Website: contoureducation.com.au | Phone: 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Polynomials [1.7]
Homework Solutions

Homework Outline:

Compulsory Questions	Pg 2 – Pg 20
Supplementary Questions	Pg 21 – Pg 37

Section A: Compulsory Questions

Sub-Section [1.7.1]: Applying Factor and Remainder Theorems

Question 1

a. State the remainder when $x^3 - 3x + 2$ is divided by $x - 3$.

We can write $x^3 - 3x + 2 = (x-3)Q(x) + r$ for some quadratic Q. Hence,

$$r = 3^3 - 3(3) + 2 = 20$$

b. State the remainder when $x^4 + 2x + 1$ is divided by $2x + 2$.

We can write $x^4 + 2x + 1 = (2x+2)Q(x) + r$ for some cubic Q. Hence,

$$r = (-1)^4 + 2(-1) + 1 = 0$$

c. Is $x - 3$ a factor of $f(x) = x^3 - 8x + 3$?

$x - 3$ is a factor of $f(x)$ if and only if $f(3) = 0$. As

$$f(3) = (3)^3 - 8(3) + 3 = 27 - 24 + 3 = 6 \neq 0,$$

$x - 3$ is not a factor of f.

Question 2

Let $f(x) = ax^3 + 4x + 1$. Find the value of a such that $f(x)$ has a factor of $2x + 1$.

We require $f\left(-\frac{1}{2}\right) = \frac{-a}{8} - 2 + 1 = 0.$

Hence $a = -8$

Question 3

a. Let $f(x) = ax^2 + 3x + c$. Find the values of a and b such that f has a factor of $2x + 5$, and when f is divided by $2x + 1$, it has a remainder of -6.

Since $2x + 5$ is a factor of f, we see that, $f\left(\frac{-5}{2}\right) = 0$.

Since $f/(2x + 1)$ yields a remainder of -6, we see that, $f\left(\frac{-1}{2}\right) = -6$. From here we construct a pair of simultaneous equations,

$$f\left(\frac{-5}{2}\right) = \frac{25a}{4} - \frac{15}{2} + c = 0 \qquad (1)$$

$$f\left(\frac{-1}{2}\right) = \frac{a}{4} - \frac{3}{2} + c = -6 \qquad (2)$$

Subtracting (2) from (1) yields,

$$\frac{24a}{4} - \frac{15-3}{2} = 6 \implies 6a = 12 \implies a = 2.$$

Substituting this value of a into (2) yields, $c = -6 + \frac{3}{2} - \frac{a}{4} = \frac{-11}{2} + \frac{1}{2} = -5$

b. Tech-Active.

Let $g(x) = ax^3 + bx^2 + cx + d$, have the following properties,

1. $g(x)$ has a factor of $x^2 - 1$.

2. $g(x)$ divided by $x - 2$ leaves a remainder of 7.

3. $g(x)$ divided by $2x + 3$ leaves a remainder of -4.

Find the values of a, b, c and d.

Since $g(x)$ has a factor of $(x^2 - 1)$, and $(x^2 - 1)$ has both ± 1 as roots, we realize that:

$$g(-1) = g(1) = 0$$

Since $\frac{g(x)}{x-2}$ leaves a remainder of 7, we realize that $g(2) = 7$

Since $\frac{g(x)}{2x+3}$ leaves a remainder of -4, we realize that $g\left(\frac{-3}{2}\right) = -4$

This leaves us with a system of 4 simultaneous equations that we can solve for a, b, c, and d using a calculator.

Thus,

$$a = \frac{166}{105} \text{ and } b = -\frac{29}{35} \text{ and } c = -\frac{166}{105} \text{ and } d = \frac{29}{35}$$

Sub-Section [1.7.2]: Finding Factored Forms of Polynomials

Question 4

Factorise the following polynomials:

a. $8x^3 + 27$.

We apply difference of cubes with $a = 2x$ and $b = 3$, thus,

$$8x^3 + 27 = (2x + 3)((2x)^2 + 2(2x)(3) + (3)^2) = (2x + 3)(4x^2 + 12x + 9)$$

Since $12^2 - 4(4)(9) < 0$ we can not factorise this expression any further.

b. $x^3 - 4x^2 - x + 4$.

$$x^3 - 4x^2 + x - 4 = x^2(x - 4) - (x - 4) = (x - 4)(x^2 - 1) = (x - 4)(x - 1)(x + 1)$$

c. $x^3 + 2x^2 + x$.

$$x^3 + 2x + x = x(x^2 + 2x + 1) = x(x + 1)^2$$

Evaluate the following expression without a calculator:

d. $7^3 - 5^3$.

We apply the difference of cubes formula, giving us,

$$7^3 - 5^3 = (7 - 5)(49 + 35 + 25) = 2(109) = 218$$

Question 5

a. Let $f(x) = x^3 - 2x^2 - 5x + 6$.

i. Show that $f(1) = 0$.

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$

ii. Hence, or otherwise, write $f(x)$ in the form $f(x) = (x-a)(x-b)(x-c)$ for integers a, b, c.

Since $f(1) = 0$ we know that $(x-1)$ is a factor of f, hence.

$$f(x) = (x-1)(ax^2 + bx + c) = ax^3 + (b-a)x^2 + (c-b)x - c$$

By comparing the x^3 coefficient, we see that $a = 1$.
By comparing the x^2 coefficient, we see that $b - a = b - 1 = -2 \implies b = -1$.
By comparing the x coefficient, we see that $c - b = c + 1 = -5 \implies c = -6$.
We can check our result with the constant term as $-c = -(-6) = 6$.
Hence $f(x) = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$
Note you should be able to do this in your head and do not need to show this working.

b. Factorise $g(x) = x^3 - 2x^2 - 9x + 18$.

$g(x) = x^2(x-2) - 9(x-2) = (x^2 - 9)(x-2) = (x-3)(x+3)(x-3)$

c. Find all of the real roots of $h(x) = x^3 + 2x^2 - 29x - 30$.

We first test ± 1 as factors.

For $x = 1$ we see that $h(1) = 1 + 2 - 29 - 30 \neq 0$, hence $x - 1$ is not a factor of h.

For $x = -1$ we see that $h(-1) = (-1)^3 + 2(-1)^2 - 29(-1) - 30 = -1 + 2 + 29 - 30 = 0$, hence $x + 1$ is a factor of h.

Now we factorise $h(x)$ using the process in part a, thus $h(x) = (x+1)(x^2 + x - 30) = (x+1)(x-6)(x+5)$

Hence all the roots of h are $x = -5, -1, 6$.

d. Tech-Active.

Factorise $P(x) = 6x^5 + 11x^4 - 49x^3 - 41x^2 + 115x - 42$.

Use the factorise function on your calculator. Hence,

$$P(x) = (x-1)(x-2)(x+3)(2x-1)(3x+7)$$

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Question 6

a. Let $f(x) = 9x^3 - 54x^2 - x + 6$.

i. According to the rational root theorem, what are the possible rational roots of f?

We first check that all coefficients of f are integers and have a GCD of 1. The first part is obviously true, whilst the second part is true as one of the coefficients of f is -1. Now the possible roots of f are of the form,

$$\pm\frac{p}{q}$$

where p divides 6 and q divides 9. Thus $p = 1, 2, 3, 6$ and $q = 1, 3, 9$ and our roots can be,

$$1, -1, 2, -2, 3, -3, 6, -6, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{9}, -\frac{1}{9}, \frac{2}{9}, -\frac{2}{9}$$

ii. Hence, or otherwise, find all of the roots of f.

$f(x) = 9x^2(x-6) - (x-6) = (x-6)(9x^2-1)$.

Hence if x is a root of x, either $x - 6 = 0$ and $x = 6$, or $9x^2 - 1 = 0$ and $x = \pm\frac{1}{3}$.

b. Show that the polynomial $P(x) = x^3 - 5$ has no rational roots.

A rational root, x of $x^3 - 5$ will be of the form $\pm\frac{p}{q}$, where p divides 5 and q divides 1. This leaves us with the following options,

$$x = -1, 1, -5, 5$$

We evaluate all of our prospective roots, specifically,

$$P(1) = 1 - 5 = -4$$
$$P(-1) = -1 - 5 = -6$$
$$P(5) = 125 - 5 = 120$$
$$P(-5) = -125 - 5 = -130.$$

As none of the roots evaluate to 0 we see that $P(x)$ has no rational roots.

c. Consider $f(x) = x^3 + \frac{7x^2}{4} + \frac{7x}{2} - 1$. It is known that f has only positive roots. Factorise $f(x)$.
Hint: To apply the rational root theorem all of your polynomial coefficients must be integers.

We will apply the rational root theorem to $g(x) = 4f(x) = 4x^3 + 7x^2 + 14x - 4$ as it has coefficients that are integers and have a GCD of 1.

By the rational root theorem along with the fact that f has only positive roots we know that our roots are within this set of numbers,

$$\left\{1, 2, 4, \frac{1}{2}, \frac{1}{4}\right\}$$

Out of these roots we can exclude $a \in \left\{1, 2, 4, \frac{1}{2}\right\}$ as $g(a) > 14a - 4 > 7 - 4 > 0$.

As $f\left(\frac{1}{4}\right) = \frac{1}{16} + \frac{7}{16} + \frac{7}{2} - 4 = 0$ we know that $4x - 1$ is a factor of g.

Hence, $g(x) = (4x - 1)(x^2 + 2x + 4)$. As $(2)^2 - 4(4)(1) < 0$ we can not factorise g further.

Thus $f(x) = \frac{1}{4}g(x) = \frac{1}{4}(4x - 1)(x^2 + 2x + 4)$

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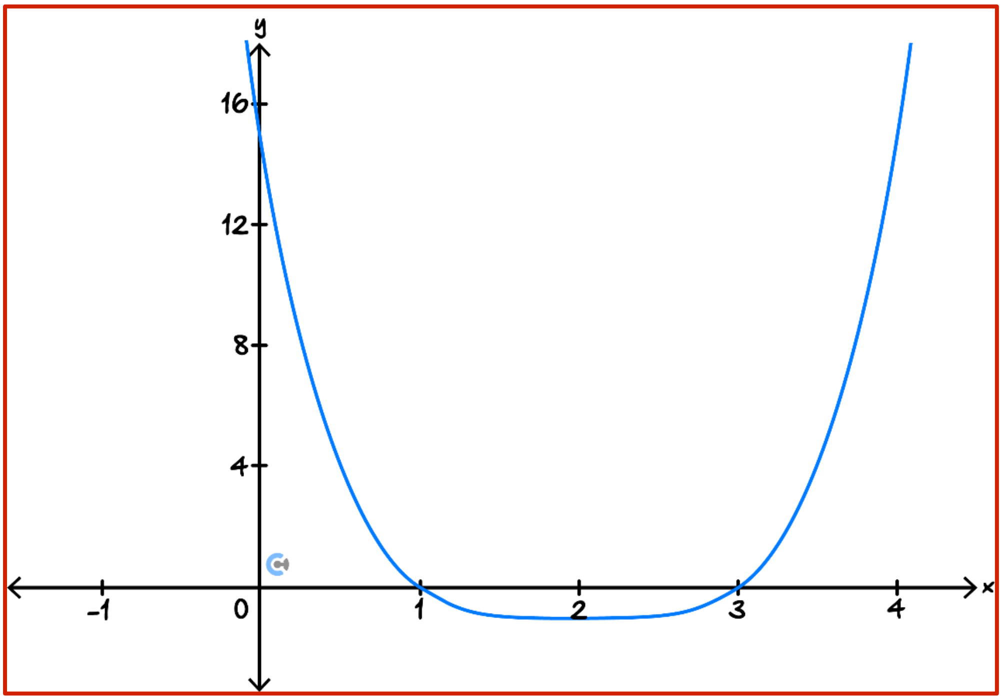
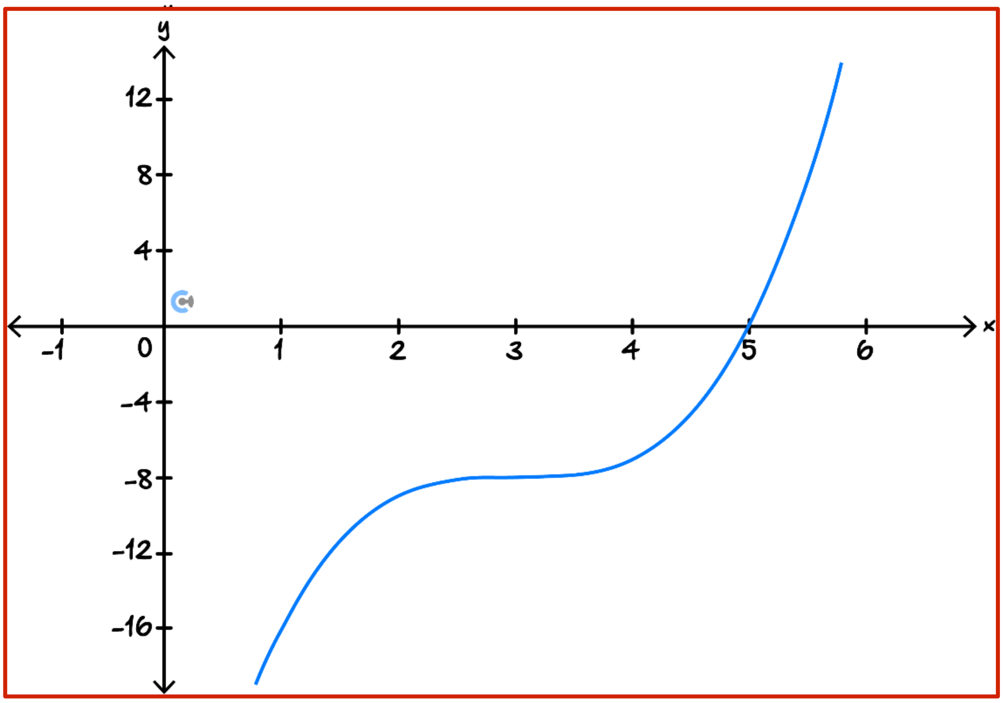
Sub-Section [1.7.3]: Graphing Factored and Unfactored Polynomials

Question 7

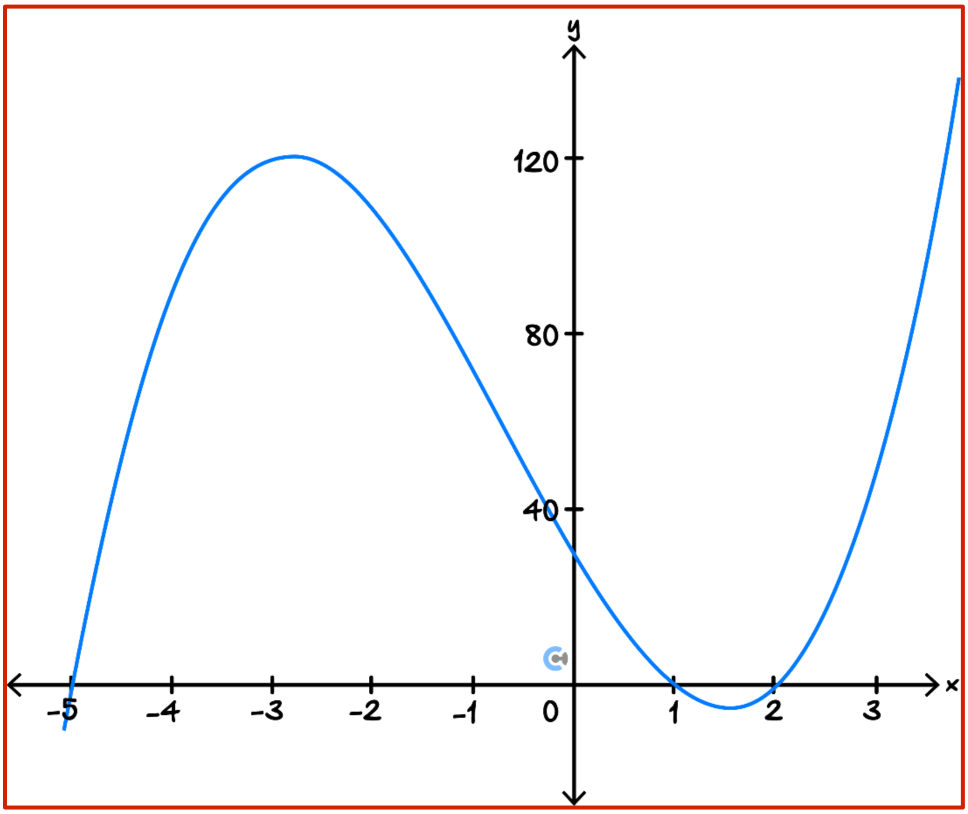
a. Sketch the graph of $y = (x-2)^4 - 1$ on the axis below.



b. Sketch the graph of $y = (x-3)^3 - 8$ on the axis below.

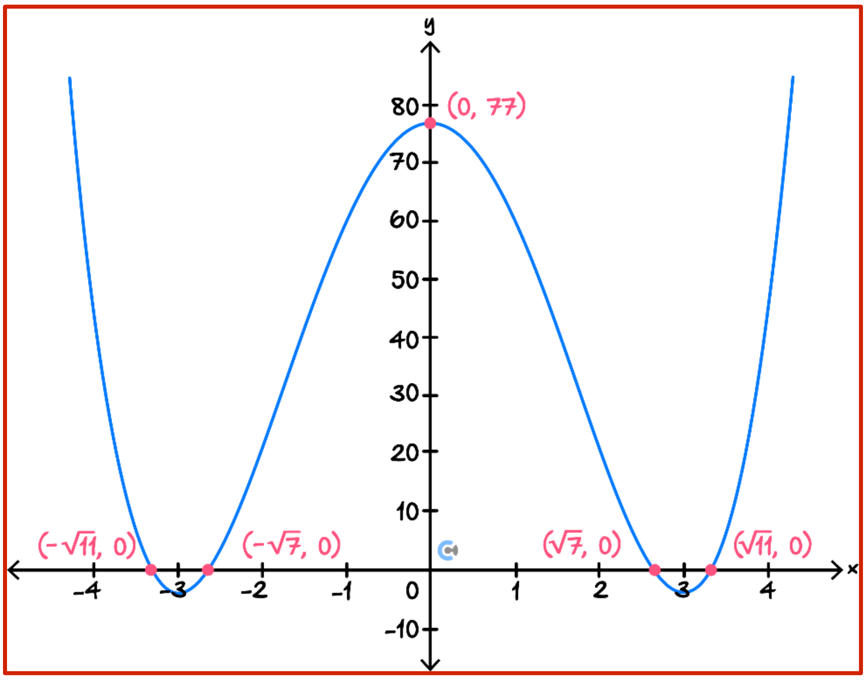


c. Sketch the graph of $y = 3(x-2)(x-1)(x+5)$ on the axis below.

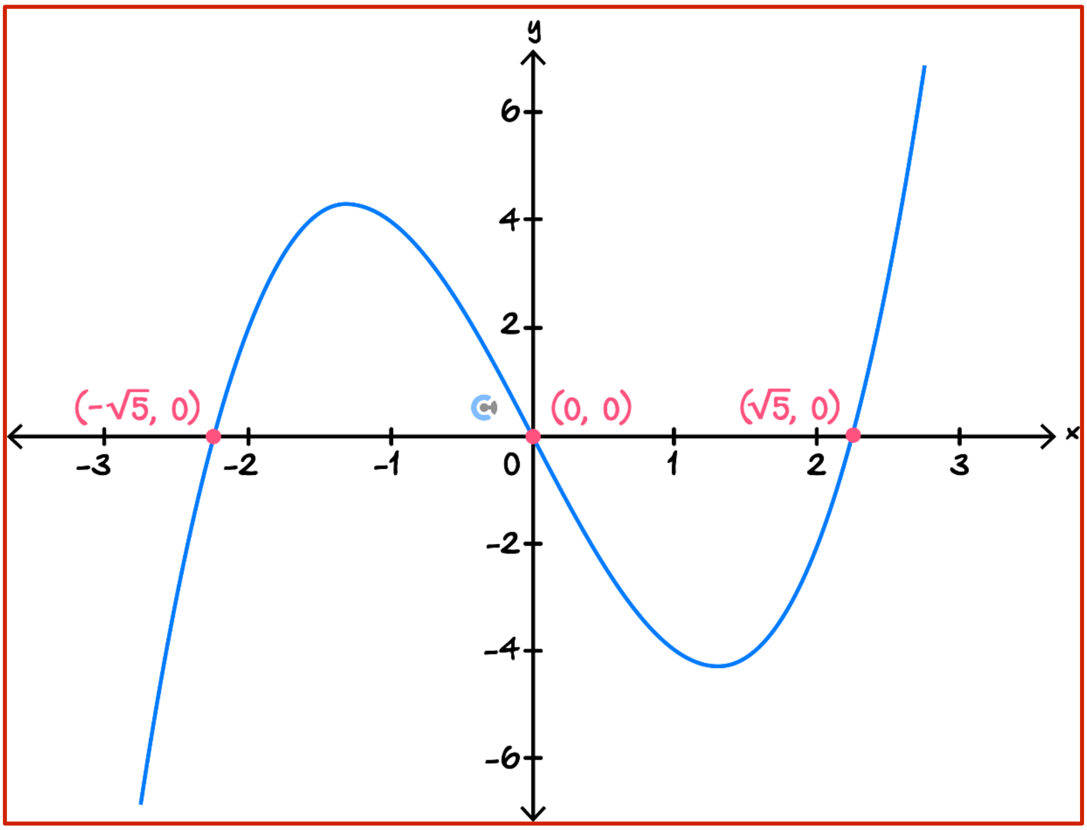


Question 8

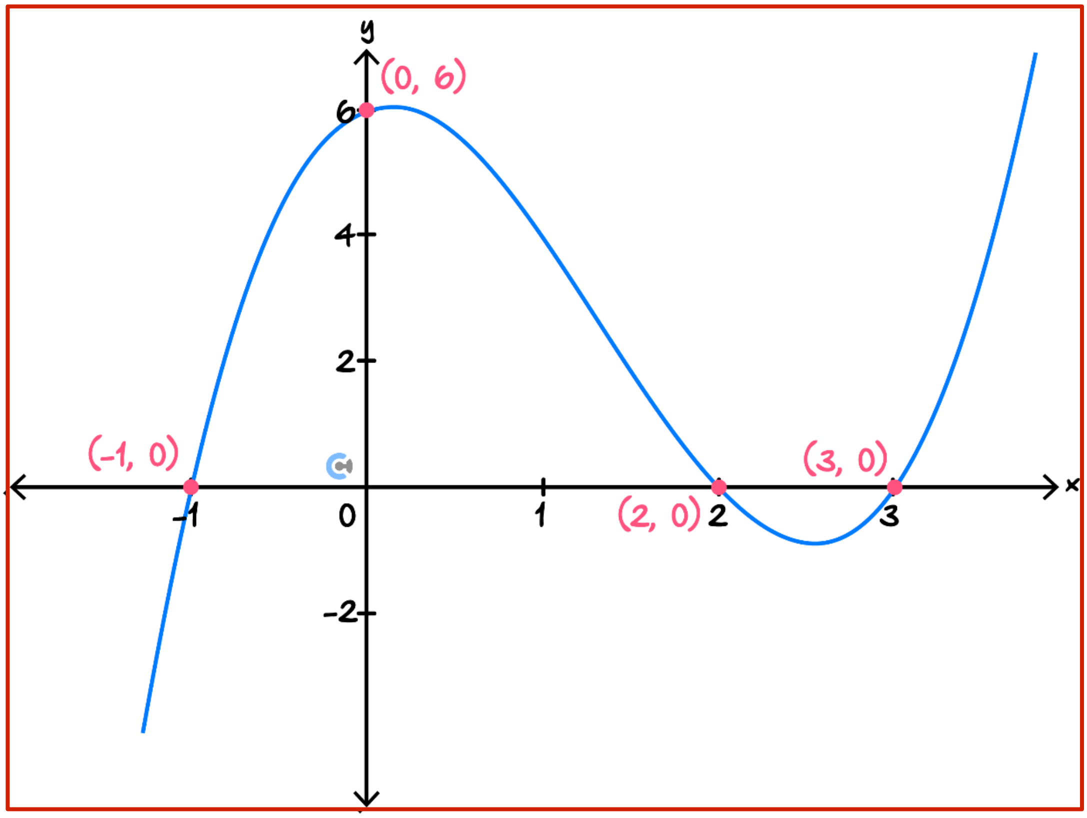
a. Sketch the graph of $y = (x^2 - 9)^2 - 4$ on the axis below, labelling axis intercepts with their coordinates.



b. Sketch the graph of $y = x(x^2 - 5)$ on the axis below, labeling axis intercepts with their coordinates.

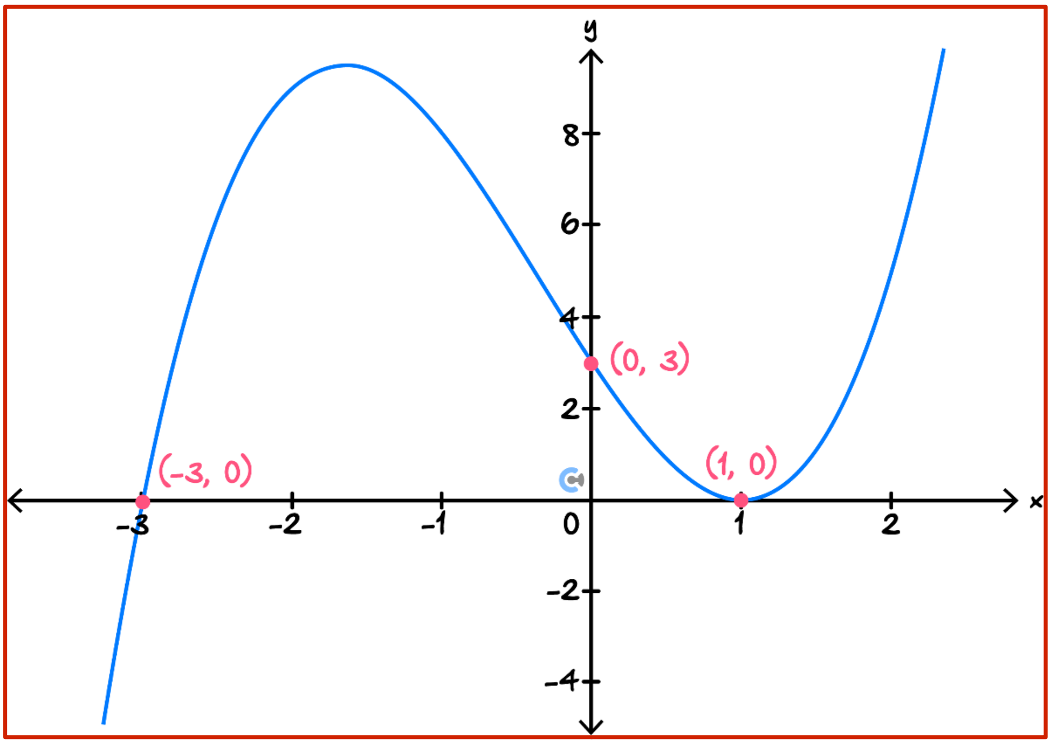


c. Sketch the graph of $y = x^3 - 4x^2 + x + 6$ on the axis below, labeling axis intercepts with their coordinates.



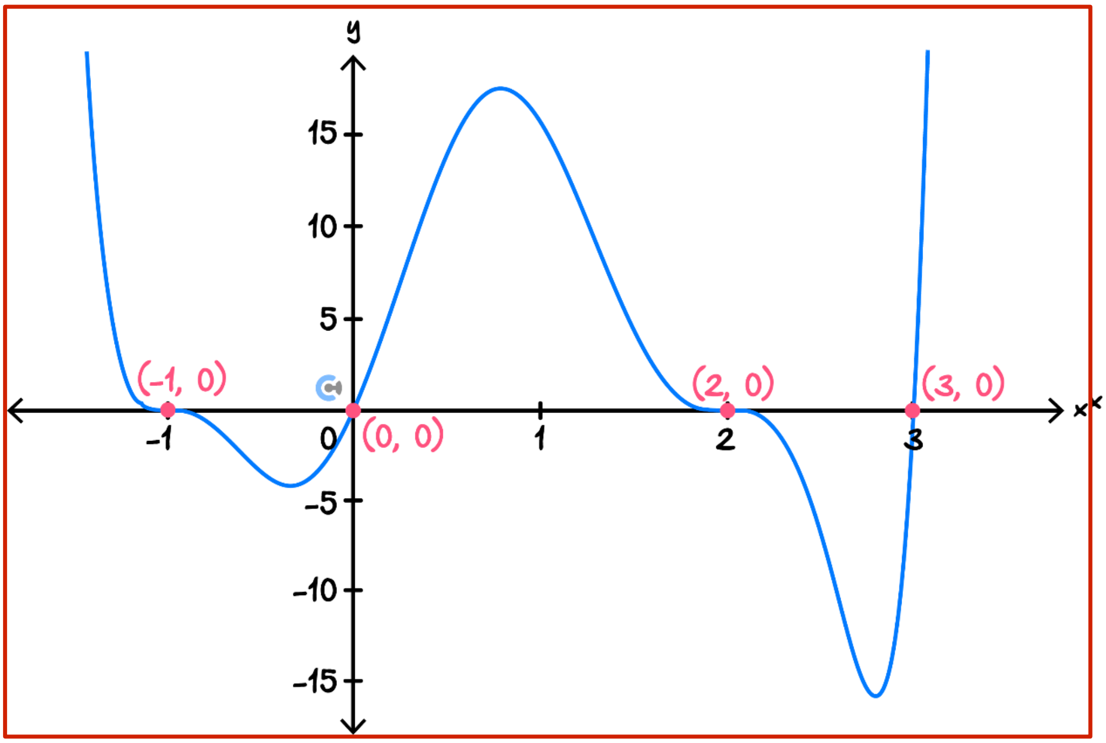
Question 9

a. Sketch the graph of $y = x^3 + x^2 - 5x + 3$ on the axis below, labeling axis intercepts with their coordinates.

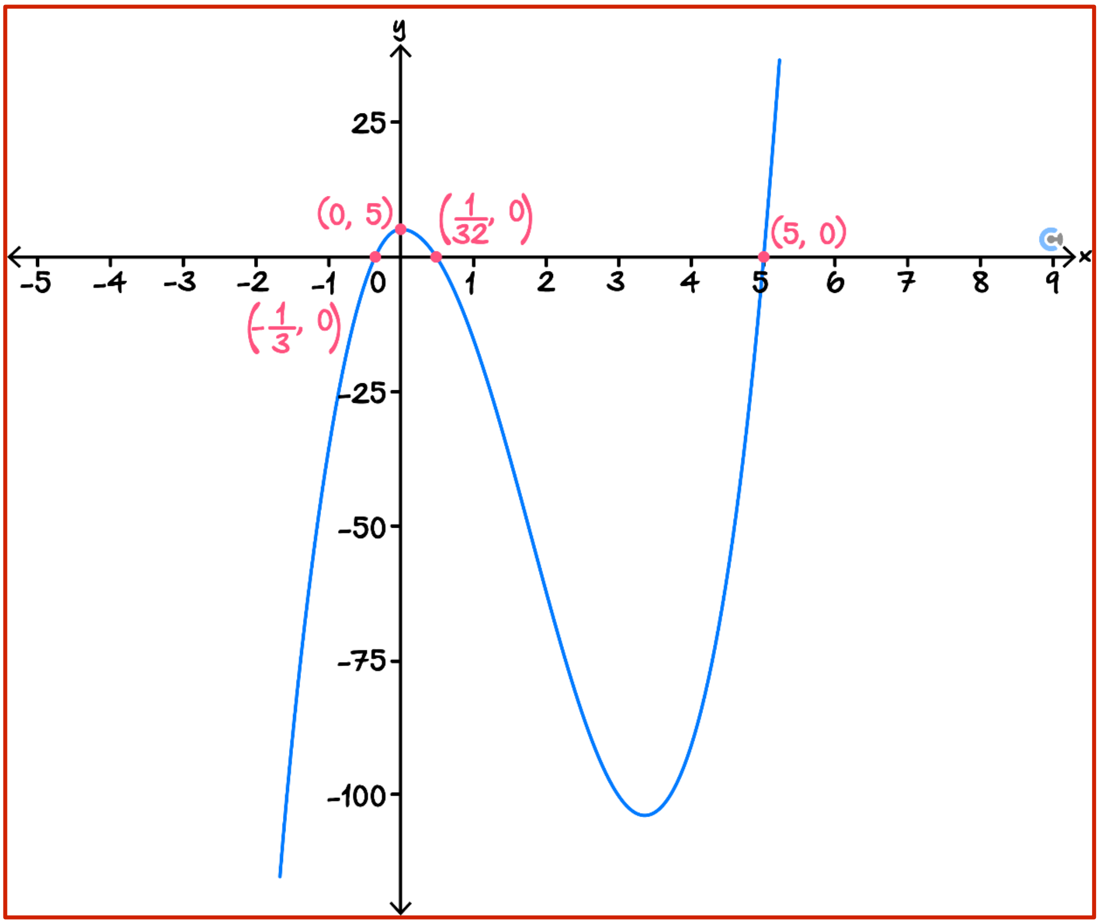


b. **Tech-Active.**

Sketch the graph of $y = x(x-2)^3(x+1)^3(x-3)$ on the axis below, labeling axis intercepts with their coordinates.



c. Sketch the graph of $y = 5 + 4x - 31x^2 + 6x^3$ on the axis below, labeling axis intercepts with their coordinates.



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Sub-Section [1.7.4]: Identify Odd and Even Functions

Question 10

a. Let $f(x)$ be an even function and $g(x)$ be an odd function.

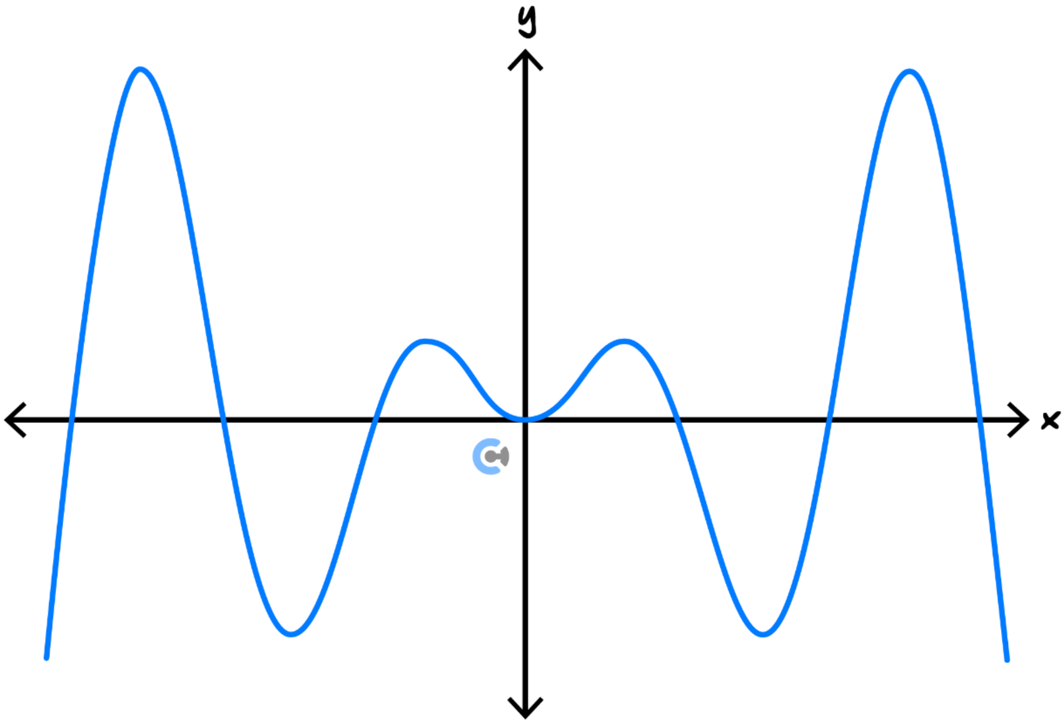
i. State whether $f(g(x))$ is an even or an odd function.

As $f(g(-x)) = f(-g(x)) = f(g(x))$, $f(g(x))$ is an even function.

ii. State whether $f(x) \times g(x)$ is an even or an odd function.

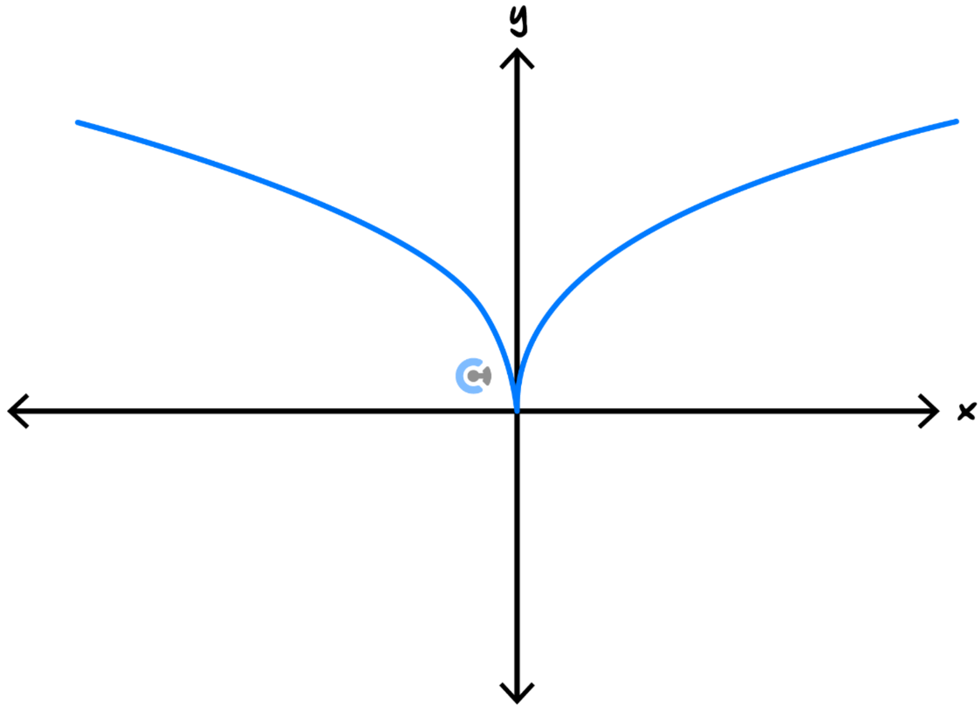
As $f(-x) \times g(-x) = f(x) \times (-g(x)) = -f(x) \times g(x)$, $f(x) \times g(x)$ is an odd function.

b. Part of the graph of $f(x)$ is drawn below. State whether f is an odd or an even function.



$f(x)$ is an even function.

c. Part of the graph of $y = x^{\frac{m}{n}}$ is drawn below where m and n are co-prime.



State whether m and n are even or odd.

As the domain of our graph is all reals, including negatives, n is odd.

As our function is even, m must be even.

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Question 11

a. Let $f(x) = (x-3)^3 + 5$.

Describe a sequence of transformations that map the graph of f onto the graph of an odd function.

We wish to transform the graph of f to the graph of $y = x^3$. Thus our transformation is,

- A translation 3 units in the negative direction of the x-axis, followed by,
- A translation of 5 units in the negative direction of the y-axis.

b. Show that $P(x) = 2(x^4 + 3x^2 - 1)^3 - 5$ is an even function.

$$P(-x) = 2((-x)^4 + 3(-x)^2 - 1)^3 - 5$$
$$= 2(x^4 + 3x^2 - 1)^3 - 5 = P(x)$$

Hence $P(x)$ is an even function.

c. Consider the function $f(x)$. It is known that $f(x+2)$ is an even function.

If $f(-1) = 3, f(7) = 5$, and $f(3) = 7$, find the value of $2f(-3)$.

$f(x+2)$is even

$$f(-x+2) = f(x+2)$$

Hence

$$f(-3) = (f(-5) + 2) = f(5+2) = f(7)$$
$$2f(-3) = 2f(7) = 2 \times 5 = 10$$

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Question 12

a. Let $f(x)$ be an even function and $g(x)$ be an odd one-to-one function.

If $f(3) = 5,\ g(1) = 3$, and $g(3) = 4$. Find $f(-3) + g^{-1}(-3)$.

As g is one to one and odd, g^{-1} is also an odd function. Hence,

$$f(-3) + g^{-1}(-3) = f(3) - g^{-1}(3) = 5 - 1 = 4$$

b. Tech-Active.

Let $f(x) = x^3 - 9x^2 + 7x$.

A transformation $T(x, y) = (x + a, y + b)$ maps the graph of $f(x)$ onto the graph of an odd function g. Find the values of a and b.

The rule for g is $g(x) = f(x - a) + b$. Thus,

$$g(x) = x^3 - (9 + 3a)x^2 + (3a^2 + 18a + 7)x - 7a - 9a^2 - a^3 + b$$

For g to be an odd function we require it's x^2 coefficient along with it's constant term to be equal to 0. Hence we solve simultaneously,

$$9 + 3a^2 = 0 \qquad \text{and} \qquad -7a - 9a^2 - a^3 + b = 0$$

This yields $a = -3$ and $b = 33$.

c. James says that he's found a function, $f(x)$ that is both odd and even.

Show that $f(x) = 0$ for all real x.

Since $f(x)$ is an even function, $f(-x) = f(x)$ for all real x.

Since $f(x)$ is an odd function, $f(-x) = -f(x)$ for all real x.

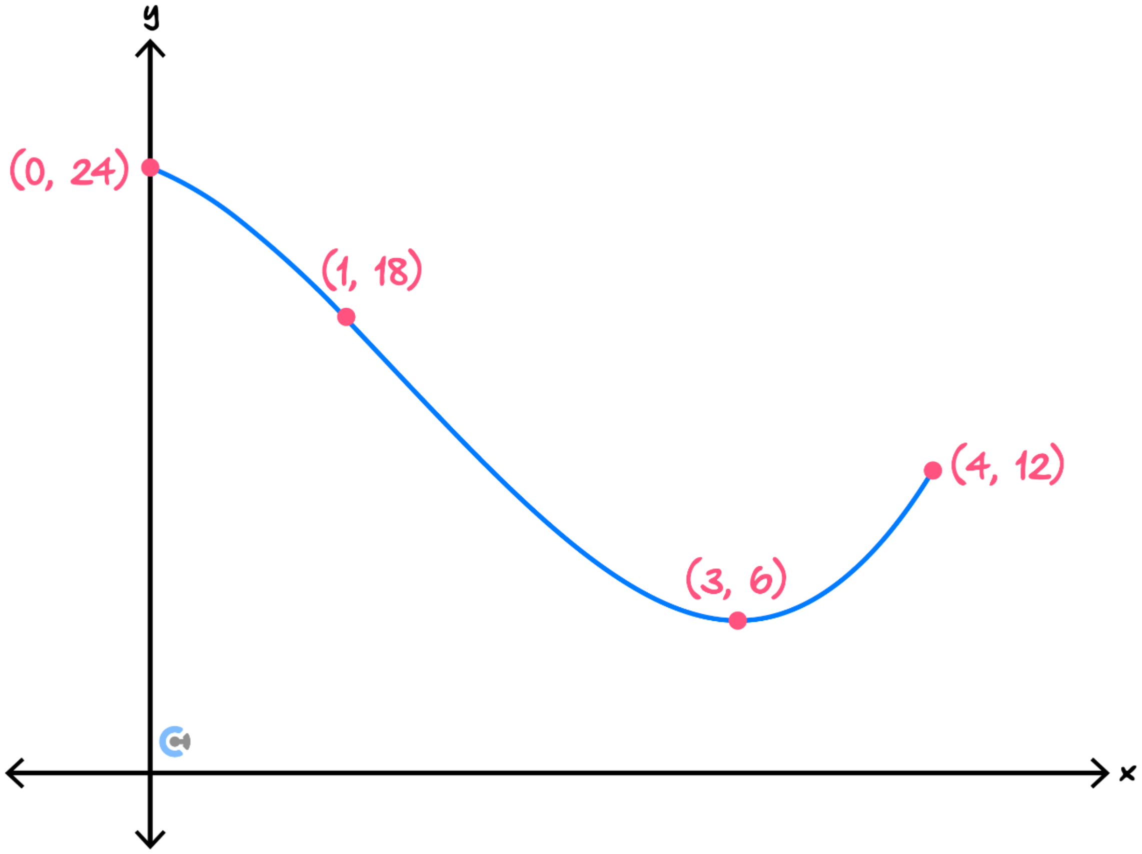
Combining these two facts we see that $f(x) = f(-x) = -f(x)$ for all real x, hence $2f(x) = 0 \implies f(x) = 0$ for all real x.

Sub-Section: Boss Question

Question 13

Samuel is building a ramp to throw students off who do not complete their homework.

The cross-section of the ramp is modelled by a function $f: [0, 4] \to \mathbb{R}, f(x) = ax^3 + bx^2 + cx + d$. The graph of f is shown below.



a. Find the values of a, b, c and d.

We can construct a system of 4 equations, specifically,

$$f(0) = d = 24$$
$$f(1) = a + b + c + d = 18$$
$$f(3) = 27a + 9b + 3c + d = 6$$
$$f(4) = 64a + 16b + 4c + d = 12$$

Solving these equations simultaneously, yields, $a = 1, b = -4, c = -3$ and $d = 24$

b. $f(x)$ can be written as $f(x) = g(x)(x-3) + r$ where r is an integer.

i. State the degree of g.

The degree of g is 2.

ii. State the value of r.

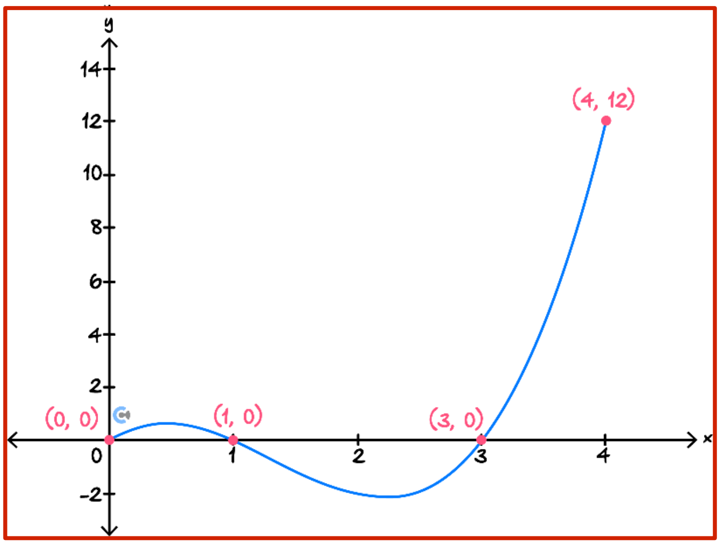
Since $f(3) = g(3)(3-3) + r = r$ we see that $r = 6$.

Samuel installs a ladder for the students to climb up to the top of the ramp. The cross-section of the ladder is given by the function $L: [0, 4] \to \mathbb{R}, L(x) = 24 - 6x$.

c. Solve $f(x) = L(x)$ for x.

$x = 0, 1, 3$

d. Sketch the graph of $f(x) - L(x)$ on the axis below, labeling axis intercepts and end-points with their co-ordinates.



Let $h(x)$ have the same rule as $f(x)$ but have a domain of all real numbers.

e. How many solutions does the equation $h(x) = 1$ have?

1 solution

f. Find a value of a such that $h(x) = a$ has exactly two solutions.

Cubic will have exactly two roots if one root is repeated

Hence $h(r) = 0$ *and* $h'(r) = 0$

$$h(x) = x^3 - 4x^2 - 3x + 24 - a$$

$$h'(x) = 3x^2 - 8x - 3$$

$h'(r) = 0$ gives , $r = 3, -\frac{1}{3}$

now we substitute the r values in $h(x) = 0$.

$$h(3) = 0$$

$$27 - 36 - 9 + 24 - a = 0$$

$$a = 6$$

Again $f\left(-\frac{1}{3}\right) = 0$

$$a = \frac{662}{27}$$

Answer $a = \frac{662}{27}$ and $a = 6$

g. Describe a sequence of translations that map the graph of h onto

Any sequence of translations is equivalent to the transformation $T(x, y) = (x + a, y + b)$ for some real numbers a, b.

The image of h under T is,

$$h(x) = x^3 - (3a + 4)x^2 + (3a^2 + 8a - 3)x + 24 + 3a - 4a^2 - a^3 + b$$

For $h(x)$ to be an odd function, both the x^2 coefficient along with the constant term must be 0. Hence we solve,

$$3a + 4 = 0 \quad \text{and} \quad 3a - 4a^2 - a^3 + b + 24 = 0$$

for a and b. This yields, $a = -\frac{4}{3}$ and $b = -\frac{412}{27}$. Hence our translations are,

1. A translation of $\frac{4}{3}$ units left
2. A translation of $\frac{412}{27}$ units down.

Space for Personal Notes

Section B: Supplementary Questions

Sub-Section [1.7.1]: Applying Factor and Remainder Theorems

Question 14

a. State the remainder when $x^2 + 5x - 3$ is divided by $x + 2$.

We can write $x^2 + 5x - 3 = (x + 2)Q(x) + r$ for some quadratic Q. Hence,

$$r = (-2)^2 + 5(-2) - 3 = -9$$

b. Is $x - 2$ a factor of $f(x) = x^4 - 16$?

$x - 2$ is a factor of $f(x)$ if and only if $f(2) = 0$. As

$$f(2) = 2^4 - 16 = 0,$$

$x - 2$ is a factor of f.

c. Is $x + 4$ a factor of $g(x) = x^3 + 4x^2 + 2$?

$x + 4$ is a factor of $g(x)$ if and only if $g(-4) = 0$. As

$$g(-4) = (-4)^3 + 4(-4)^2 + 2 = -64 + 64 + 2 = 2 \neq 0$$

$x + 4$ is not a factor of g.

Question 15

Let $f(x) = 2x^3 + ax^2 + ax + 3$. Find the value of a such that $f(x)$ has a factor of $2x + 3$.

We require $f\left(-\frac{3}{2}\right) = -\frac{27}{4} + \frac{9a}{4} - \frac{3a}{2} + 3 = 0$. Multiplying our expression by 4 yields,

$$-27 + 9a - 6a + 12 = 0 \implies 3a = 15 \implies a = 5$$

Question 16

Let $f(x) = x^2 + ax + b$. Find the values of a and b such that f has a factor of -1, and when f is divided by $2x - 3$, it has a remainder of -5.

Since -1 is a root of f, we see that, $f(-1) = 0$.

Since $f/(2x-3)$ yields a remainder of -5, we see that, $f\left(\frac{3}{2}\right) = -5$. From here we construct a pair of simultaneous equations,

$$f(-1) = 1 - a + b = 0 \quad (1)$$

$$f\left(\frac{3}{2}\right) = \frac{9}{4} + \frac{3}{2}a + b = -5 \quad (2)$$

Subtracting (2) from (1) yields,

$$-\frac{5}{4} - \frac{5a}{2} = 5 \implies -10a = 25 \implies a = \frac{-5}{2}.$$

Substituting this value of a into (1) yields, $b = a - 1 = \frac{-7}{2}$

Question 17

A cubic polynomial, $g(x)$ has the following properties.

1. $g(x) - 3$ has a factor of $(x-2)^2$.

2. $g(x)$ divided by $x^2 - 1$ leaves a remainder of 2.

Find the rule for $g(x)$.

The first statement implies that $g(x) - 3 = a(x-2)^2(x-b)$ for some real numbers a and b.

Since $x^2 - 1 = (x-1)(x+1)$, the second statement implies that $g(1) = g(-1) = 2$.

Hence, $a(1-2)^2(1-b) = a(1-b) = 2 - 3 = -1$ and $a(-1-2)^2(-1-b) = -9a(1+b) = 2 - 3 = -1$. Equating these two expressions yields,

$$a - ab = -9a - 9ab \implies 10a = -8ab \implies b = -\frac{5}{4}$$

Substituting this value into $a(1-b) = -1$ yields, $\frac{9a}{4} = -1 \implies a = -\frac{4}{9}$. Hence,

$$g(x) = -\frac{4}{9}(x-2)^2\left(x + \frac{5}{4}\right) + 3$$

Sub-Section [1.7.2]: Finding Factored Forms of Polynomials

Question 18

Factorise the following polynomials:

a. $x^3 - 8$.

We apply difference of cubes with $a = x$ and $b = 2$, thus,

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

As $2^2 - 4 \times 4 < 0$ we cannot factorise our expression any further.

b. $x^3 - 7x^2 + 10x$.

$$x^3 - 7x^2 + 10x = x(x^2 - 7x + 10) = x(x - 2)(x - 5)$$

c. $x^3 + 3x^2 - 4x - 12$.

$$x^3 + 3x^2 - 2x - 6 = x^2(x + 3) - 4(x + 3) = (x + 3)(x^2 - 4) = (x + 3)(x - 2)(x + 2)$$

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Question 19

a. Factorise $f(x) = x^3 + x^2 - 17x + 15$.

By the rational root theorem, our possible roots are $\pm 1, \pm 3, \pm 5$ and ± 15.
After some testing we see that $f(1) = 0$ hence $x - 1$ is a factor.
Thus $f(x) = (x-1)(x^2 + 2x + 15) = (x-1)(x+5)(x-3)$

b. Factorise $g(x) = x^3 - 4x^2 + x + 6$.

$g(x) = (x+1)(x-2)(x-3)$

c. Find all of the real roots of $h(x) = x^3 - 3x^2 + 4$.

By the rational root theorem, possible rational roots of h are $\pm 1, \pm 2, \pm 4$.
After some testing we see that -1 is a root hence $x + 1$ is a factor of h.
Thus $h(x) = (x+1)(x^2 - 4x + 4) = (x+1)(x-2)^2$.
Hence the real roots of $h(x)$ are $-1, 2$.

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Question 20

a. Factorise $f(x) = x^3 - 5x^2 - 29x + 105$.

$(x-7)(x-3)(x+5)$

b. Factorise $g(x) = 18x^3 - 3x^2 - 28x - 12$.

$(2+3x)^2(2x-3)$

c. Factorise $h(x) = 2x^3 + 14x^2 - 10x - 150$.

$$2(x-3)(x+5)^2$$

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Question 21

Let $f(x) = ax^2 + bx + c$ with a, b, c being co-prime non-zero integers, and assume that $\frac{p}{q}$ is a root of f with p and q co-prime and both non-zero.

a. Show that p divides c.

We know that $f\left(\frac{p}{q}\right) = a\frac{p^2}{q^2} + b\frac{p}{q} + c = 0$.

After subtracting c from both sides and multiplying both sides by q^2 we have that,

$$-cq^2 = p(ap + bq)$$

Hence p is a factor of cq^2. Since q is coprime to p it follows that p divides c.

b. Show that q divides a.

Like in part **a.** we can rearrange $f\left(\frac{p}{q}\right) = 0$ to get the equation,

$$-ap^2 = q(bp + cq)$$

Thus we see that q divides ap^2. Since p is coprime to q it follows that q divides a.

c. If a, b, c are not co-prime integers, where would your arguments for parts a and b breakdown?

In the equation $-cq^2 = p(ap + bq)$ we assume that $ap + bq$ is an integer, hence p is a factor of $-cq^2$. If $ap + bq$ is not an integer, this may no longer be the case.

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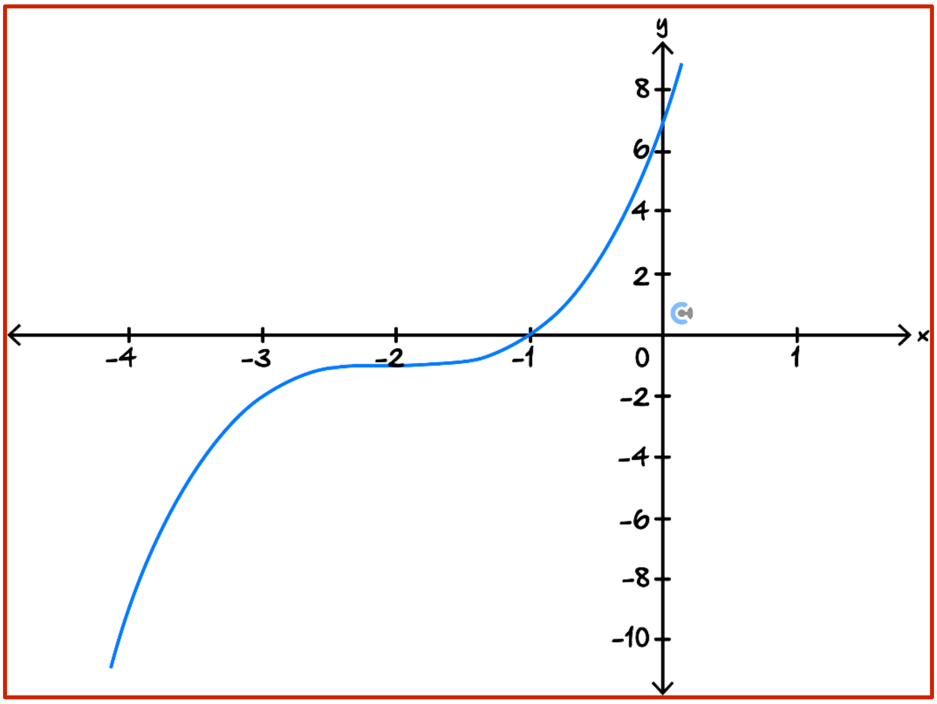
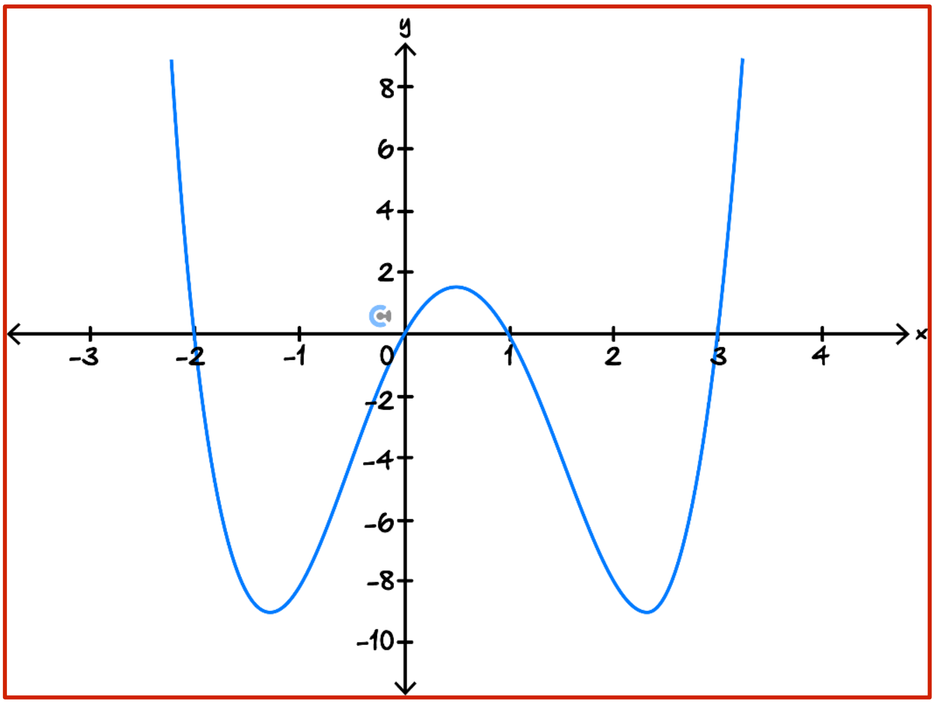
Sub-Section [1.7.3]: Graphing Factored and Unfactored Polynomials

Question 22

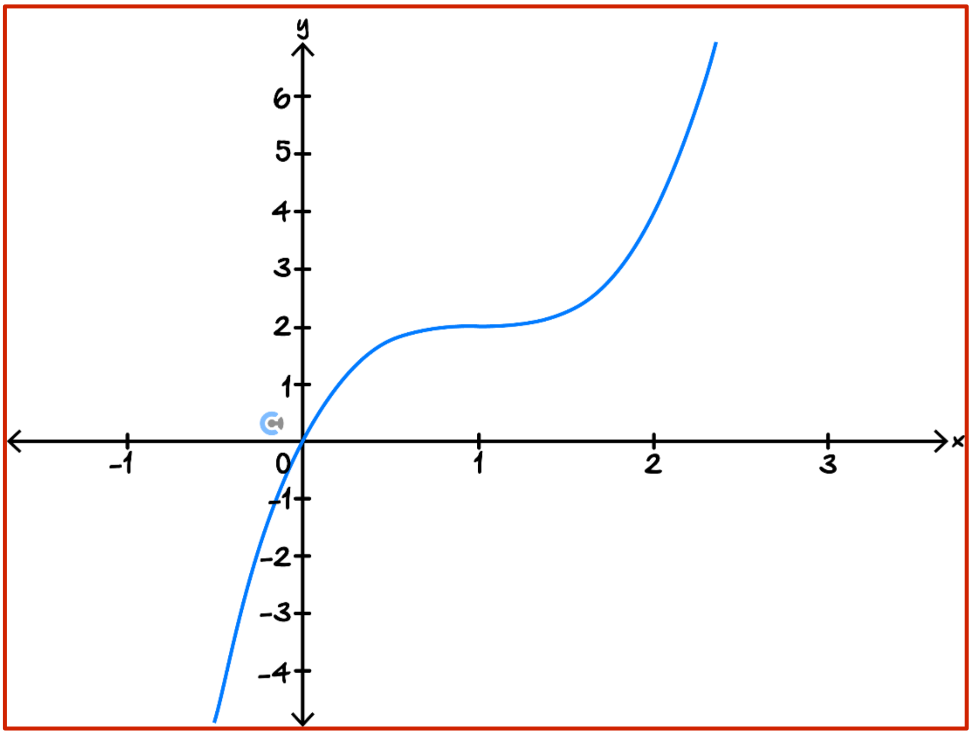
a. Sketch the graph of $y = (x+2)^3 - 1$ on the axis below.



b. Sketch the graph of $y = x(x-1)\,(x+2)(x-3)$ on the axis below.

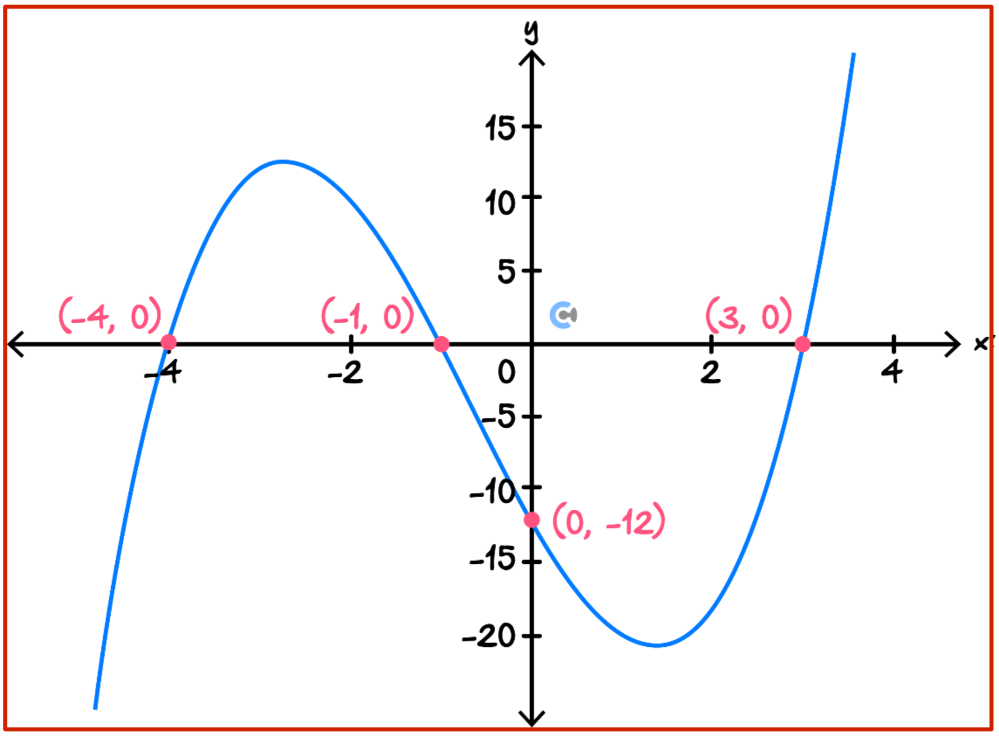


c. Sketch the graph of $y = 2(x-1)^3 + 2$ on the axis below.

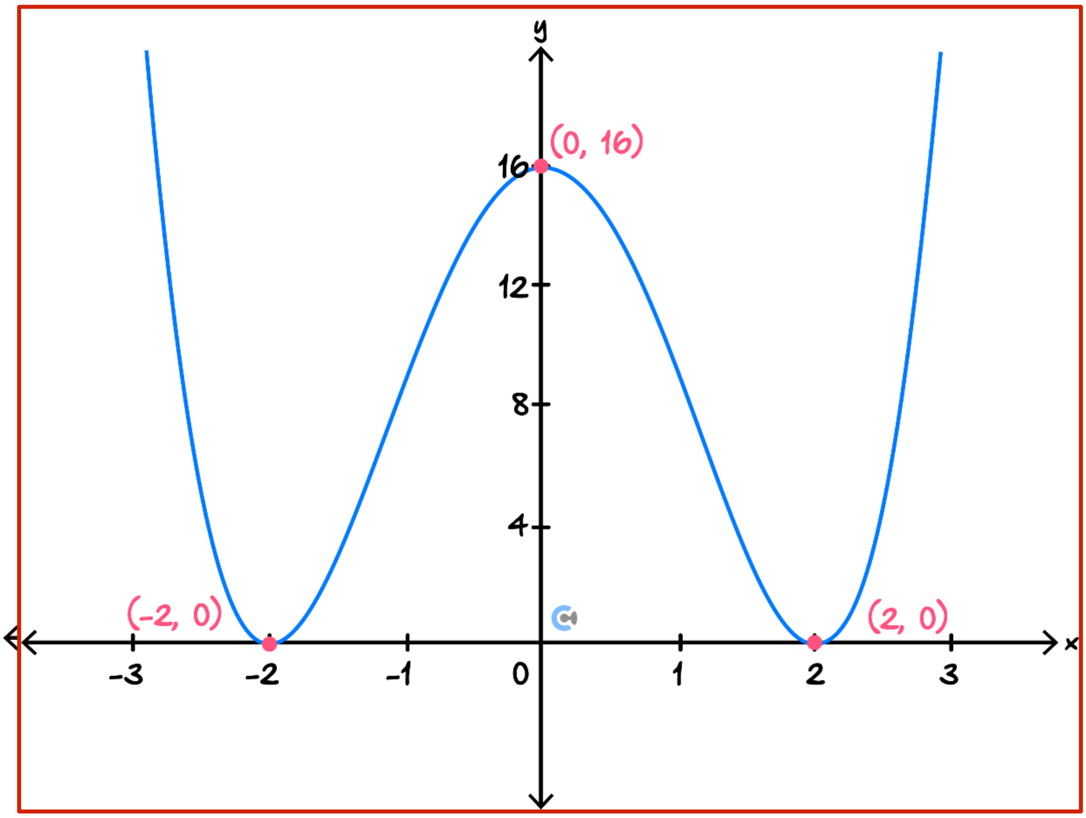


Question 23

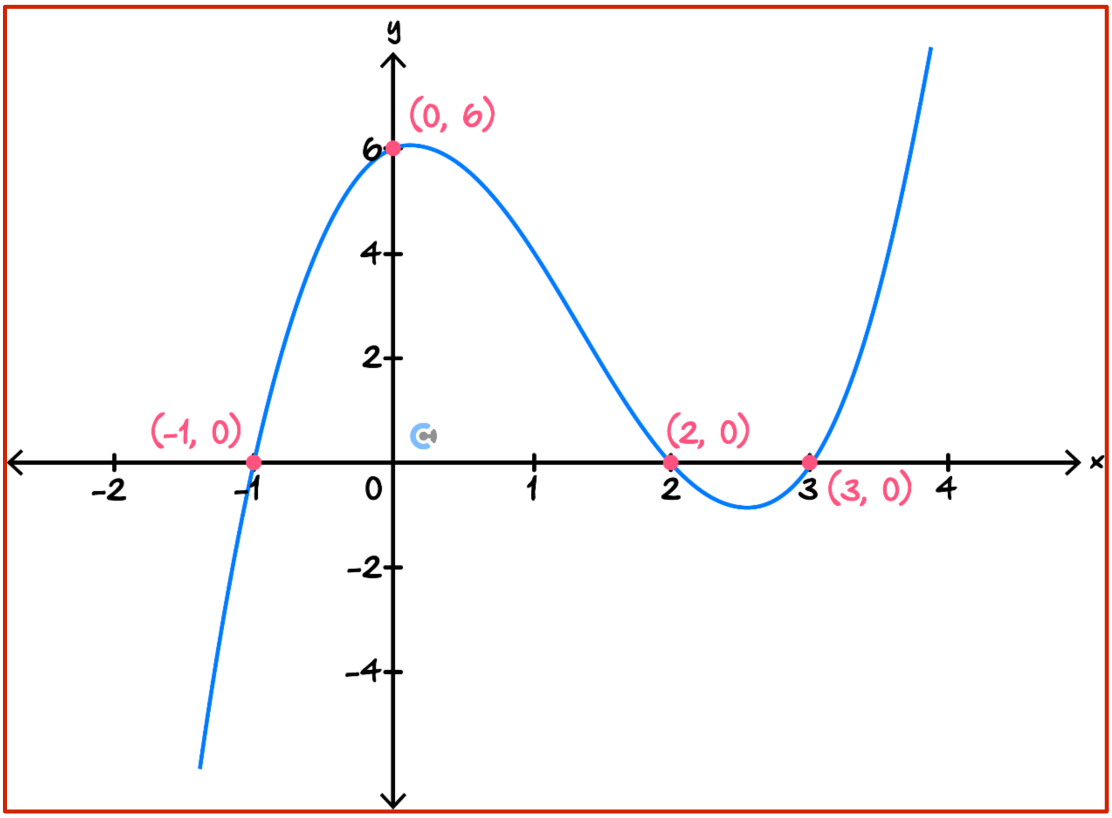
a. Sketch the graph of $y = x^3 + 2x^2 - 11x - 12$ on the axis below, labeling axis intercepts with their coordinates.



b. Sketch the graph of $y = x^4 - 8x^2 + 16$ on the axis below, labeling axis intercepts with their coordinates.

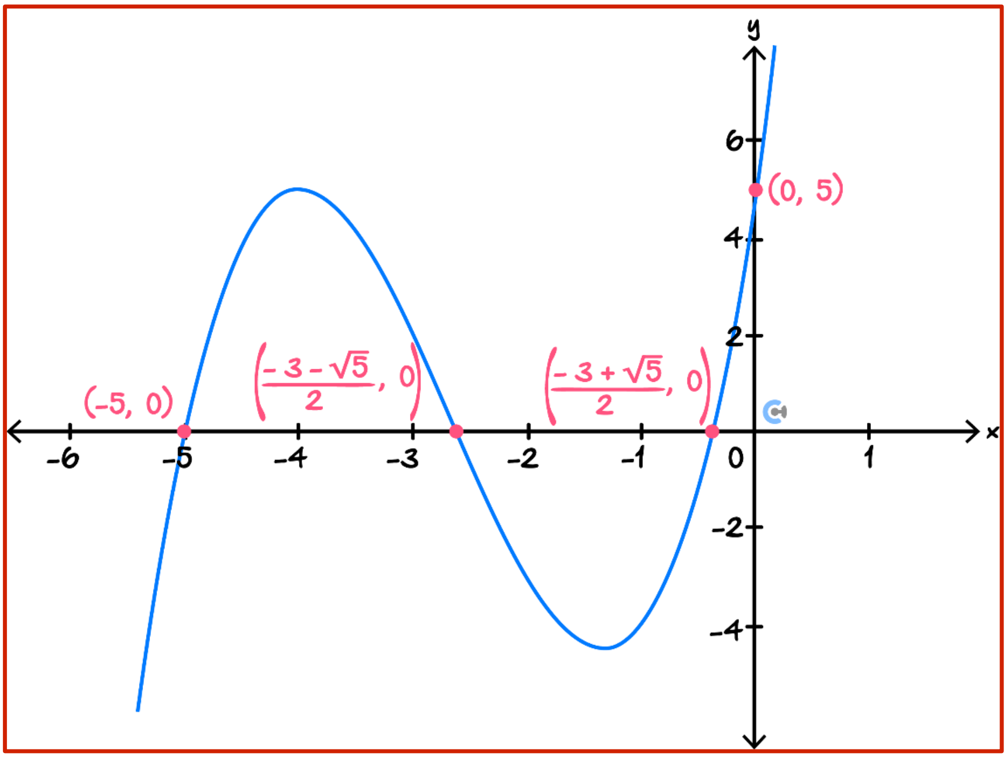


c. Sketch the graph of $y = x^3 - 4x^2 + x + 6$ on the axis below, labeling axis intercepts with their coordinates.

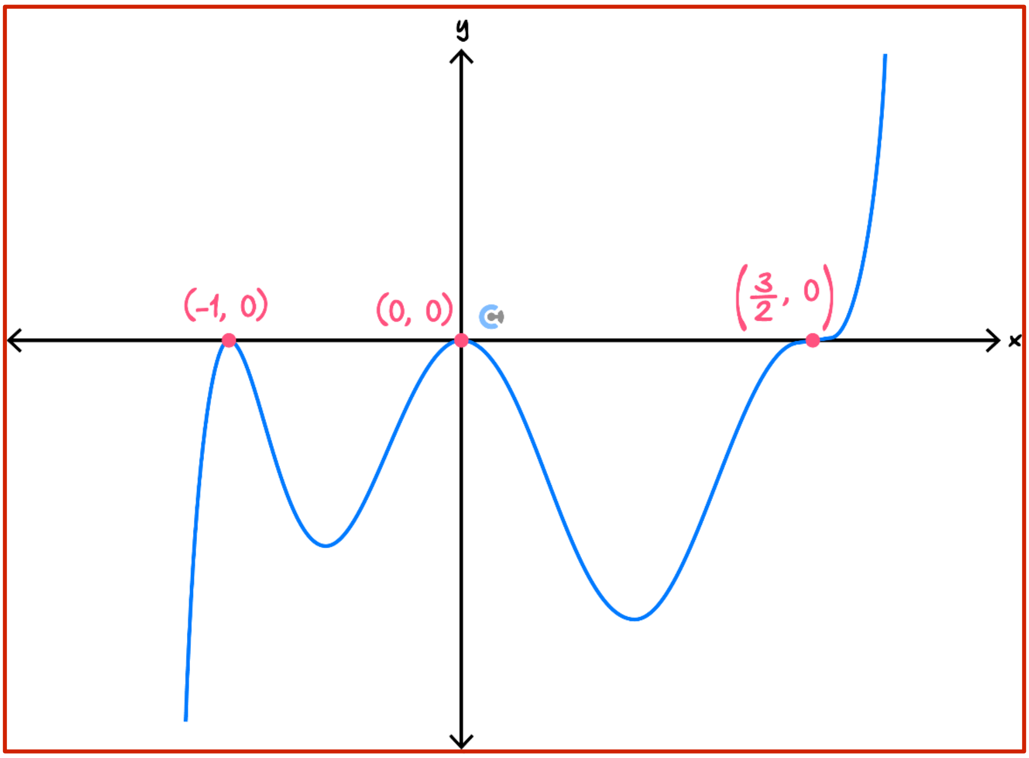


Question 24

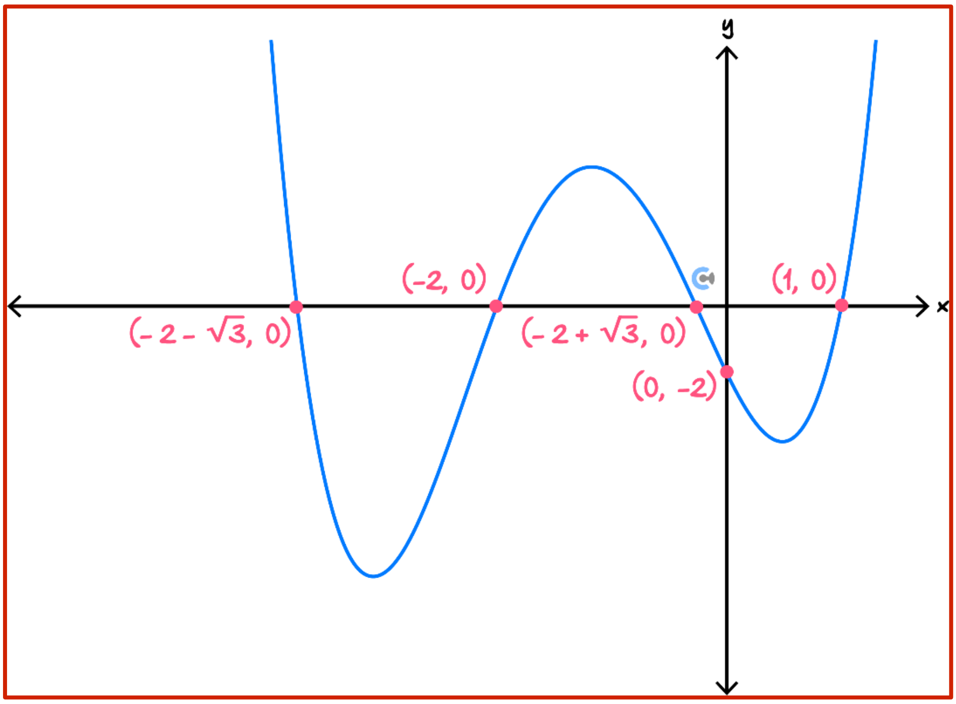
a. Sketch the graph of $y = x^3 + 8x^2 + 16x + 5$ on the axis below, labeling axis intercepts with their coordinates.



b. Sketch the graph of $y = x^2(2x-3)^3(x+1)^2$ on the axis below, labeling axis intercepts with their coordinates.

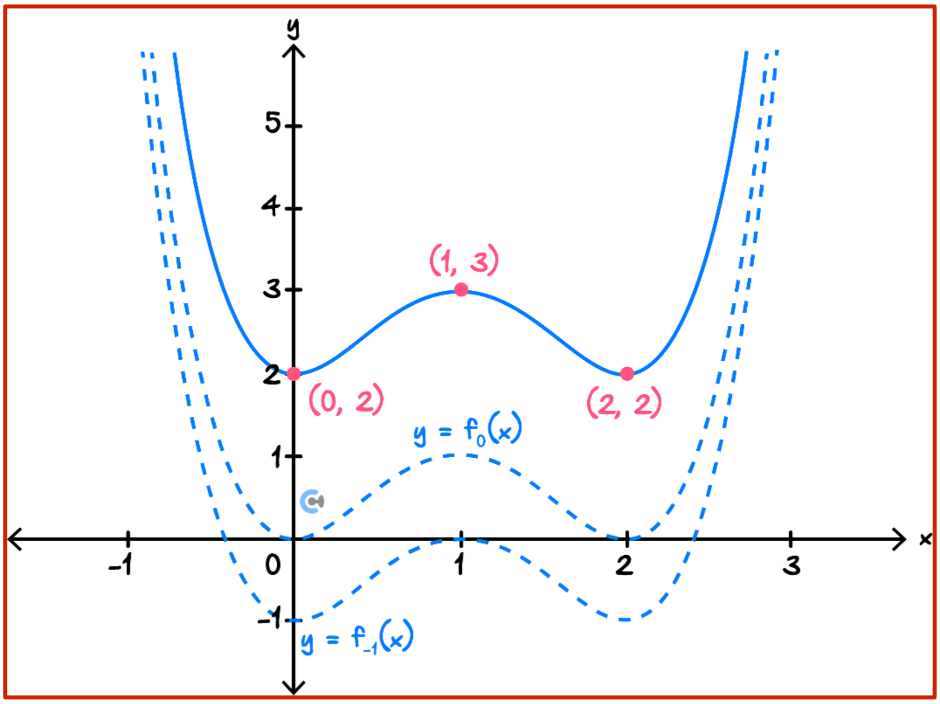


c. Sketch the graph of $y = x^4 + 5x^3 + 3x^2 - 7x - 2$ on the axis below, labeling axis intercepts with their coordinates.



Question 25

Let $f_k(x) = x^4 - 4x^3 + 4x^2 + k$. By considering f_0 and f_{-1}, sketch the graph of f_2 on the axis below, labeling axis intercepts and turning points with their coordinates.



Sub-Section [1.7.4]: Identify Odd and Even Functions

Question 26

a. Let $f(x)$ and $g(x)$ both be an odd functions.

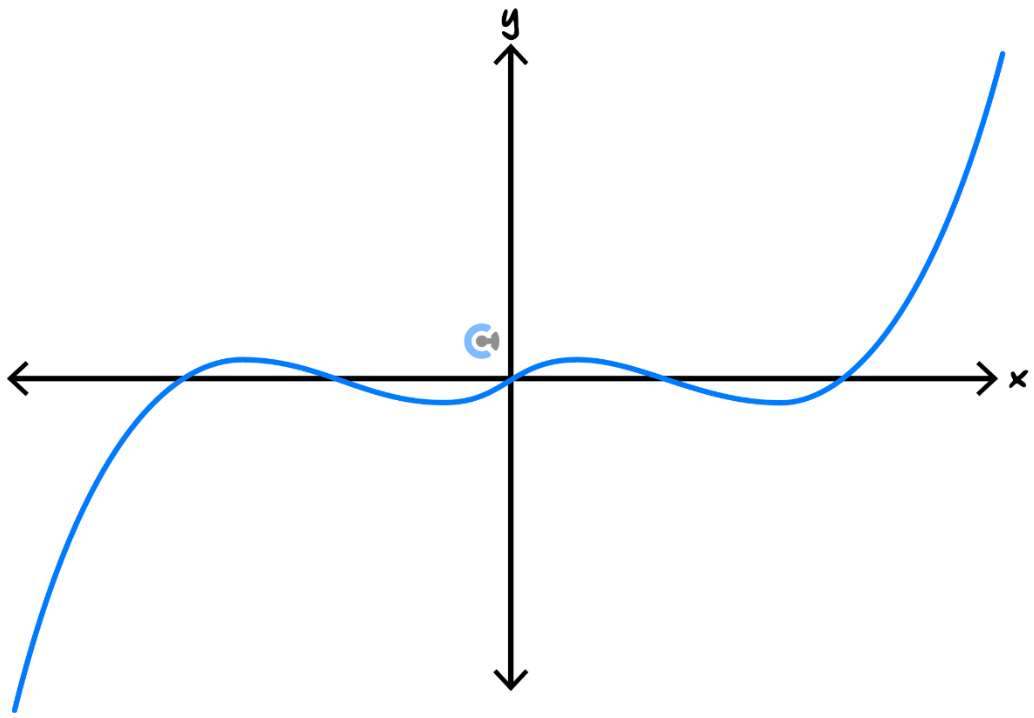
i. State whether $f(x) + g(x)$ is an even or an odd function.

As $f(-x) + g(-x) = -f(x) - g(x) = -[f(x) + g(x)]$, $f(x) + g(x)$ is an odd function.

ii. State whether $(f(x))^2 + 2f(x)g(x) + \left(g(x)\right)^2$ is an even or an odd function.

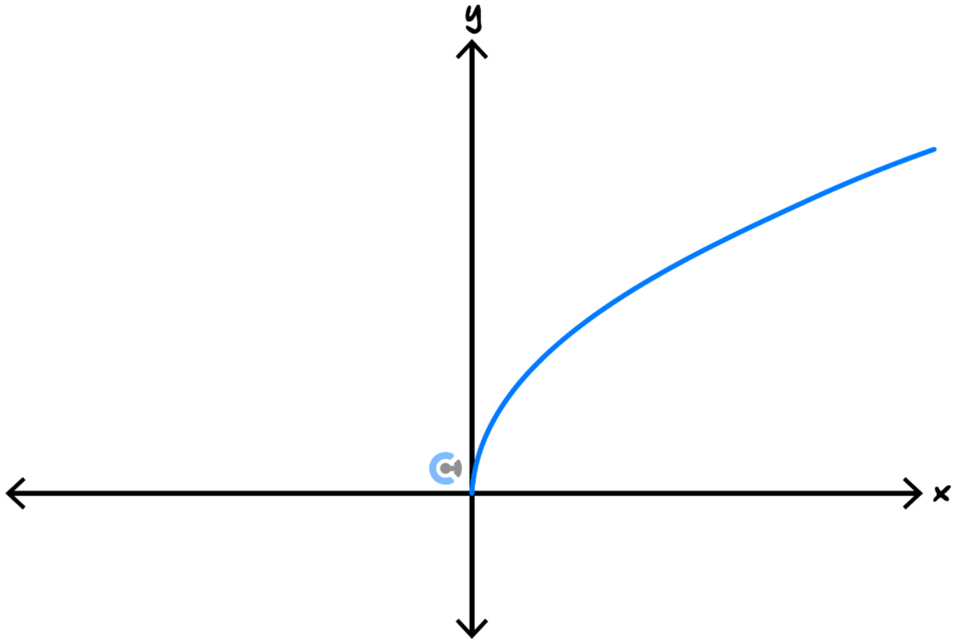
$(f(x))^2 + 2f(x)g(x) + (g(x))^2 = (f(x) + g(x))^2$ and is hence an even function.

b. Part of the graph of $f(x)$ is drawn below. State whether f is an odd or an even function.



$f(x)$ is an odd function.

c. Part of the graph of $y = x^{\frac{m}{n}}$ is drawn below where m and n are co-prime.



State whether m and n are even or odd.

As the domain of our graph only positive numbers, n is even.

As m and n are co prime, m is thus odd.

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Question 27

a. Show that $f(x) = x^4 - 2x^3$ is neither an even nor an odd function.

Observe that $f(-x) = x^4 + 2x^3 \neq x^4 - 2x^3$, hence f is not an even function.
Observe that $f(-x) = x^4 + 2x^3 \neq -x^4 + 2x^3$, hence f is not an odd function.

b. Describe a translation that maps the graph of $y = x^2 + 6x + 7$ onto the graph of an even function.

$x^2 + 6x + 7 = (x+3)^2 - 2$. Thus we want to map our graph onto the graph of $y = x^2 - 2$, hence we simply need to translate our graph 3 units to the right.

c. Consider the function $f(x)$. It is known that $f(2x+3)$ is an odd function.

If $f(5) = 4$ and $f(-1) = -3$, find the value of $f(1)$.

Let $g(x) = f(2x+3)$. Then,

$$f(1) = g(-1) = -g(1) = -f(5) = -4.$$

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Question 28

a. Let $f(x)$ be a strictly increasing function with $f(0) = 0$.

If $(f(x))^2$ is an even function, show that $f(x)$ is an odd function.

Since $(f(x))^2 = (f(-x))^2$ we know that for all $x > 0$ that $f(x) = \pm f(-x)$.
However since f is an increasing function it must be one to one, and thus $f(x) \neq f(-x)$ for $x > 0$.
Hence $f(x) = -f(-x)$ for $x > 0$, which implies that $f(x) = f(-x)$ for $x \neq 0$. Lastly we see that $f(0) = -f(-0) = 0$.

b. Let $f(x) = x^4 + 2x^3 + x^2$.

Describe a transformation that maps the graph of f onto the graph of an even function.

Observe that $f(x) = x^2(x+1)^2$.
We can map the graph of f onto the graph of $y = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{2}\right)^2$ which is an even function by translating it $\frac{1}{2}$ units right.

c. Let $f(x)$ be an even function.
The function,

$$g(x) = \begin{cases} f(x) + c & x \geq 0 \\ -f(x) + d & x < 0 \end{cases}$$

is an odd function.

Find the values of c and d.

We require $g(0) = -g(-0) = g(0) = f(0) + c = 0$ for g to be an odd function.
Hence $c = -f(0)$.
Now for $x > 0$ we require that $f(x) + c = g(x) = -g(-x) = f(-x) - d = f(x) - d$.
Hence $d = -c = f(0)$.

Question 29

Let $f(x) = x^4 - 4x^3 + x^2 + 6x + k$, where k is a real number.
The function $g(x) = f(x - h)$ is an even function.
Find the value of h.

We observe that as k represents a vertical translation, it does not affect whether or not g is an even function.
Thus for simplicity we set $k = 0$ and factorise f.
Thus $f(x) = x(x^3 - 4x^2 + x + 6) = x(x + 1)(x - 2)(x - 3)$.
Hence $f(x + 1) = (x + 1)(x + 2)(x - 1)(x - 2)$ is an even function and thus $h = -1$.

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