Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Coordinate Geometry [1.5]
Workbook

Outline:

Learning Objectives:

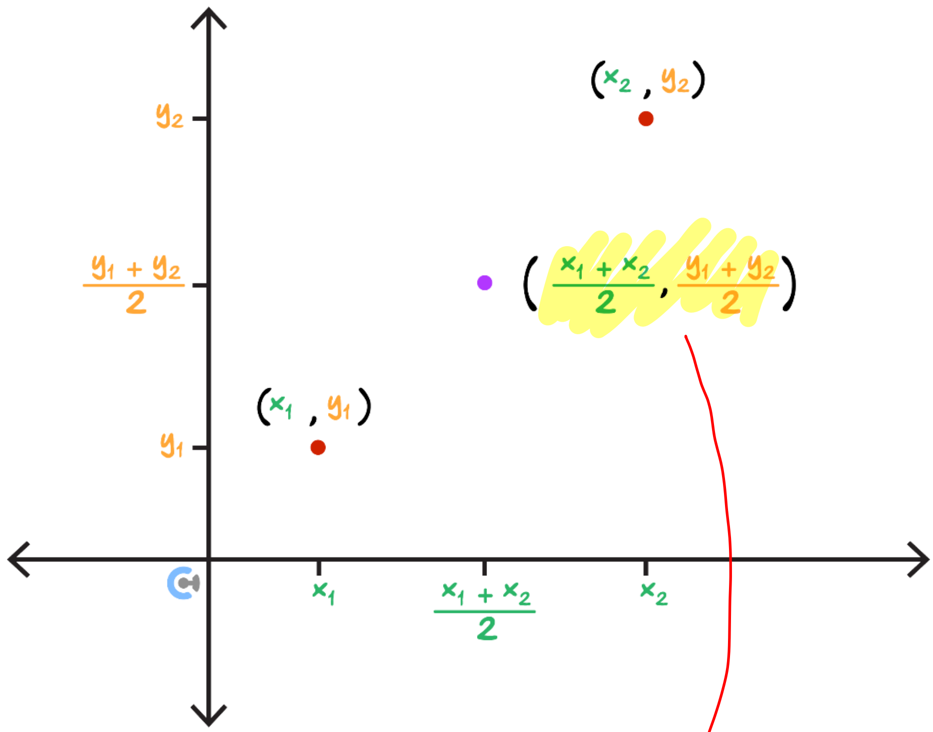
- ☐ **MM34 [1.5.1]** - Find Midpoint, Distance (Horizontal & Vertical) Between Two Points Or Functions
- ☐ **MM34 [1.5.2]** - Find Parallel and Perpendicular Lines
- ☐ **MM34 [1.5.3]** - Find the Angle Between a Line and x-axis or Two Lines
- ☐ **MM34 [1.5.4]** - Find The Unknown Value for Systems of Linear Equations
- ☐ **MM34 [1.5.5]** - Sketching the sum of two function's graph by using the addition of ordinates

Section A: Simple Geometry

Sub-Section: Midpoint

Discussion: How might we find a midpoint between two points?

Midpoint



- The midpoint, M, of two points A and B is simply the point halfway between A and B.

$$M(x_m, y_m) = \left(\qquad \qquad \qquad \right)$$

- The midpoint can be found by taking the ___Average___ of the x-coordinate and y-coordinate of the two points.

Sub-Section: Distance Between Two Points

Distance Between Two Points

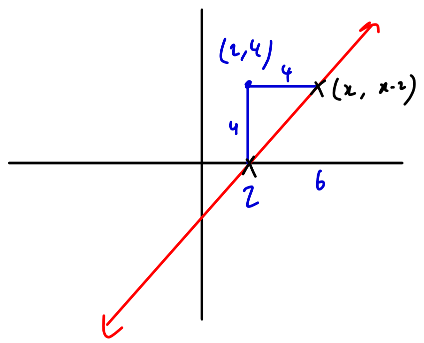
Definition

- The distance between two points (x_1, y_1) and (x_2, y_2) can be found using Pythagoras' theorem:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Question 1 Walkthrough.

Find the points on the line $y = x - 2$ which have a distance of 4 from the point (2,4).



$$\sqrt{(x-2)^2 + (x-2-4)^2} = 4$$

$$(x-2)^2 + (x-6)^2 = 16$$

$$x^2 - 4x + 4 + x^2 - 12x + 36 = 16$$

$$2x^2 - 16x + 24 = 0$$

$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

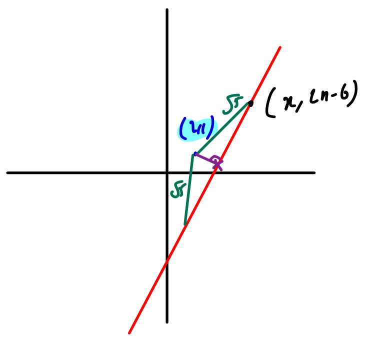
$$\therefore x = 2, 6$$

$$\therefore (2,0), (6,4)$$

Space for Personal Notes

Question 2

a. Find the points on the line $y = 2x - 6$ which has a distance of $\sqrt{5}$ from the point $(2,1)$.



$$\sqrt{(x-2)^2 + (2x-7)^2} = \sqrt{5}$$

$$x^2 - 4x + 4 + 4x^2 - 28x + 49 = 5$$

$$5x^2 - 32x + 48 = 0$$

$$x = \frac{12}{5}, 4$$

$$\therefore \left(\frac{12}{5}, -\frac{6}{5}\right), (4,2)$$

b. Give a reason as to why there are more than 1 points found in **part a.**

$\sqrt{5}$ is not $\perp$ dist

Space for Personal Notes

Question 3 Extension.

Find the points on the line $y = 2x - 3$ which have a distance of $2\sqrt{2}$ from the point $(3,5)$.

$x = 5, \frac{13}{5}$

$y = 7, \frac{11}{5}$

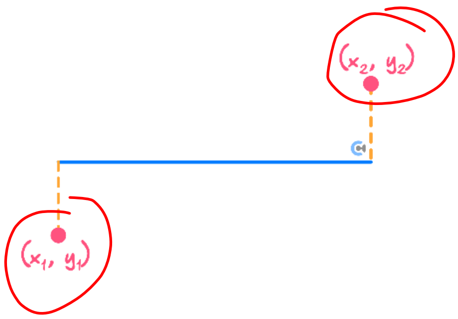
TIP: Don't hesitate to define a point by letting its y value be the function (linear in the above question!)

Space for Personal Notes

Sub-Section: Vertical Distance vs Horizontal Distance

Discussion: How can we find a horizontal distance between two points?

Horizontal Distance



$$\text{Horizontal Distance} = x_2 - x_1 \text{ where } x_2 > x_1$$

- Find the difference between their x-values.

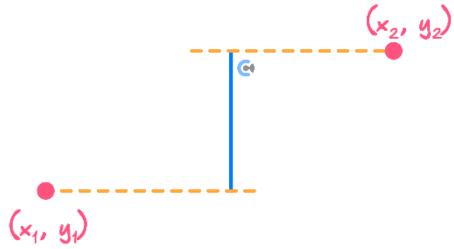
Question 4

Find the horizontal distance between the two points $(7,3)$ and $(11,4)$.

4

What about vertical distance then?

Vertical Distance



$$\text{Vertical Distance} = y_2 - y_1 \text{ where } y_2 > y_1$$

- Find the difference between their y values.

Question 5

Find the vertical distance between the two points $(7,4)$ and $(1,9)$.

5

Key Takeaways

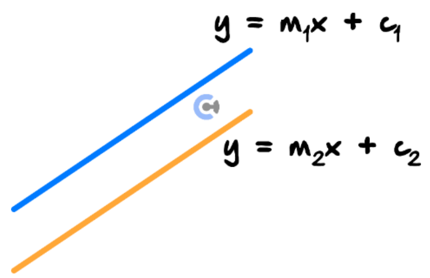
- Midpoint is simply an average point.
- Midpoint can be used to find the point reflected around any axis.
- Distance between two points is derived from Pythagoras theorem.
- Horizontal distance is simply a difference in their x values.
- Vertical distance is simply a difference in their y values.

Section B: Line Geometry

Sub-Section: Parallel and Perpendicular Lines

Discussion: What do we need for the two lines to be parallel?

Parallel Lines



- Parallel lines have the Same gradient.

$$m_1 = m_2$$

Question 6

Find a line that is parallel to $y = 5x - 1$ passing through the point $(-2, -6)$.

$y = 5x + c$

Sub $(-2, -6)$

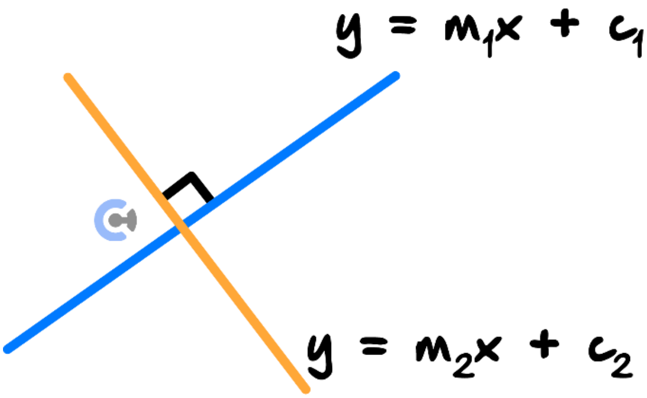
$-6 = -10 + c$

$\therefore c = 4$

$\therefore y = 5x + 4$

Discussion: What about perpendicular lines?

Perpendicular Lines



$$m_1 = -\frac{1}{m_2}$$

Question 7

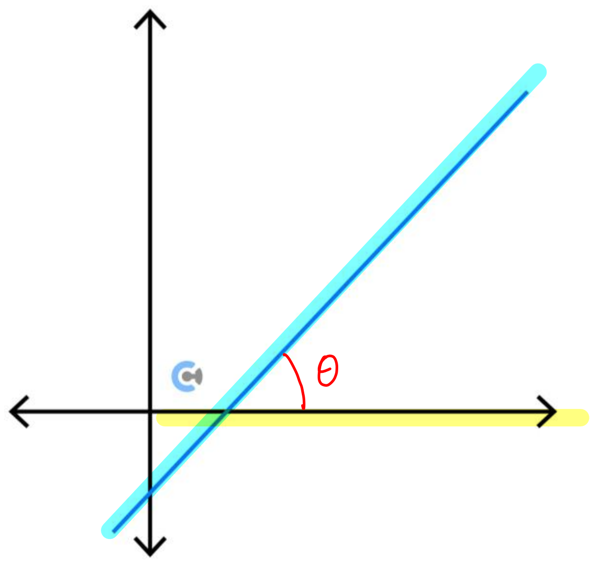
Find a line that is perpendicular to $y = 5x - 1$ passing through the point (1,0).

$y = -\frac{1}{5}x + \frac{1}{5}$

Sub-Section: Angle Between a Line and the x-axis

How do we find the angle between a line and the x-axis?

Angle between a Line and the x-axis

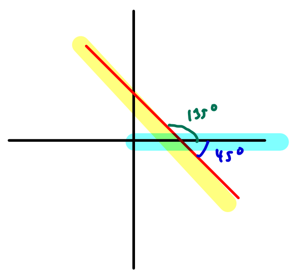


- The angle between a line and the positive direction of the x-axis (anticlockwise) is given by

$$\tan(\theta) = m$$

Question 8

Find the angle made between the line $y = -x + 2$ and the x-axis measured in the anticlockwise direction.



$\tan(\theta) = -1$

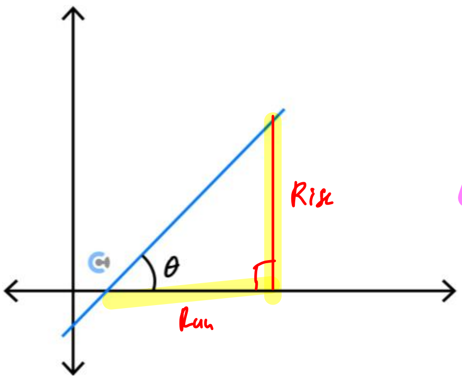
$\theta = -\frac{\pi}{4}$

$= -45°$

How does this formula work?

<u>Exploration</u>: Angle between a line and x-axis.

- Consider a line in the visual below.



$\tan(\theta) = \frac{OPP}{ADJ} = \frac{Rise}{Run} = m$

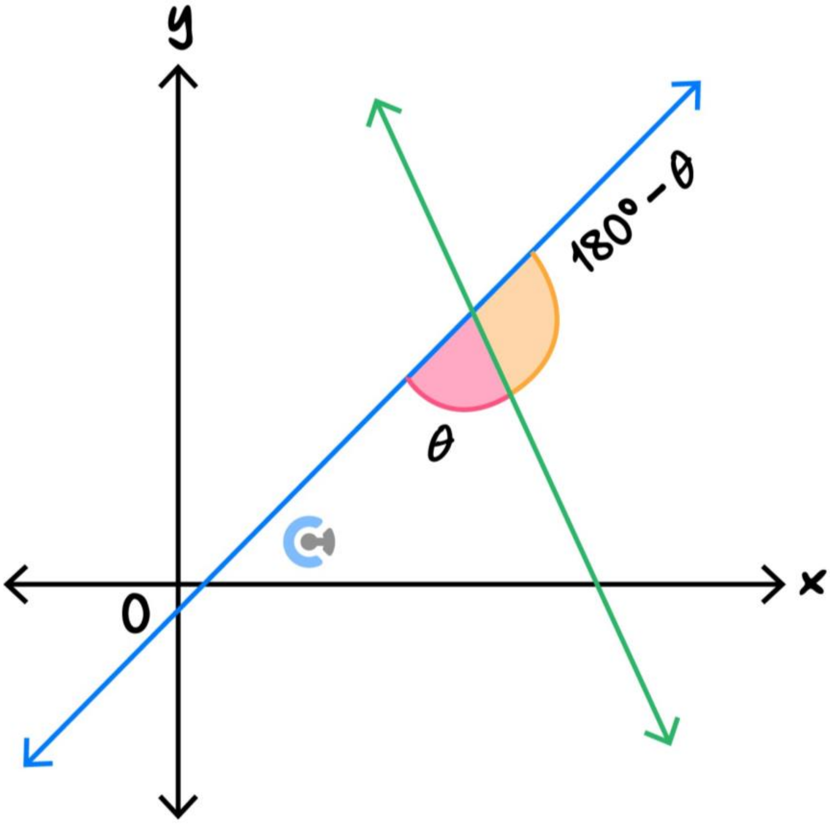
- Construct a right-angle triangle with the angle θ.

- Consider the opposite and adjacent sides of the right-angle triangle. What can we call them?

- Hence, what does $\tan(\theta)$ equal to given that $\tan = opposite/adjacent$?

Sub-Section: Angle Between Two Lines

Acute Angle Between Two Lines



$$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)|$$

- **Alternatively:**

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

For your understanding, note that this formula is derived from the $\tan$ compound angle formula covered in SM34.

NOTE: $|x|$ just takes the positive value of x.

Question 9 Tech-Active.

$y = \frac{x}{3} - \frac{2}{3}$

Find the acute angle between the lines $x - 3y = 2$ and $y = \frac{4}{5}x - 2$. Give your answer in degrees correct to two decimal places.

$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)|$

$= |\tan^{-1}(\frac{1}{3}) - \tan^{-1}(\frac{4}{5})|$

$= 20.22°$

TIP: Make sure your CAS is in degrees.

Let's see if the formula is consistent with parallel lines!

Exploration: Understanding parallel lines using the angle between two lines formula

- When two lines are parallel, what must be the angle θ between them?

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

0

- Let's substitute the value of θ and see what we get!

$0 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$m_1 - m_2 = 0$

$\boxed{m_1 = m_2}$

This looks rather familiar!

And now perpendicular lines!

Exploration: Understanding perpendicular lines using the angle between two lines formula

- When two lines are perpendicular, what must be the angle θ between them?

$$\tan(90) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$\text{undef} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

- Let's substitute the value of θ and see what we get! (Note: $\tan(90) = Undef$)

$1 + m_1 m_2 = 0$

$m_1 m_2 = -1$

$m_1 = -\frac{1}{m_2}$

- This looks rather familiar, doesn't it?

Key Takeaways

- Parallel lines have the same gradient.
- Perpendicular lines have negative reciprocal gradients.
- Angle between a line and x-axis is given by $\tan^{-1}(m)$.
- Angle between two lines is given by $\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.
- The parallel lines and perpendicular lines formula is consistent with the angle between the two lines formula.

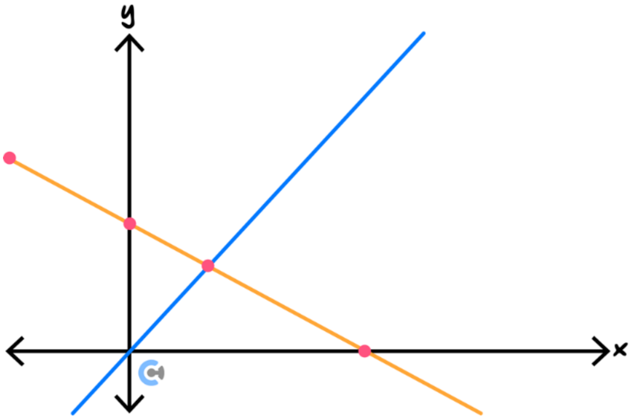
Section C: Simultaneous Equations

Sub-Section: Systems of Linear Equations

Exploration: Geometry of the number of solutions between linear graphs

- Unique Solution

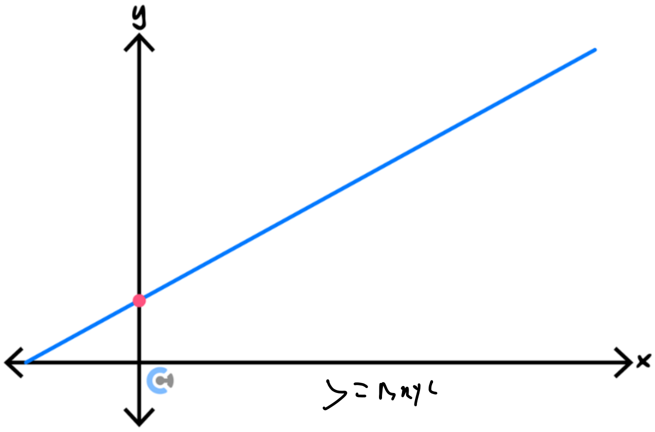
$$m_1 \neq m_2$$



- They just need to have diff gradient.

- Infinite Solutions

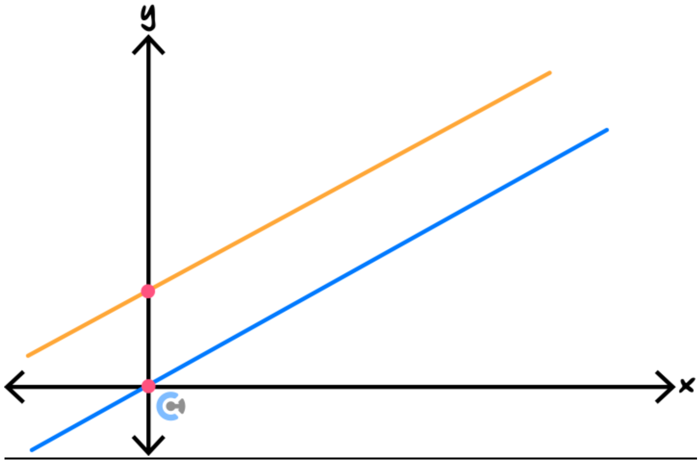
$$m_1 = m_2 \; AND \; c_1 = c_2$$



- They just need to have the same gradient and the same + c.
- In other words, they have to be the same line.

- **No Solutions**

$$m_1 = m_2 \ AND \ c_1 \neq c_2$$



- They need to have the same m but different $+c$.
- They have to be two different Parallel lines.

General Solutions of Simultaneous Linear Equations

- Two linear equations are either:
  - The same line is expressed in a different form. In this case, they have infinitely many solutions.
  - Unique lines which are parallel. In this case, they have no solutions.
  - Unique lines which are not parallel. In this case, they have exactly one solution.

Space for Personal Notes

Question 10 Walkthrough.

Consider the following pair of simultaneous equations in terms of $a \in R\backslash\{0\}$:

$$(a+1)x + 2y = 1$$

$$6x + (a-3)y = 1$$

$y = \frac{-(a+1)x}{2} + \frac{1}{2}$

$y = \frac{-6x}{a-3} + \frac{1}{a-3}$

a. Find the value(s) of a for which there are no solutions to the simultaneous equations.

$m_1 = m_2$

$\frac{a+1}{2} = \frac{6}{a-3}$

$a^2 - 2a - 3 = 12$

$a^2 - 2a - 15 = 0$

$(a-5)(a+3) = 0$

$\therefore a = -3, 5$

$c_1 \neq c_2$

$\frac{1}{2} \neq \frac{1}{a-3}$

$a - 3 \neq 2$

$a \neq 5$

$\boxed{\therefore a = -3}$

b. Find the value(s) of a for which there is a unique solution to the simultaneous equations.

$m_1 \neq m_2$

$a \neq -3, 5$

$a \in R \backslash \{-3, 0, 5\}$

c. Find the value(s) of a for which there are infinite solutions to the simultaneous equations.

$a = 5$

TIP: Substitute your answer back into the equations to see if the criteria are met for each part.

Question 11

Consider the following pair of simultaneous equations in terms of $a \in R\backslash\{0\}$:

$$ax + 3y = 1$$

$$2x + (a + 1)y = 1$$

$y = -\frac{ax}{3} + \frac{1}{3}$

$y = -\frac{2x}{a+1} + \frac{1}{a+1}$

a. Find the value(s) of a for which there are no solutions to the simultaneous equations.

$-\frac{a}{3} = -\frac{2}{a+1}$

$a^2 + a = 6$

$a^2 + a - 6 = 0$

$(a+3)(a-2) = 0$

$\therefore a = -3, 2$

$a + 1 \neq 3$

$a \neq 2$

$\boxed{\therefore a = -3}$

b. Find the value(s) of a for which there is a unique solution to the simultaneous equations.

$a \neq -3, 2$

$\therefore a \in R \backslash \{-3, 0, 2\}$

c. Find the value(s) of a for which there are infinite solutions to the simultaneous equations.

$a = 2$

Space for Personal Notes

Question 12 Extension.

Consider the following pair of simultaneous equations in terms of $a \in R\backslash\{0\}$:

$$(1-2a)x - 3y = a + 1$$

$$-2x + (a+3)y = 2a + 2$$

a. Find the value(s) of a for which there are no solutions to the simultaneous equations.

$m_1 = m_2$ | $c_1 \neq c_2$

$a = -\frac{1}{2}, -1$ | $a \neq -9, -1$

$\therefore a = -\frac{1}{2}$

b. Find the value(s) of a for which there is a unique solution to the simultaneous equations.

$a \in R \backslash \{-\frac{1}{2}, -1, 0\}$

c. Find the value(s) of a for which there are infinite solutions to the simultaneous equations.

$a = -1$

Sub-Section: Finding Simultaneous Equation for Three Variables

Solving Systems of Linear Equations with Parameters

- Occurs when solving for three variables with two equations. We simply,

$$\text{Let } x = k, \text{ or}$$

$$\text{Let } y = k, \text{ or}$$

$$\text{Let } z = k$$

- And solve simultaneously.

Question 13 Walkthrough.

Solve the following system of linear equations with the parameter of k.

$$2x - y = 4 \quad (1) \qquad 2k - y = 4$$

$$x + z = 3 \quad (2) \qquad k + z = 3$$

Let $x = k$

$\therefore y = 2k - 4$

$z = 3 - k$, $k \in R$

NOTE: We can let any variable equal to k.

<u>Discussion:</u> **How many solutions did we find in the question above?**

Infinite

Question 14

Solve the following system of linear equations with the parameter of k.

$$x + 3z = 1$$

$$x + y = 2$$

Let $x = k$

$y = 2 - k$

$z = \frac{1-k}{3}$

Discussion: **Why did we get infinite solutions?**

Not enough info.

Key Takeaways

- Simultaneous equations can be solved using elimination or substitution methods.
- Two lines can have either unique, no or infinite solutions.
- For unique solutions, we just need different gradients.
- For no solutions, we need the same gradient but different c value.
- For infinite solutions, we need the same gradient and same c value.
- When solving two simultaneous equations for three variables, we can let any variable equal to the parameter of k.
- When we have infinite solutions, it is also called general solutions.
- We always get infinite solutions when there are fewer equations than a number of variables.

Space for Personal Notes

<u>Section D:</u> Addition of Ordinates

<u>Sub-Section</u>: Addition of Ordinates

<u>REMINDER</u>: Don't forget Function is always equal to its __y____ value.

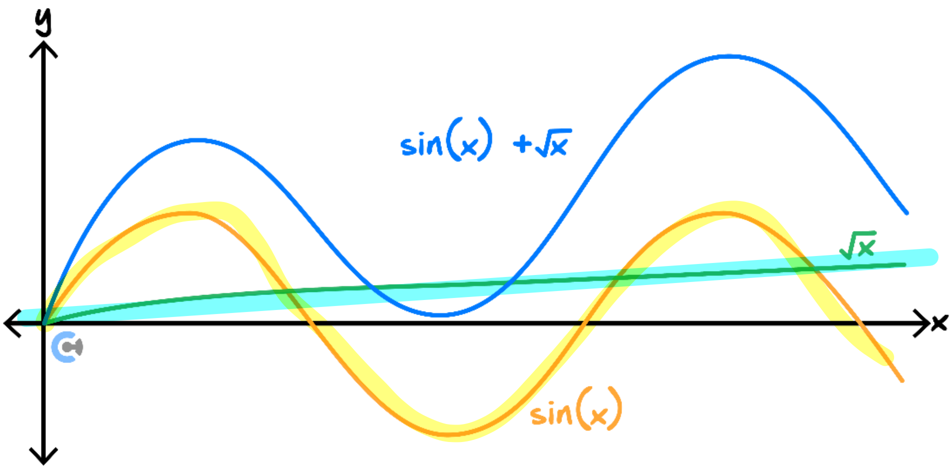
<u>Discussion:</u> How can we sketch $\sin(x) + \sqrt{x}$?

<u>Addition of Ordinates</u>

sco

- **Definition:**
  - Technique used to graph the sum/difference of two functions.

$$\text{e.g. } y = \sin(x) + \sqrt{x}$$



- Addition of ordinates involves adding the __y values____ of two functions.

$$\textbf{Add two } y \textbf{ values}$$

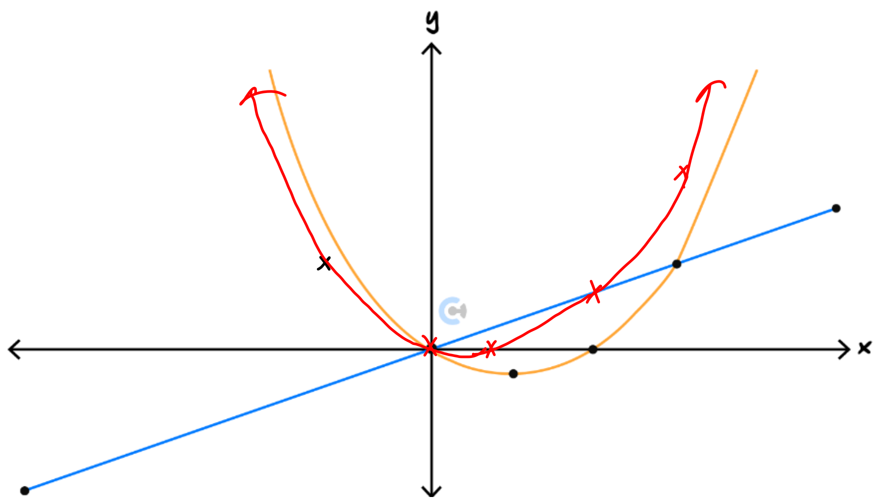
- **Steps to sketching $f(x) + g(x)$**

1. Sketch $f(x)$ and $g(x)$ on the same axes.
2. Plot points for $f(x) + g(x)$ by adding the **y values** of $f(x)$ and $g(x)$.
   - At x-intercepts, the sum equals to the other function. **Why?**
   - At intersections, the sum equals to double the y value. **Why?**
   - When functions are equidistance from x-axis, sum equals to 0. **Why?**
3. Join the plotted points.

Question 15 Walkthrough.

Two functions, f and g, are shown below.

Sketch the function $f(x) + g(x)$ on the same axes, without finding or using the rule for either function.



NOTE: We always add their y values.

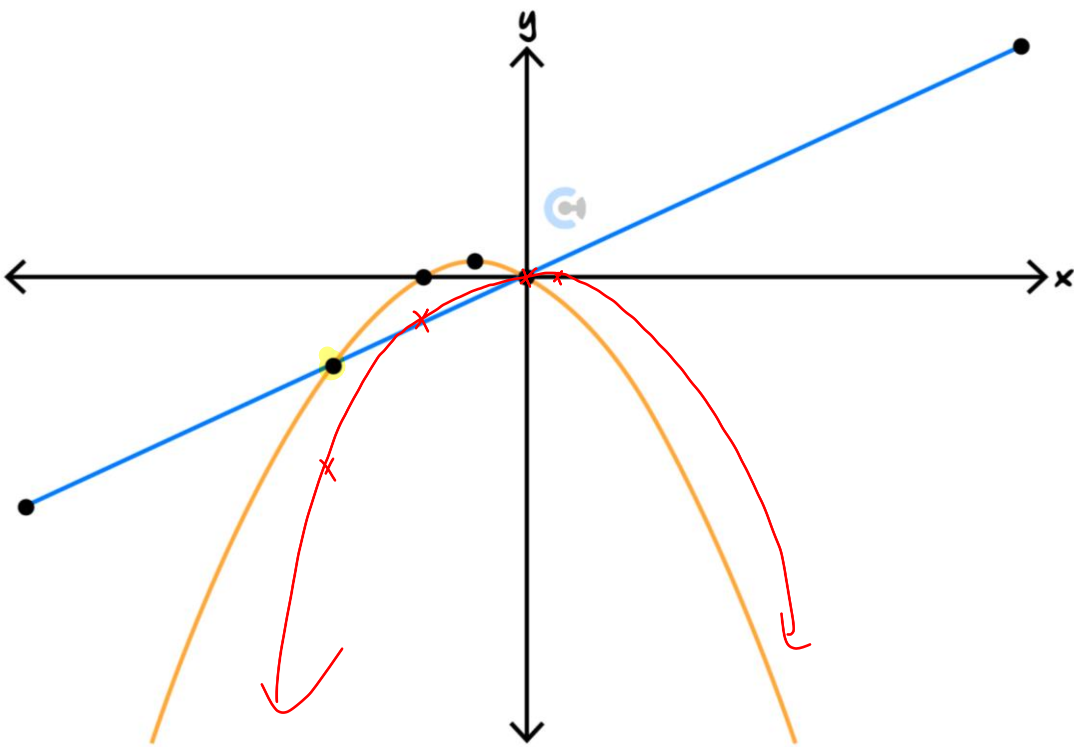
Your turn!

Active Recall: Steps to sketching $f(x) + g(x)$

1. Sketch $f(x)$ and $g(x)$ on the same axes.
2. Plot points for $f(x) + g(x)$ by adding the ______________ of $f(x)$ and $g(x)$.
   - At x intercepts, the sum equals to the ___________________________.
   - At intersections, the sum equals to _____________ the y value.
   - When functions are equidistance from x-axis, sum equals to ____.
3. Join the plotted points.

Question 16

Plot the sum of the two functions given below, using the addition of ordinates.



Key Takeaways

- Addition of Ordinates is used to sketch the sum of two functions.
- We always add their y values.
- When we have an x-intercept for one graph, the sum graph intersects the other graph.
- When we have an intersection between two graphs, the sum graph equals to double their y value.
- When we have an equidistance from the x-axis, the sum graph has an x-intercept.

Space for Personal Notes

Contour Check

Learning Objective: [1.5.1] - Find Midpoint, Distance (Horizontal & Vertical) Between Two Points Or Functions

Key Takeaways

- Midpoint is simply the Avg of 2 points.
- Distance formula is derived from Pythag.
- Horizontal distance is the distance between x values.
- Vertical distance is the distance between y values.

Learning Objective: [1.5.2] - Find Parallel and Perpendicular Lines

Key Takeaways

- Parallel lines have the Same gradient.
- Perpendicular lines have negative reciprocal gradient.

Learning Objective: [1.5.3] - Find the Angle Between a Line and x-axis or Two Lines

Key Takeaways

- To find the angle between a line and the x-axis we can use equation $m = \tan(\theta)$.
- To find the angle between two lines we can use $\theta = \left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$ or $\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Learning Objective: [1.5.4] - Find The Unknown Value for Systems of Linear Equations

Key Takeaways

- Two linear equations have unique solution if they have __diff__ gradients.
- Two linear equations have infinitely many solutions when they have __Same__ gradient and __Same__ constant.
- Two linear equations have no solution when they have __Same__ gradient and __diff__ constant.

Learning Objective: [1.5.5] - Sketching the sum of two function's graph by using the addition of ordinates

Key Takeaways

- Addition of Ordinates is used to sketch the __Sum__ of two functions.
- We always add their __y__ values.
- When we have an x intercept for one graph, sum graph __equals__ the other graph.
- When we have an intersection between two graphs, the sum graph equals to __double__ their __y__ value.
- When we have an equidistance from the x-axis, sum graph has an __x__ intercept.

CONTOUREDUCATION

Website: contoureducation.com.au | Phone: 1800 888 300 | Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾

Free 1-on-1 Consults

What Are 1-on-1 Consults?

- **Who Runs Them?** Experienced Contour tutors (45+ raw scores and 99+ ATARs).
- **Who Can Join?** Fully enrolled Contour students.
- **When Are They?** 30-minute 1-on-1 help sessions, after school weekdays, and all day weekends.
- **What To Do?** Join on time, ask questions, re-learn concepts, or extend yourself!
- **Price?** Completely free!
- **One Active Booking Per Subject:** Must attend your current consultation before scheduling the next :)

SAVE THE LINK, AND MAKE THE MOST OF THIS (FREE) SERVICE!

Booking Link

bit.ly/contour-methods-consult-2025