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VCE Mathematical Methods ¾
Coordinate Geometry Exam Skills [1.6]
Workbook

Outline:

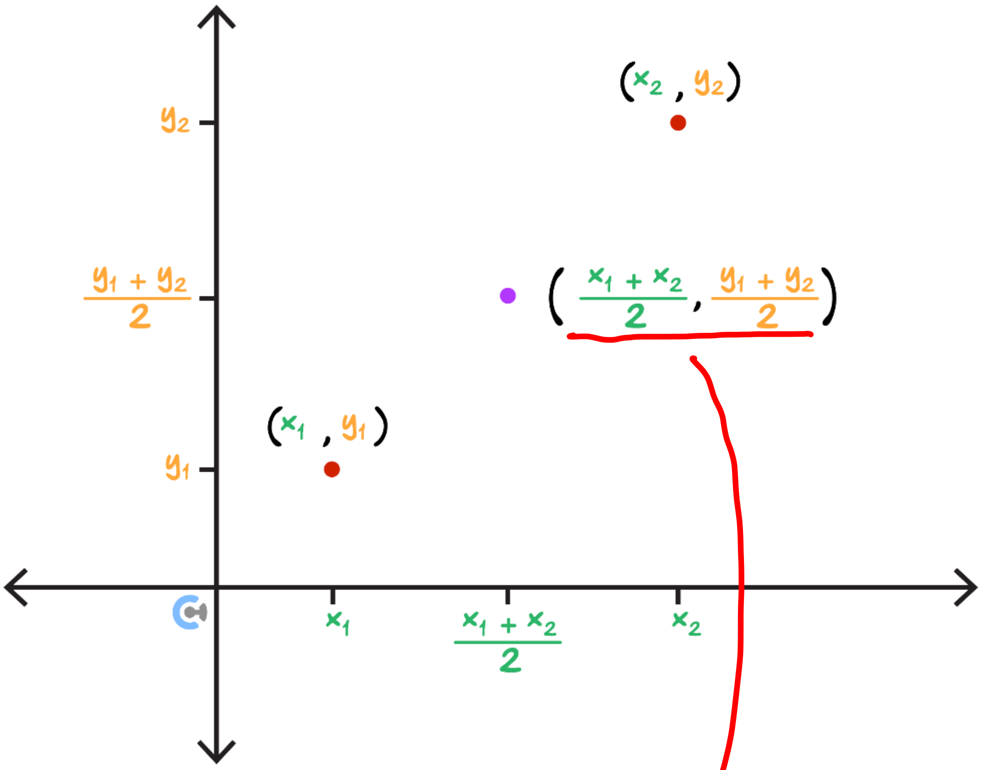
Learning Objectives:

- ☐ **MM34 [1.6.1]** – Apply midpoint to find a reflected point.
- ☐ **MM34 [1.6.2]** – Find the angle between a line and x-axis or two lines.

Section A: Recap

All the students who were here last week, skip to section B: Warm-Up Test!

Midpoint



- The midpoint, M, of two points A and B is simply the point halfway between A and B.

$$M(x_m, y_m) = \left(\qquad \qquad \qquad \qquad \right)$$

- The midpoint can be found by taking the Average of the x-coordinate and y-coordinate of the two points.

Distance Between Two Points

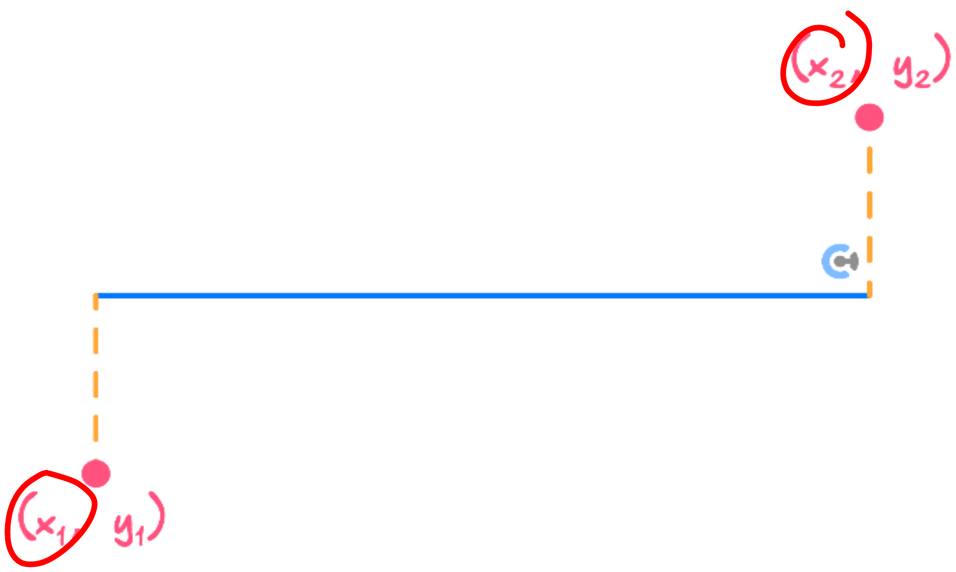
- The distance between two points (x_1, x_2) and (y_1, y_2) can be found using Pythagoras' theorem:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Question 1

Find the points on the line $y = 2x - 6$ which has a distance of $\sqrt{5}$ from the point $(2, 1)$.

Horizontal Distance



Horizontal Distance = $x_2 - x_1$ where $x_2 > x_1$

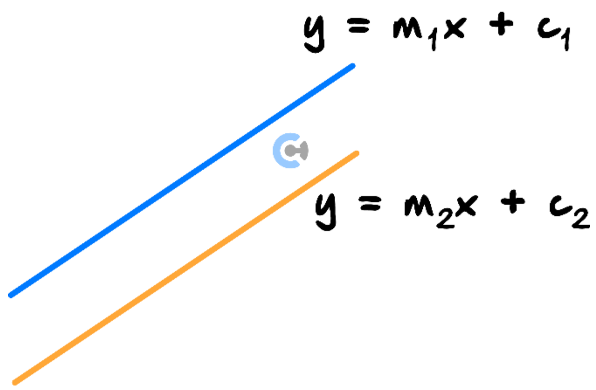
- Find the difference between their x-values.

Vertical Distance

Vertical Distance$=$ $y_2 - y_1$ where $y_2 > y_1$

- Find the difference between their y-values.

Parallel Lines



- Parallel lines have the same gradient.

$$m_1 = m_2$$

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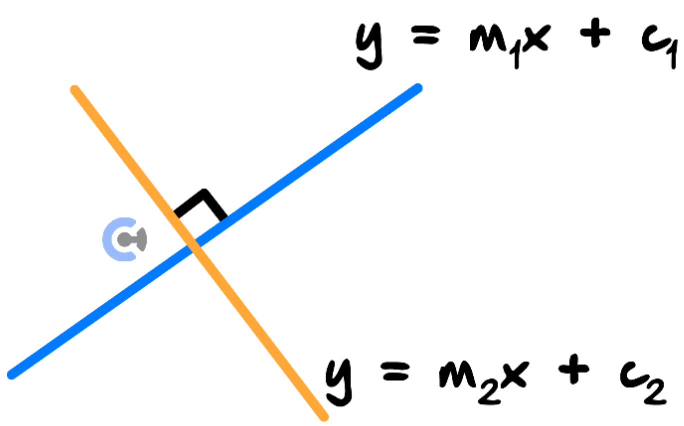
Question 2

Find a line that is parallel to $y = 3x - 1$ passing through the point $(-2, 6)$.

Perpendicular Lines

Definition

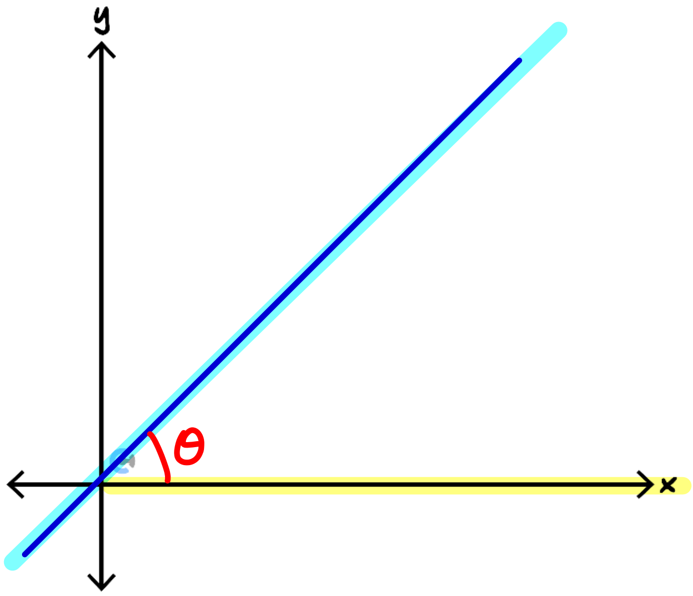


$$m_{\perp} = -\frac{1}{m}$$

Question 3

Find a line that is perpendicular to $y = 3x - 1$ passing through the point $(1, 0)$.

<u>Angle Between a Line and the x-axis</u>



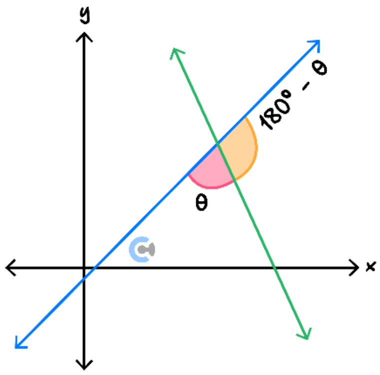
- The angle between a line and the positive direction of the x-axis (anticlockwise) is given by:

$$\tan(\theta) = m$$

Question 4

Find the angle made between the line $y = -x + 2$ and the x-axis measured in the anticlockwise direction.

Acute Angle Between Two Lines



$$\theta = \left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$$

- **Alternatively:**

Board Ref.

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Mod

$|-5| = 5$

$|5| = 5$

For your understanding, note that this formula is derived from the tan compound angle formula covered in SM34.

NOTE: $|x|$ just takes the positive value of x.

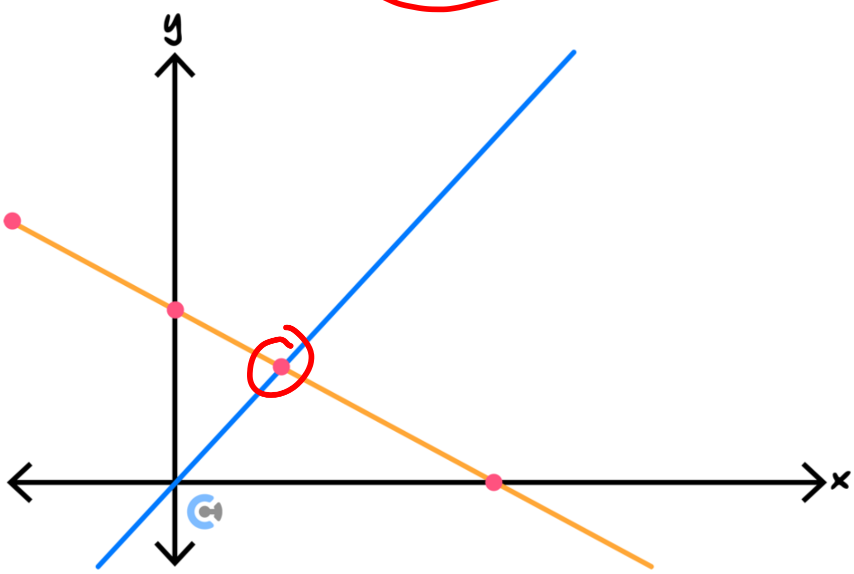
Question 5 Tech-Active.

Find the acute angle between the lines $x - 3y = 2$ and $y = \frac{4}{5}x - 2$. Give your answer in degrees correct to two decimal places.

Exploration: Geometry of the Number of Solutions Between Linear Graphs

- Unique Solution

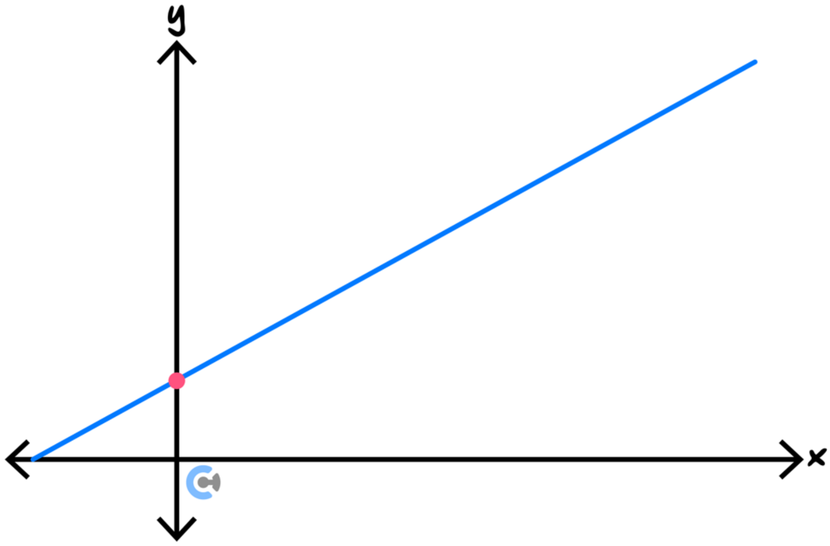
$m_1 \neq m_2$



They just need to have different m's.

- **Infinite Solutions**

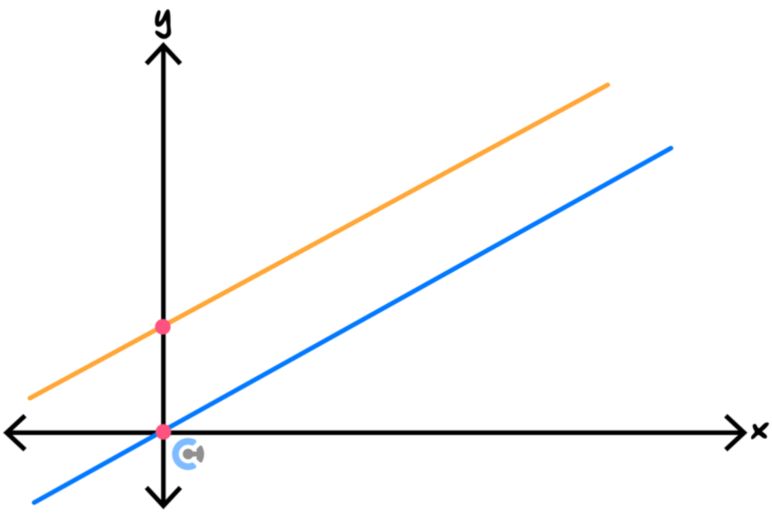
$$m_1 = m_2 \text{ AND } c_1 = c_2$$



- They just need to have the same m and the same +c.
- In other words, they have to be the same line.

- **No Solutions**

$$m_1 = m_2 \text{ AND } c_1 \neq c_2$$



- They need to have the Same m but diff $+c$.
- They have to be two different parallel lines.

General Solutions of Simultaneous Linear Equations

- Two linear equations are either:
  - The same line is expressed in a different form. In this case, they have infinitely many solutions.
  - Unique lines that are parallel. In this case, they have no solutions.
  - Unique lines which are not parallel. In this case, they have exactly one solution.

Space for Personal Notes

Question 6

Consider the following pair of simultaneous equations in terms of $a \in \mathbb{R}\backslash\{0\}$:

$$ax + 3y = 1$$

$$2x + (a + 1)y = 1$$

a. Find the value of a for which there are no solutions to the simultaneous equations.

b. Find the value(s) of a for which there is a unique solution to the simultaneous equations.

c. Find the value of a for which there are infinite solutions to the simultaneous equations.

Solving Systems of Linear Equations with Parameters

- Occurs when solving for three variables with two equations. We simply,

1 $\textit{Let } x = k$, or

2 $\textit{Let } y = k$, or

3 $\textit{Let } z = k$

- And solve simultaneously.

Space for Personal Notes

Question 7

Solve the following system of linear equations with the parameter of k.

$$x + 3z = 1 \quad (1)$$

$$x + y = 2 \quad (2)$$

Let $x = k$

$y = 2 - k$

$z = \frac{1-k}{3}, \quad k \in R$

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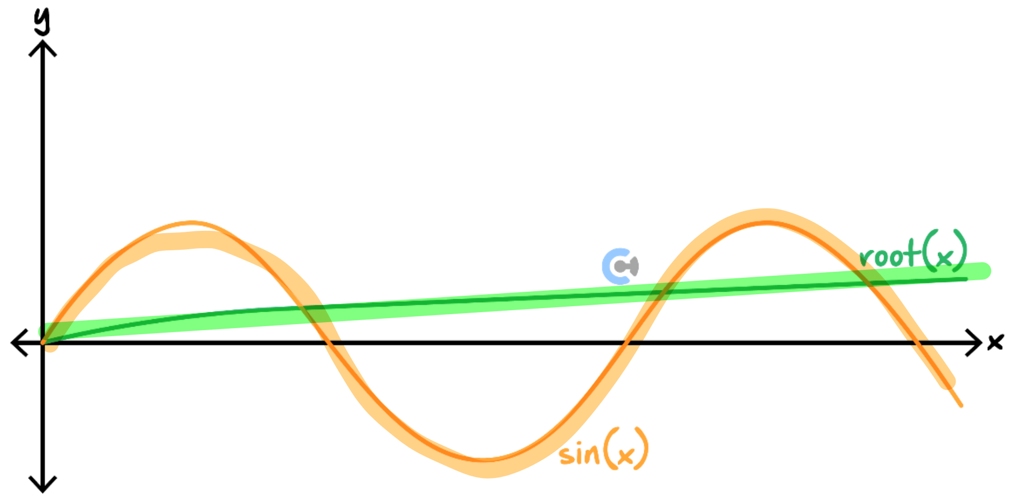
Definition

Addition of Ordinates

➤ **Definition:**

- Technique used to graph the sum/difference of two functions.

$$e.g.\ y = \boxed{\sin(x) + \sqrt{x}}$$



➤ The addition of ordinates involves adding the y values of two functions.

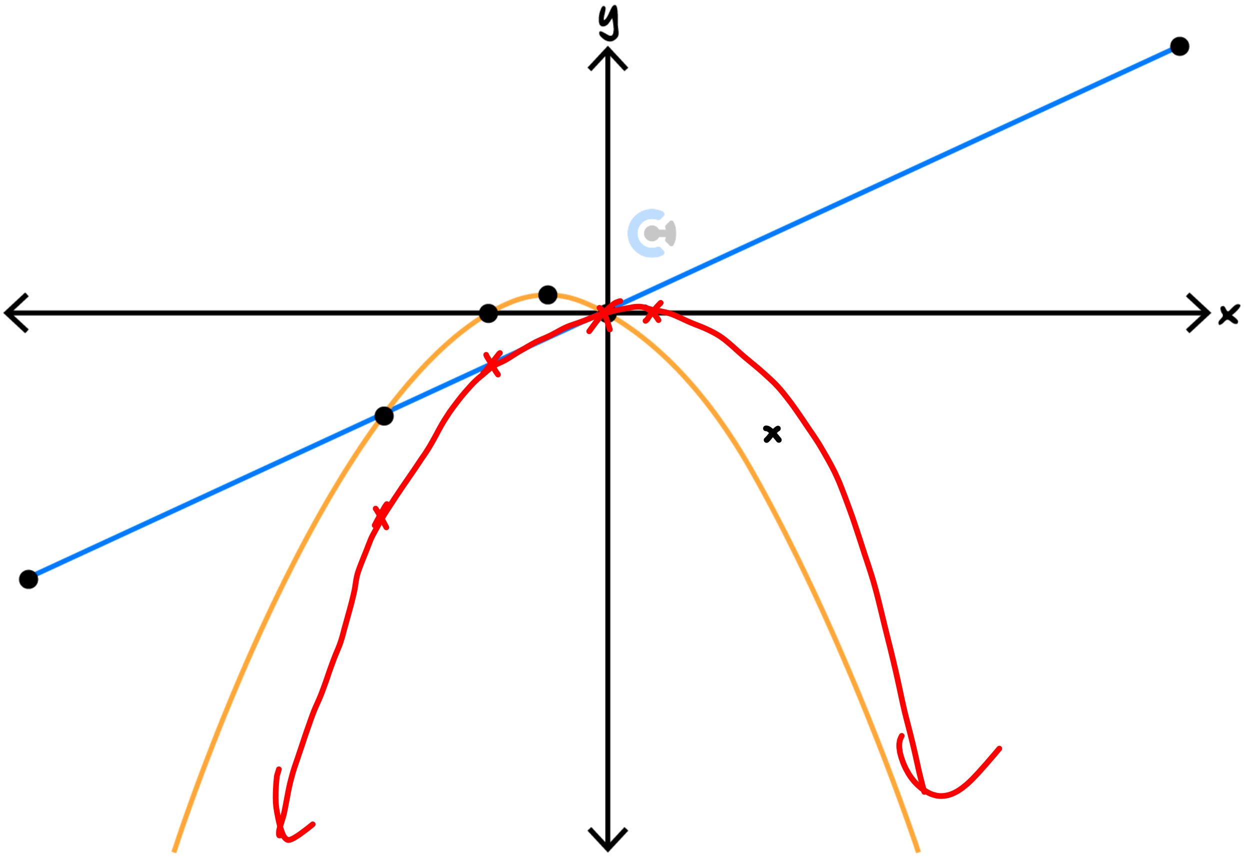
$$\textit{Add } \boxed{\textit{two y-values}}$$

➤ **Steps to sketching $f(x) + g(x)$:**

1. Sketch $f(x)$ and $g(x)$ on the same axes.
2. Plot points for $f(x) + g(x)$ by adding the $\boxed{\text{y-values}}$ of $f(x)$ and $g(x)$.
   - At x-intercepts, the sum equals to the other function. **Why?**
   - At intersections, the sum equals to double the y-value. **Why?**
   - When functions are equidistant from x-axis, the sum equals to 0. **Why?**
3. Join the plotted points.

Question 8

Plot the sum of the two functions given below, using the addition of ordinates.



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Section B: Warm-Up Test (15 Marks)

INSTRUCTION: 15 Marks. 15 Minutes Writing.

Question 9 (3 marks)

Given that the distance between point $A(3,4)$ and point $B(m,2)$ is 3 units, find the possible values of m.

$\sqrt{(m-3)^2+(2-4)^2} = 3$

$m^2 - 6m + 9 + 4 = 9$

$m^2 - 6m + 4 = 0$

$m = \frac{6 \pm \sqrt{36 - 4 \cdot 4}}{2}$

$= \frac{6 \pm \sqrt{20}}{2}$

$= \frac{6 \pm 2\sqrt{5}}{2}$

$= 3 \pm \sqrt{5}$

Space for Personal Notes

Question 10 (3 marks)

Find the equation of the line that passes through $(2, 1)$ and is perpendicular to a line that makes an angle of $60°$ with the positive direction of the x-axis.

$m = \tan(60)$

$= \sqrt{3}$

$m_\perp = -\frac{1}{\sqrt{3}}$

$y = -\frac{1}{\sqrt{3}}x + c$

Sub $(2,1)$

$\therefore y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} + 1$

Space for Personal Notes

Question 11 (4 marks)

Sarah is standing at point $Q(7, 3)$ and wants to walk to the road, which is described by $y = 2x - 5$. But Sarah wants to reach the road by covering the least amount of distance possible.

a. Find the equation of the line that is perpendicular to $y = 2x - 5$ and passes through the point $Q(7, 3)$. (2 marks)

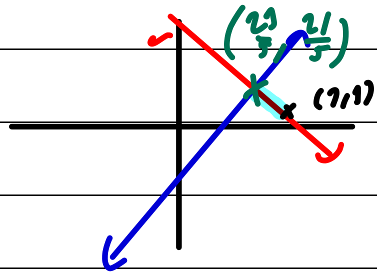
$y = -\frac{1}{2}x + \frac{13}{2}$

b. Hence, find the shortest distance that Sarah can travel to reach the road. (2 marks)

$2x - 5 = -\frac{1}{2}x + \frac{13}{2}$

$\frac{5x}{2} = \frac{23}{2}$

$x = \frac{23}{5}, \quad y = \frac{21}{5}$



$d = \sqrt{\left(7 - \frac{23}{5}\right)^2 + \left(3 - \frac{21}{5}\right)^2}$

$= \sqrt{\left(\frac{12}{5}\right)^2 + \left(-\frac{6}{5}\right)^2}$

$= \sqrt{\frac{144}{25} + \frac{36}{25}}$

$= \sqrt{\frac{180}{25}} = \boxed{\frac{6\sqrt{5}}{5}}$

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Question 12 (5 marks)

Consider the simultaneous linear equations:

$$kx + 4y = 6$$

$$2x + (k-2)y = 3$$

$y = -\frac{kx}{4} + \frac{3}{2}$

$y = -\frac{2x}{k-2} + \frac{3}{k-2}$

Where k is a real constant.

a. Find the values of k for which there is a unique solution to the simultaneous equations. (2 marks)

$m_1 \neq m_2$

$k \neq -2, 4$

$\therefore k \in R \setminus \{-2, 4\}$

b. Find the values of k for which there are infinitely many solutions. (2 marks)

$m_1 = m_2$ $c_1 = c_2$

$k = -2, 4$

$k = 4$

c. Find the values of k for which there are no solutions. (1 mark)

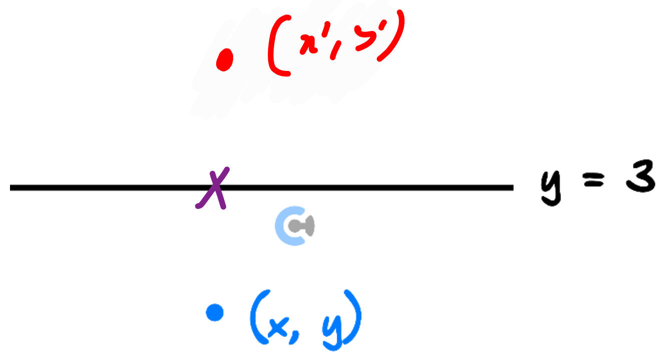
$\boxed{k = -2}$

Section C: Coordinate Geometry Exam Skills

Sub-Section: Reflect a Point Around a Vertical/Horizontal Line

Exploration: Reflection of a Point Around a Vertical/Horizontal Line

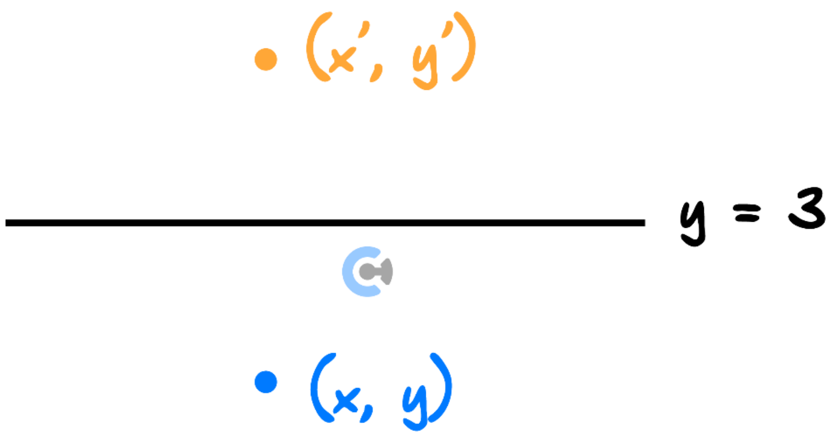
- Consider a point reflected around $y = 3$.



- What do you notice about their midpoint?
- What equation can we construct?

$$\frac{y+y'}{2} = 3$$

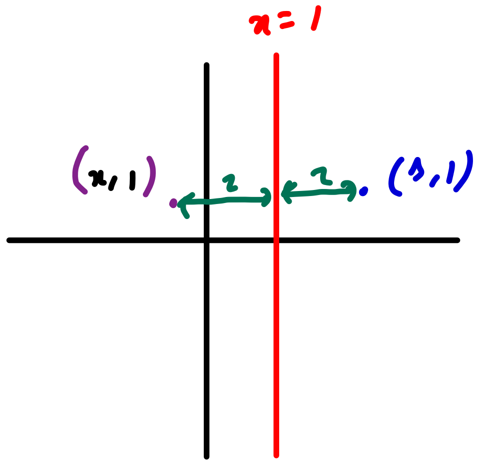
Reflection of a Point Around a Vertical/Horizontal Line



- Midpoint must be on the line of reflection.

Question 13

Find the reflection of $(3, 1)$ around $x = 1$.



$\frac{x+3}{2} = 1$

$x + 3 = 2$

$x = -1$

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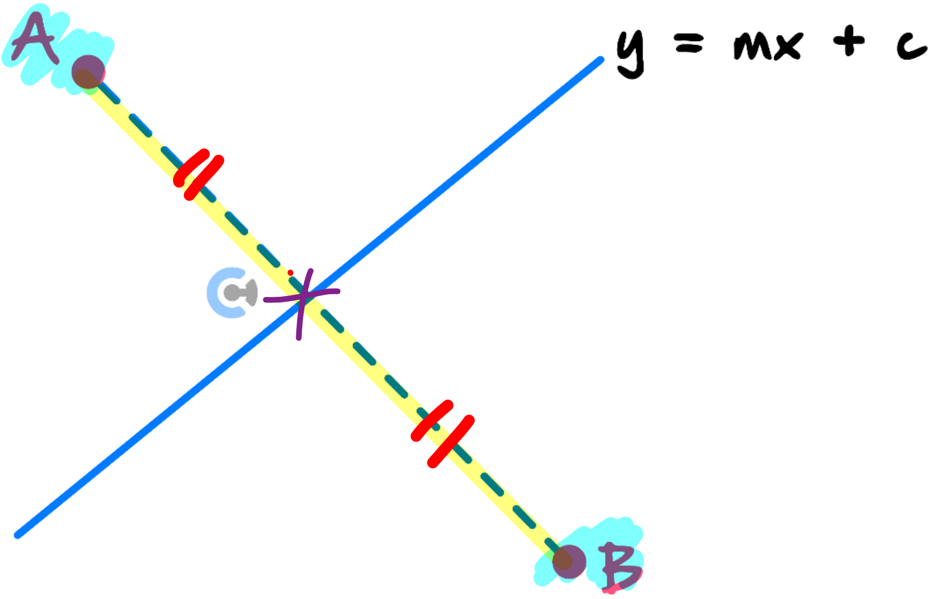
Sub-Section: Reflect a Point Around a Line

How about non-vertical/horizontal lines?

Exploration: Reflection of a Point in a Line

- Consider point A which is reflected across the line $y = mx + c$ to point B, the line AB is perpendicular to the mirror.



- The distance between A and the line is equal to the distance between the line and point B.
- Where would the midpoint of A and B lie?

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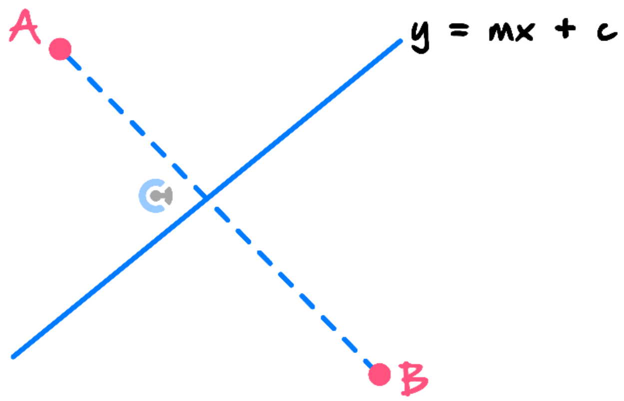
Definition

Finding the Reflection of a Point in a Line:

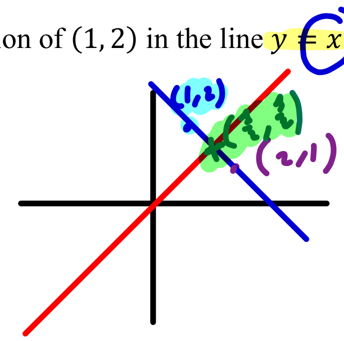


➤ **Steps:**

1. Find the perpendicular line passing through the point.
2. Find the intersection between the original line and the perpendicular line.
3. Find the reflected point (x, y) by treating the intersection from **2.** as the midpoint between the original and reflected point.

Question 14 Walkthrough.

Find the reflection of $(1, 2)$ in the line $y = x$.



Perpendicular Line;

$y = -x + c$

Sub $(1,2)$

$2 = -1 + c$

$c = 3$

$\therefore y = -x + 3$

Intersection:

$x = -x + 3$

$2x = 3$

$x = \frac{3}{2}$

$\left(\frac{x+1}{2}, \frac{y+2}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}\right)$

$\therefore x = 2, y = 1$

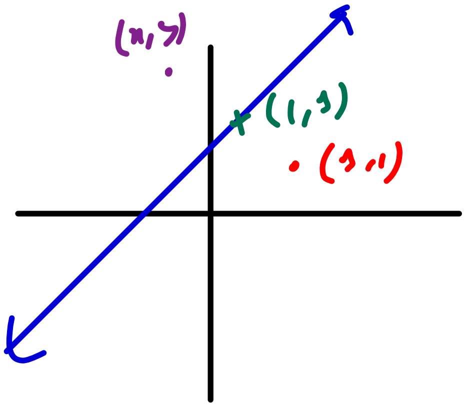
$\therefore (2, 1)$

Active Recall: Steps for Finding the Reflection of a Point in a Line

1. Find the ________________________ line passing through the point.
2. Find the ______________________ between the original line and the perpendicular line.
3. Find the reflected point (x, y) by treating the intersection from **2.** as the __________________ between the original and reflected point.

Question 15

Find the reflection of $(3, 1)$ in the line $y = x + 2$.



$y = -x + c$

sub $(3,1)$

$1 = -3 + c$

$\therefore c = 4$

$y = -x + 4$

$-x + 4 = x + 2$

$2x = 2$

$x = 1$

$\frac{x+3}{2} = 1$

$\frac{y+1}{2} = 3$

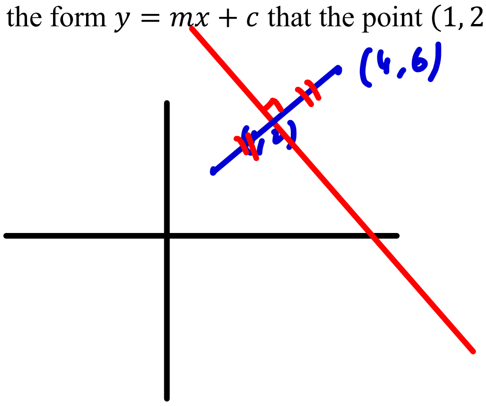
$\therefore x = -1$

$\therefore y = 5$

$\therefore (-1, 5)$

Question 16 Extension.

Find the equation of the line in the form $y = mx + c$ that the point $(1, 2)$ is reflected in the point $(4, 6)$.



$\therefore y = -\frac{3}{4}x + \frac{47}{8}$

Space for Personal Notes

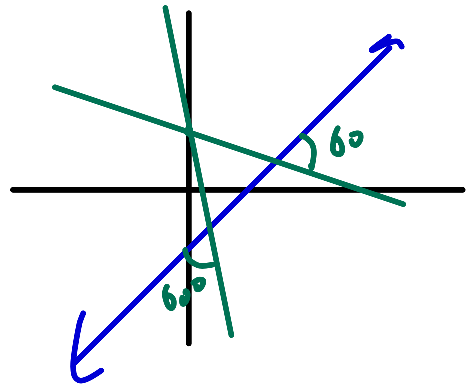
Sub-Section: Application of Angle Between Two Lines

① $\theta = \left| \tan^{-1}(m_1) - \tan^{-1}(m_2) \right|$

② $\tan(\theta) = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

Question 17 Walkthrough.

It is known that the angle between $y = x - 1$ and $y = mx + 1$ is given by $60°$. Find the value(s) of m.



$\tan(60) = \left| \dfrac{1-m}{1+m} \right|$

$\therefore m = -2 - \sqrt{3}$

or

$m = -2 + \sqrt{3}$

Question 18

It is known that the angle between $y = 4x - 3$ and $y = mx + 5$ is given by $45°$. Find the value(s) of m.

$m = -\frac{5}{3}, \frac{3}{5}$

Section D: Exam 1 Questions (16 Marks)

Question 19 (5 marks)

Consider the simultaneous linear equations:

$$(k+1)x + 3y = 6$$

$$4x + (k-3)y = 4$$

$y = -\frac{(k+1)x}{3} + 2$

$y = -\frac{4x}{k-3} + \frac{4}{k-3}$

Where k is a real constant.

a. Find the values of k for which there is a unique solution to the simultaneous equations. (2 marks)

$M_1 \neq M_2$

$k \neq -1, 5$

$\therefore k \in R \setminus \{-1, 5\}$

b. Find the value of k for which there are infinitely many solutions. (2 marks)

$k = 5$

c. Find the value of k for which there are no solutions. (1 mark)

$k = -1$

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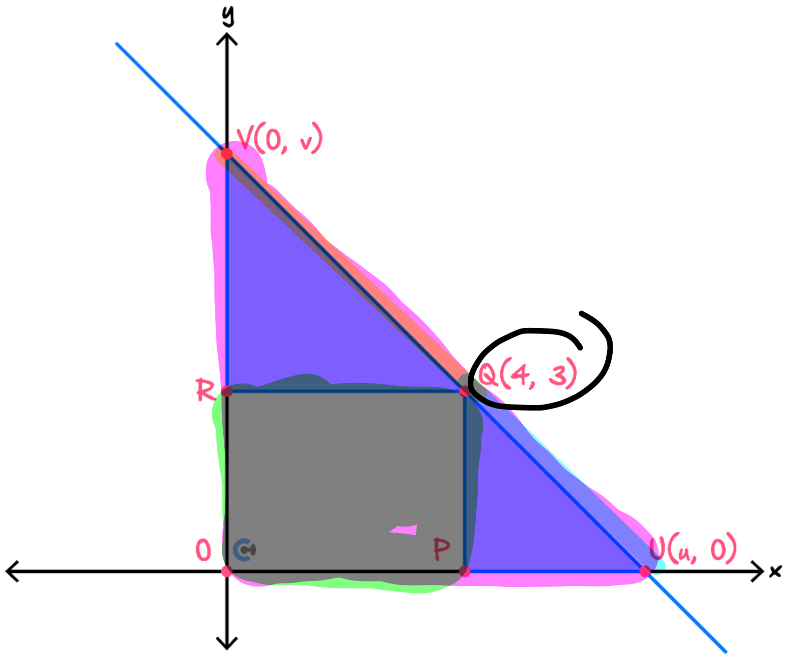
Question 20 (3 marks)

Consider the diagram below:



The rectangle $OPQR$ has a vertex $Q(4,3)$ on the line that passes through U and V.

a. Find an expression for v in terms of u. (1 mark)

$$\frac{3-v}{4} = \frac{3}{4-u}$$

$$3-v = \frac{12}{4-u}$$

$$v = 3 - \frac{12}{4-u}$$

$$= \frac{3(4-u)}{4-u} - \frac{12}{4-u}$$

$$\boxed{v = -\frac{3u}{4-u}}$$

b. Hence, find an expression for the shaded area in terms of u. (2 marks)

$$A = \frac{1}{2} \times u \times \frac{-3u}{4-u} - 12$$

$$= -\frac{3u^2}{8-2u} - 12$$

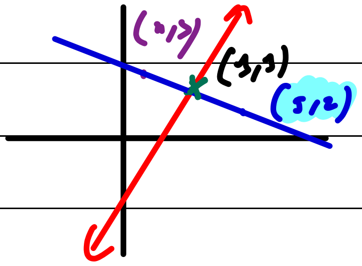
Question 21 (6 marks)

The point $P(5,2)$ is reflected in the line $y = 2x - 3$ to become the point P'.

a. Find the coordinates of P'. (3 marks)

$y = -\frac{1}{2}x + c$

$\therefore y = -\frac{1}{2}x + \frac{9}{2}$



$2x - 3 = -\frac{1}{2}x + \frac{9}{2}$

$\frac{5x}{2} = \frac{15}{2}$

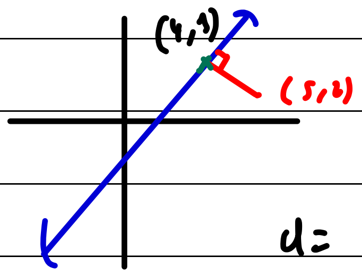
$\therefore x = 3$

$\frac{x+5}{2} = 3, \quad \frac{y+2}{2} = 3$

$\therefore x = 1, \quad y = 4$

$\therefore P'(1,4)$

b. Find the minimum distance between the point P and the line $y = x - 1$. (3 marks)



$y = -x + 7$

$x = 4$

$d = \sqrt{(5-4)^2 + (2-3)^2}$

$= \boxed{\sqrt{2}}$

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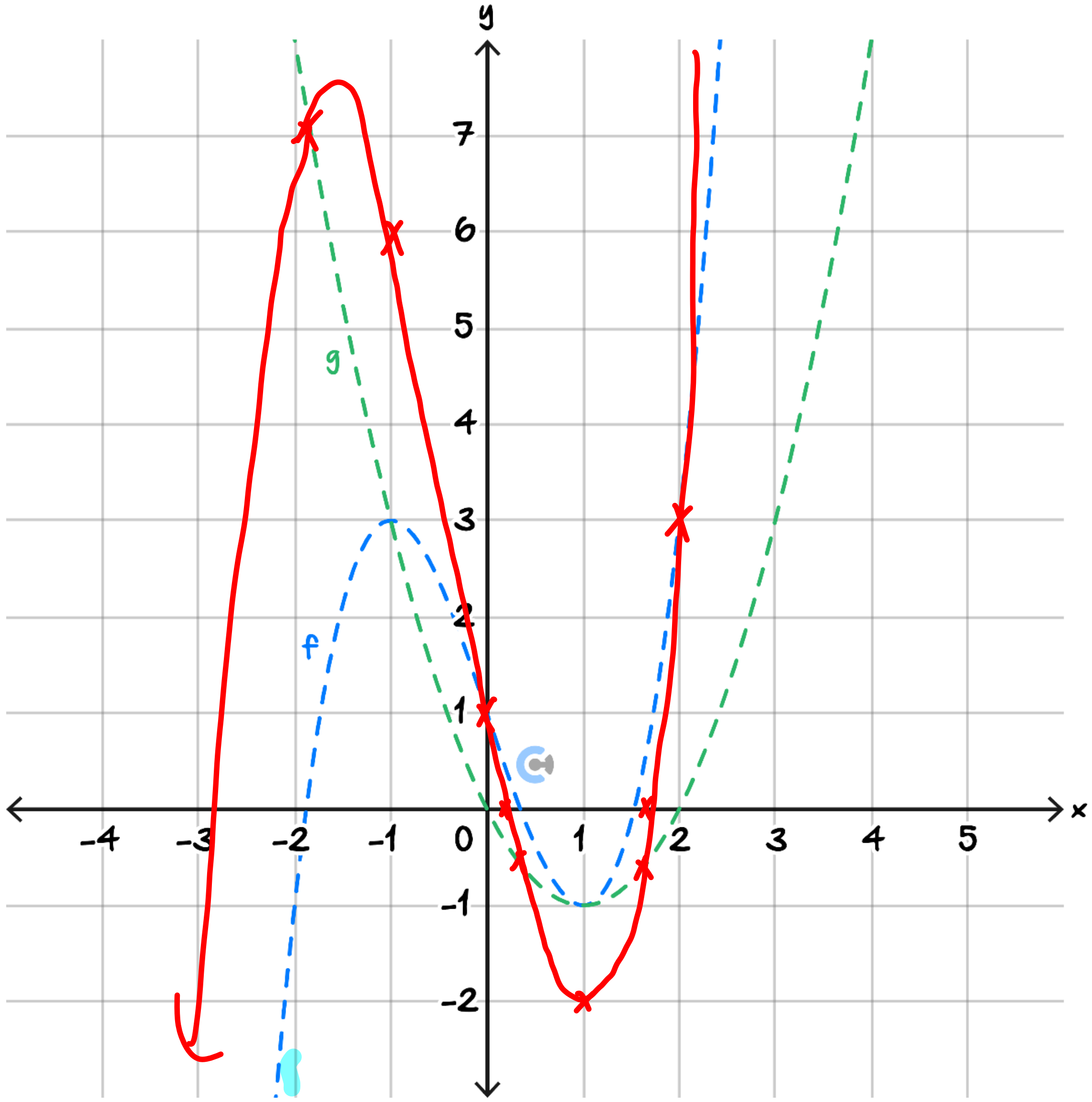
Question 22 (2 marks)

The graphs of f and g are sketched on the axes below. Sketch the graph of $f + g$ on the same axes.



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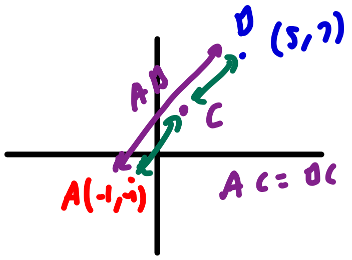


Section E: Tech-Active Exam Skills

Calculator Commands: Simultaneous Equations on CAS

System of Linear Equations

Example:

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have no solution for:

A. $a = 3$

B. $a = -3$

C. Both $a = 3$ and $a = -3$.

D. $a \in R\backslash\{3\}$

E. $a \in R\backslash[-3, 3]$

$\text{system\_solve}(a\cdot x-3\cdot y=5, 3\cdot x-a\cdot y=8-a, a)$

- Solving: $\begin{bmatrix} a\cdot x-3\cdot y=5 \\ 3\cdot x-a\cdot y=8-a \end{bmatrix}$
- Unique Solution: $a \neq -3$ and $a \neq 3$
- No Solutions: $a=-3$
- Infinite Solutions: $a=3$

Overview:

This program takes two linear equations and a parameter and finds the parameter values for the system to obtain a unique solution, no solution, or infinite solutions.

Input:

$\boldsymbol{system\_solve}(< \boldsymbol{equation\ 1} >, < \boldsymbol{equation\ 2} >, < \boldsymbol{parameter} >)$

Other Notes:

The program can only handle one parameter.

➤ Or menu – 3 – 7.

- UDF line functions:

Normal Line

$normal\_line(x^3-x,x,2)$

- Derivative: $3\cdot x^2-1$
- Gradient: 11
- Perpendicular Gradient: $\frac{-1}{11}$
- Passes Through: $[2 \quad 6]$
- x-Intercept: $[68 \quad 0]$
- Vertical Intercept: $\left[0 \quad \frac{68}{11}\right]$
- Normal Line:

$\frac{68}{11}-\frac{x}{11}$

Overview:

This program will find all the necessary information related to a normal line at a point on a function, which includes:

- The derivative.
- The gradient and perpendicular gradient.
- The point on the function the normal line passes through.
- The axis intercepts of the normal line.
- The equation of the normal line.

Input:

$\boldsymbol{normal\_line}(<\ \boldsymbol{function}\ >, <\ \boldsymbol{variable}\ >, <\ \boldsymbol{x\,point}\ >)$

Tangent Line

$tangent\_line(x^3-x,x,2)$

- Derivative: $3\cdot x^2-1$
- Gradient: 11
- Passes Through: $[2 \quad 6]$
- x-Intercept: $\left[\frac{16}{11} \quad 0\right]$
- Vertical Intercept: $[0 \quad -16]$
- Tangent Line:

$11\cdot x-16$

Overview:

This program will find all the necessary information related to a tangent line at a point on a function, which includes:

- The derivative.
- The gradient of the tangent line.
- The point on the function the tangent line passes through.
- The axis intercepts of the tangent line.
- The equation of the tangent line.

Input:

$\boldsymbol{tangent\_line}(<\ \boldsymbol{function}\ >, <\ \boldsymbol{variable}\ >, <\ \boldsymbol{x\,point}\ >)$

Calculator Commands: Finding the Angle Between Two Lines

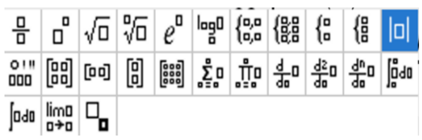
- The angle between two lines with gradients m_1 and m_2 respectively is

$$\theta = \left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$$

- **Mathematica**
  - Use the Abs[] function.

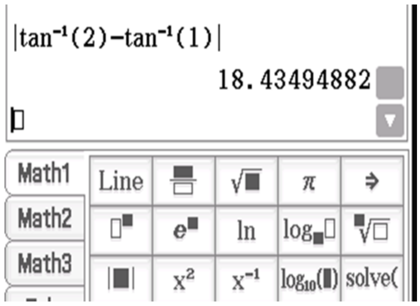
```
In[126]:= Abs[ArcTan[2] - ArcTan[1]] / Degree // N
Out[126]= 18.4349
```

- **TI-Nspire**
  - Find the modulus sign.



$|\tan^{-1}(2) - \tan^{-1}(1)|$ 18.4349

- **Casio Classpad**
  - Modulus sign under Math1.



Calculator Commands: Finding the Gradients of Lines Given the Angle They Make

- If we know the angle and one of the gradients m_1 or m_2 then we can find the other gradient by Solving,

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

- e.g. Find the gradient of the line that makes an angle of 60° with $y = -x$.

- **Mathematica**

- **TI-Nspire**
  - Find the modulus sign.

- **Casio Classpad**
  - Modulus sign under Math1.

Section F: Exam 2 Questions (13 Marks)

Question 23 (1 mark)

The perpendicular bisector of the points $(3, 6)$ and $(7, -4)$ is:

$m = -\frac{5}{2}$

$m_L = \frac{2}{5}$

A. $y = -x + 2$

B. $y = \frac{2}{5}x - 1$

C. $y = -\frac{5}{2}x + 1$

D. $y = x + \frac{2}{5}$

Question 24 (1 mark)

It is known that the lines $y = mx + 4$ and $y = 3x - 5$ make an angle of $45°$ when they intersect. The possible values of m are:

$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

A. $m = -\frac{1}{2}$ only

B. $m = 2$ only

C. $m = -2, \frac{1}{2}$

D. $m = -2, -\frac{1}{2}$

Space for Personal Notes

Question 25 (1 mark)

The simultaneous linear equations:

$$2x - (k+3)y = 8$$

$$(2-k)x - 2y = 3$$

Where k is a real constant that has no solutions for:

$m_1 = m_2, \quad c_1 \neq c_2$

A. $k = 1$ only

B. $k = -2$ only

C. $k = -2, 1$

D. $k \in \mathbb{R}\backslash\{-2, 1\}$

Question 26 (1 mark)

The acute angle made between the lines $y = 2x - 3$ and $y = -x + 1$ correct to the nearest degree is:

$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)|$

A. 108

B. 70

C. 72

D. 51

Space for Personal Notes

Question 27 (1 mark)

The simultaneous linear equations:

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$$(k+1)x - 3y = 1, 0 \leq x \leq 1$$

$$4x - (k+5)y = 2, 0 \leq x \leq 1$$

$x = -\frac{1}{-7-k}$

Where k is a real constant always has exactly one solution for:

unique sol,

$0 \leq -\frac{1}{-7-k} \leq 1$

$M_1 \neq M_2$

$k \neq -7, 1$

$k \geq -6$

A. $k > 1$

B. $k > -6$

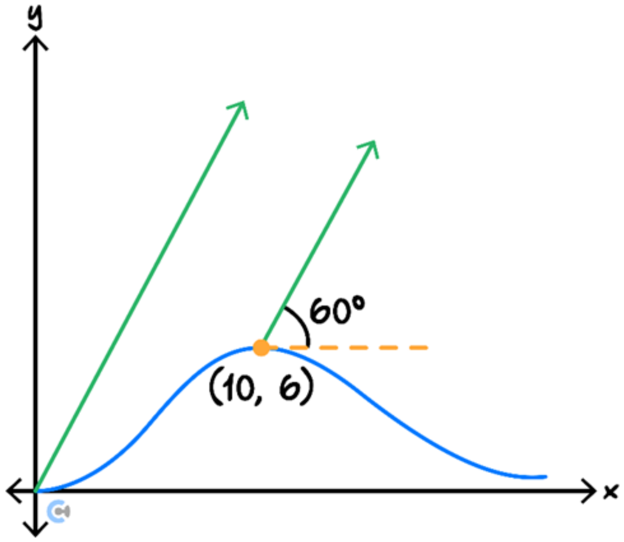
C. $k \in [-6, \infty) \setminus \{1\}$

D. $k \geq -7$

Space for Personal Notes

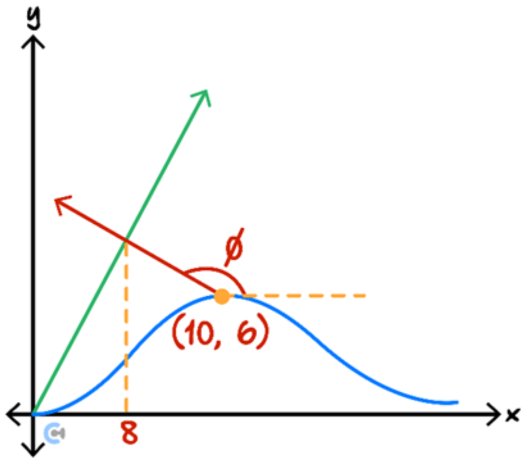
Question 28 (8 marks)

Emma is riding a bicycle on a hill, which starts from the origin as shown in the diagram. When she reaches the point $(10, 6)$, she fires a laser at an angle of $\theta = 60°$ above the horizontal. Meanwhile, David fires a laser from the origin that is parallel to Emma's laser such that both lasers travel together through space.



a. Find the shortest distance between the two laser paths. Give your answer correct to two decimal places. (3 marks)

Emma now changes the direction in which she fires her laser.



b. At what angle, $\emptyset$, should Emma have fired her laser such that the two lasers would intersect at $x = 8$? Give your answer correct to two decimal places. (3 marks)

c. Calculate the acute angle between these two laser paths when they intersect. Express your answer in degrees correct to two decimal places. (1 mark)

d. Give your answer correct to two decimal places. (1 mark)

Space for Personal Notes

Contour Check

Learning Objective: [1.6.1] – Apply Midpoint to Find a Reflected Point.

Key Takeaways

- The line between a point and its reflection is ______________________ to the line it is reflected in.
- The ___________________ of a line and its reflection lies on the line it is reflected in.
- **Steps** for Finding the Reflection of a Point in a Line
  - Find the _________________________ line passing through the point.
  - Find the _______________________ between the original line and the perpendicular line.
  - Find the reflected point (x, y) by treating the intersection from **2.** as the __________________ between the original and reflected point.

Learning Objective: [1.6.2] – Find the Angle Between a Line and x-axis or Two Lines.

Key Takeaways

- To find the angle between a line and the x-axis, we can use equation $m =$ __________.
- To find the angle between two lines, we can use $\theta =$ ___________________________________ or $\tan(\theta) =$ ____________________.