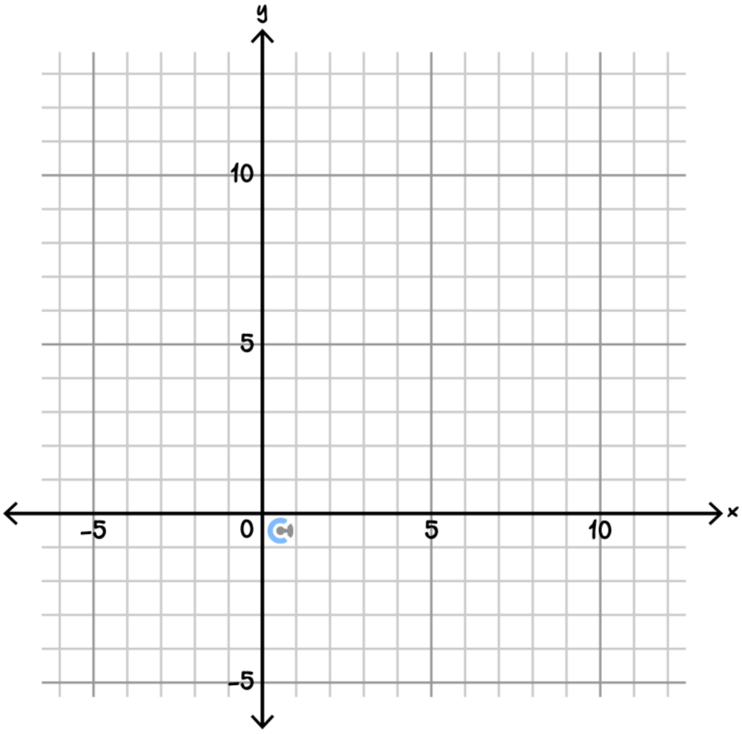
1. Plot ________________.
2. Determine whether the polynomial is positive or ______________.
3. Use the repeated factors to deduce the shape:
   - Non – Repeated: Only ________________.
   - Even Repeated: x-intercept and a ________________________.
   - Odd Repeated: x-intercept and a ______________________________________.

Space for Personal Notes

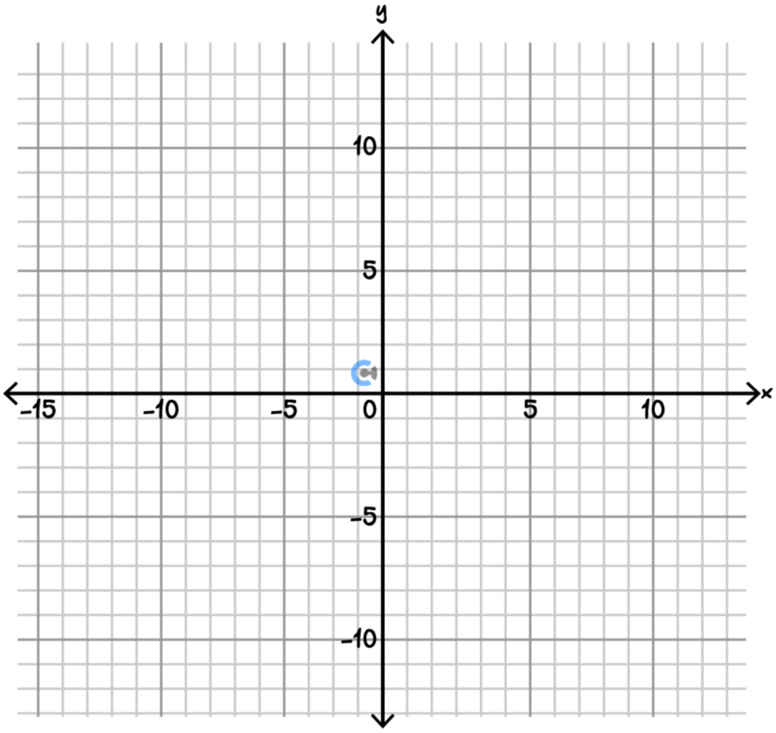
Question 19

Sketch the graphs of the following functions on the axes provided. Ignore the y-axis scale.

a. $y = (1+x)(6-x)^2$



b. $y = (x+8)^3(x-3)(x-7)$

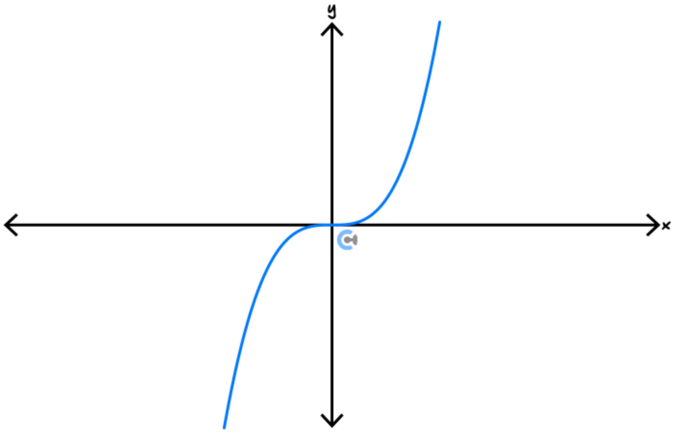


Section C: Odd and Even Functions

Sub-Section: Odd Functions

Let's consider odd functions!

Odd Functions



- E.g.: $x^3, x^5, x^7 - x^3$... see the pattern?

$$f(-x) = -f(x)$$

- Property: Reflecting around the ______________ is the same as reflecting around the ______________.

Question 20

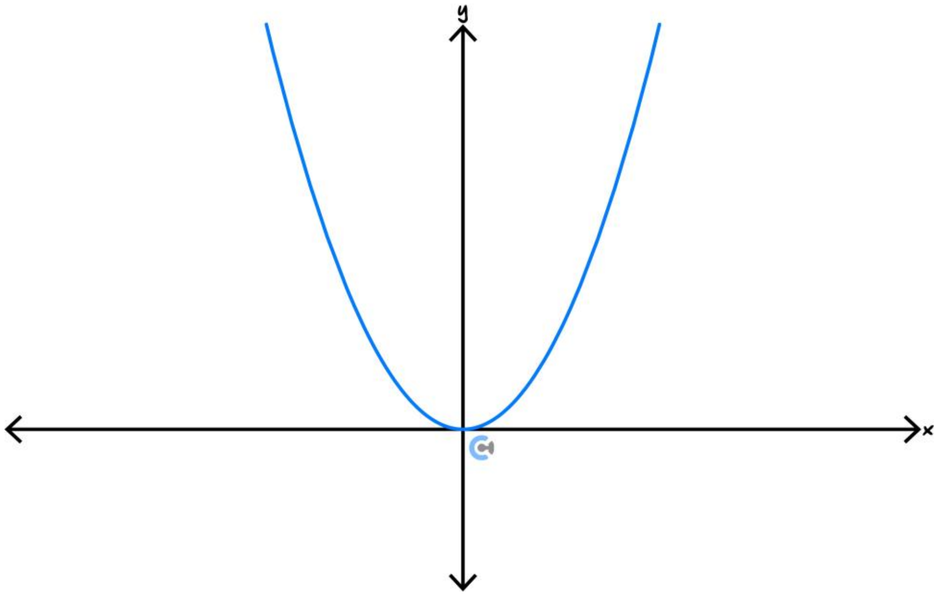
Show that the following function is an odd function.

$$h(x) = -x^3 + 5x$$

Sub-Section: Even Functions

What about even functions?

Even Functions



- E.g.: $x^2, x^4, -x^{10}, x^4 - 4$... see the pattern?

$$f(-x) = f(x)$$

- Property: It is symmetrical around the ______________.

Space for Personal Notes

Question 21

Identify whether the function $f(x) = 2x^2 - 4x$ is an even function, odd function or neither.

Space for Personal Notes

Section D: Power Functions

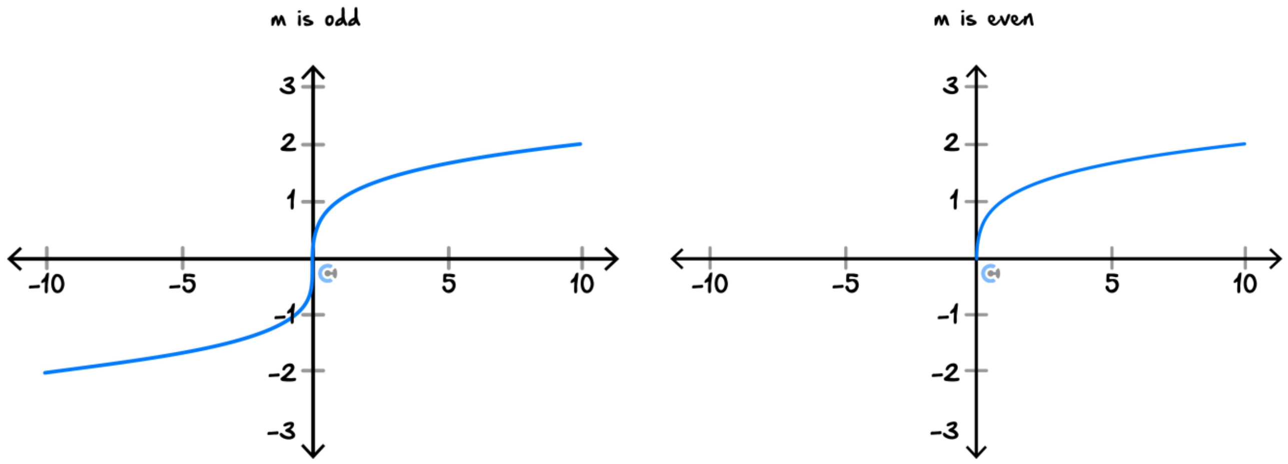
Sub-Section: Power Functions

How do we generalise power functions?

Exploration: Power Functions

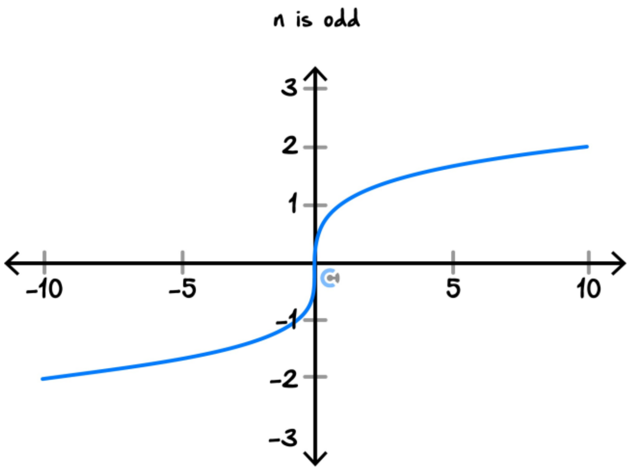
$$y = x^{\frac{n}{m}}$$

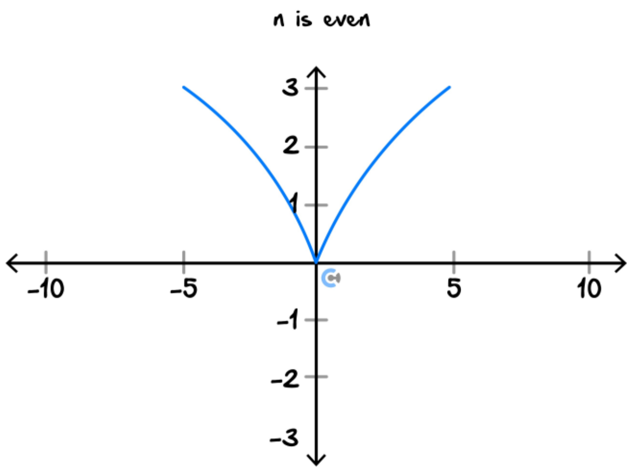
- m: Dictates the number of **tails**.



- **Why?**

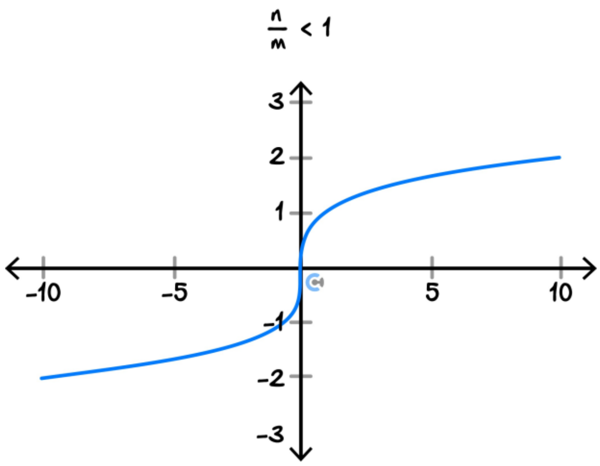
- n: Dictates the **range**.

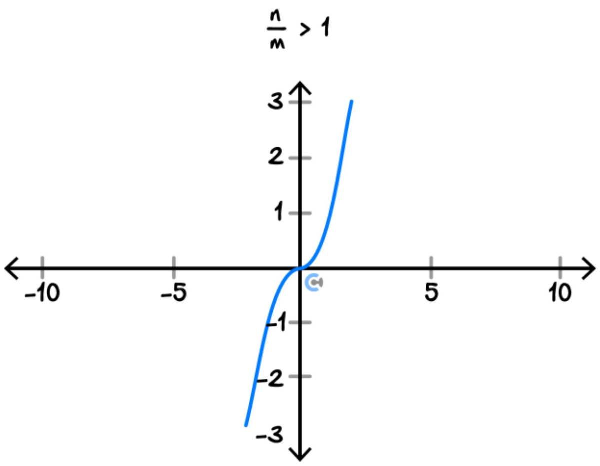




Why?

- $\frac{n}{m}$(Power): Dictates **concavity** (i.e., does the function look like a polynomial or a root?)





Space for Personal Notes

Definition

Power Functions

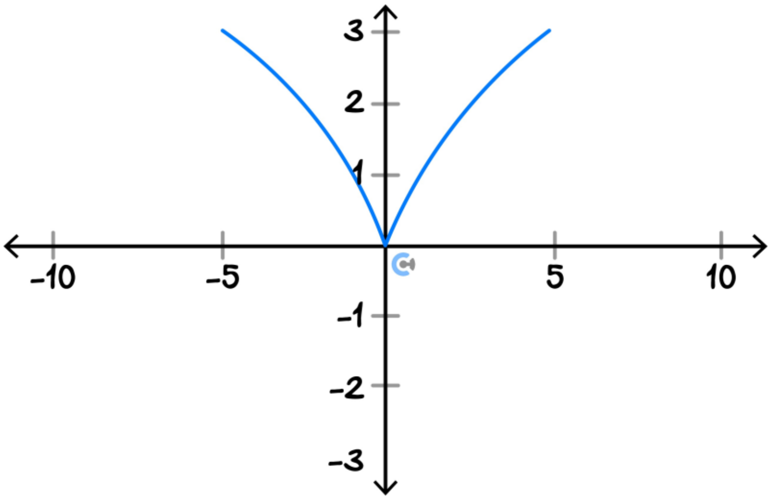
$$y = x^{\frac{n}{m}}$$

- m: Dictates the number of **tails**.
  - **Odd m = Two tails.**
  - **Even m = One tail.**
- n: Dictates the **range**.
  - Odd n: Range could be all real.
  - Even n: Range must be non-negative.
- $\frac{n}{m}$ $(Power)$:
  - Power > 1: Looks like a polynomial function.
  - Power < 1: Looks like a root function.

Space for Personal Notes

Question 22 Walkthrough.

Find a possible rule for the given function out of the 4 options below.



A. $x^{\frac{2}{3}}$

B. $x^{\frac{3}{2}}$

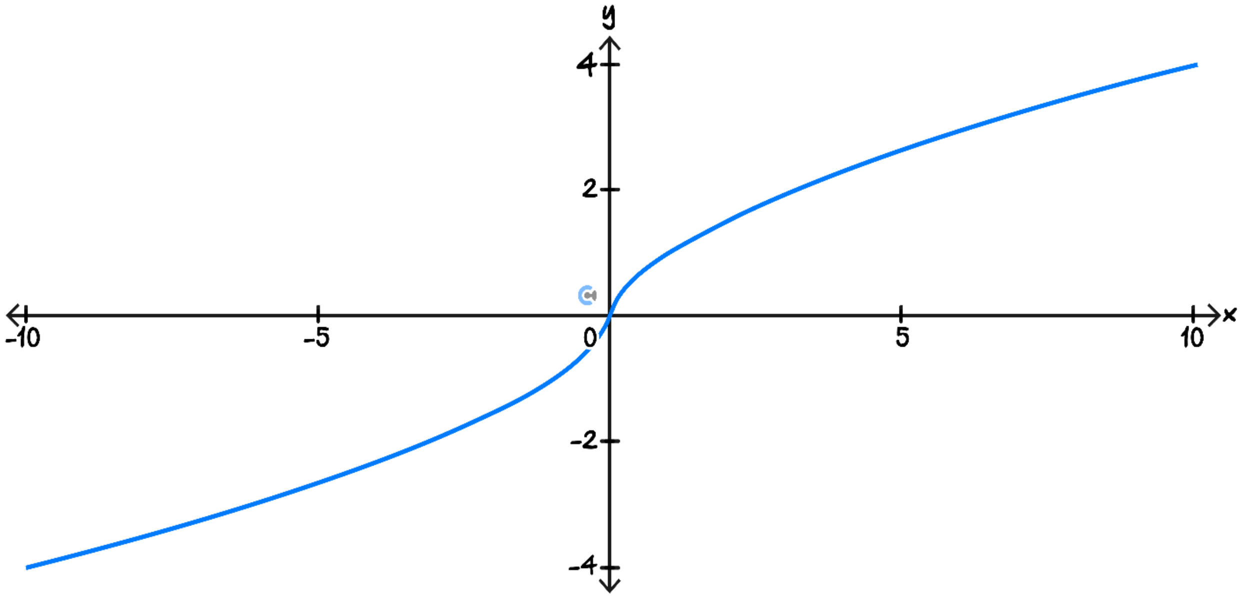
C. $x^{\frac{1}{5}}$

D. $-x^{\frac{2}{7}}$

Space for Personal Notes

Question 23

Find a possible rule for the given function out of the 4 options below.



A. $x^{\frac{2}{3}}$

B. $x^{\frac{3}{5}}$

C. $x^{\frac{2}{5}}$

D. $-x^{\frac{3}{5}}$

Space for Personal Notes

Contour Check

Learning Objective: [1.7.1] – Apply Factor Theorem and Remainder Theorem to identify the roots, remainders and find unknown of a function.

Key Takeaways

- [ ] The degree of a polynomial is the polynomial's ________________________ power.
- [ ] The roots of a polynomial are its _________________________.
- [ ] For polynomial long division:

$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient} + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- [ ] When $P(x)$ is divided by $(x - \alpha)$, the remainder is ________.
- [ ] If $P(\alpha) = 0$, then $(x - \alpha)$ is a _________________ of $P(x)$.

Learning Objective: [1.7.2] - Find factored form of polynomials.

Key Takeaways

- Steps to factor a cubic polynomial are:
  - Find a single root by trial and error.

    (Factor Theorem: Substitute into the function and see if we get ________________.)
  - Use __________________ to find the quadratic factor.
  - Factorise the quadratic.
- Rational Root Theorem **narrows down** the possible roots. If the roots are rational numbers, it must be that any:

$$\text{Potential root} = \pm \frac{\text{Factors of } \_\_\_\_\_\_\_\_\_\_ \; a_0}{\text{Factors of } \_\_\_\_\_\_\_\_\_\_ \; a_n}$$

- Sum and difference of cubes:

$$a^3 + b^3 = (\_\_\_\_\_\_\_\_\_\_)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (\_\_\_\_\_\_\_\_\_\_)(a^2 + ab + b^2)$$

Learning Objective: [1.7.3] - Graph factored and unfactored polynomials.

Key Takeaways

- Graphs of $a(x-h)^n + k$, where n is an odd positive integer that is not equal to 1:
  - The point (h, k) gives us the stationary point of ______________.
- Graphs of $a(x-h)^n + k$, where n is an even positive integer:
  - The point (h, k) gives us the ____________________.
  - These graphs look like a __________________.
- Steps to graphing factorised polynomials:
  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape:
     - Non-Repeated: Only ___________________.
     - Even Repeated: x-intercept and a ________________________.
     - Odd Repeated: x-intercept and a _____________________________________.

Learning Objective: [1.7.4] - Identify odd, even functions and correct power functions.

Key Takeaways

- **Odd functions:**

$$f(-x) = -f(x)$$

  - Property: Reflecting around the _____________ is the same as reflecting around the _____________.

- **Even functions:**

$$f(-x) = f(x)$$

  - Property: It is symmetrical about the ______________.

- **Power Functions:**

$$y = x^{\frac{n}{m}}$$

  - m: Dictates the number of **tails**.
    - **Odd m = ____________ tails.**
    - **Even m = ____________tail.**
  - n: Dictates the **range**.
    - Odd n: Range could be _______________.
    - Even n: Range must be ________________.
  - Power > 1: Looks like a ____________________ function.
  - Power < 1: Looks like a ______________ function.

CONTOUR EDUCATION

Website: contoureducation.com.au | **Phone:** 1800 888 300 | **Email:** hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make The Most of These (Free) Services!

- Experienced Contour tutors (45+ raw scores, 99+ ATARs).
- For fully enrolled Contour students with up-to-date fees.
- After school weekdays and all-day weekends.

1-on-1 Video Consults	Text-Based Support
➤ Book via bit.ly/contour-methods-consult-2025 (or QR code below). ➤ One active booking at a time (must attend before booking the next).	➤ Message +61 440 138 726 with questions. ➤ Save the contact as “Contour Methods”.

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726