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VCE Mathematical Methods ¾
Transformations Exam Skills [1.4]
Homework Solutions

Homework Outline:

Compulsory	Pg 2 – Pg 29
Supplementary	Pg 30 – Pg 61

Section A: Compulsory

Sub-Section [1.4.1]: Apply Quick Method to Find Transformations

Question 1

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the following sequence of transformations:

- A dilation by a factor of 3 from the y-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the x-axis, followed by,
- A translation 3 units upwards and 2 units left.

T maps the graph of $f(x) = \sqrt{x}$ onto the graph of g. Find the rule for g.

The transformations pertaining to x are,

- A dilation by a factor of 3 from the y-axis ($x \mapsto 3x$), followed by,
- A translation of 2 units left ($x \mapsto x - 2$).

Thus we apply them oppositely and in opposite order to get $\sqrt{\frac{x+2}{3}}$.

Now we simply apply the transformations pertaining to y in regular order to get $g(x) = \frac{1}{2}\sqrt{\frac{x+2}{3}} + 3$.

Question 2

A transformation, $T(x, y) = (ax + b, cx + d)$ maps the graph of $y = f(x)$ onto the graph of $y = 4 - 2f(3 - x)$. Find the values of a, b, c and d.

Under T we know that $x = 3 - x'$. Thus $x' = 3 - x$.
We can read the transformations of y straight off, getting $y' = 4 - 2y$.
Hence $a = -1, b = 3, c = -2$ and $d = 4$.

Question 3

Describe a sequence of transformations that maps the graph of $y = e^{2x+3} + 2$ onto the graph of $y = 1 - 3e^{x}$.

We first isolate the x part of our transformations, noting that $2x + 3 = x'$.
Thus our desired transformations are the reverse (including the order) of the transformations that will map $2x + 3$ onto x, specifically.

- A dilation by a factor of 2 from the y-axis, followed by,
- A translation of 3 units right.

Now we look at y. Observing that $-3(e^{x} + 2) + 7 = 1 - 3e^{x}$ we can read off our transformations to be,

- A dilation by a factor of 3 from the x-axis, followed by,
- A reflection in the x-axis, followed by,
- A translation of 7 units upwards.

Applying the dot points in order gives our transformation.

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Sub-Section [1.4.2]: Apply Transformations of Functions to Find its Domain and range

Question 4

The function $f\ [-1,3) \to \mathbb{R}$ has a range of $(-3,5]$.

Find the domain and range of $g(x) = -2f\left(\frac{x}{2} - 1\right)$.

Observe that x is in the domain of g if and only if, $\frac{x}{2} - 1$ is in the domain of f.
Thus $\frac{x}{2} - 1 \geq -1 \implies x \geq 0$ and $\frac{x}{2} - 1 < 3 \implies x < 8$.
Hence the domain of g is $[0,8)$.
Similarly, y is in the range of f if and only if $-2y$ is in the range of g.
As $y > -3$ we see that $-2y < 6$, and as $y \leq 5$ we see that $-2y \geq -10$.
Hence the range of g is $[-10,6)$

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Question 5

The function $f:[0,1]\to\mathbb{R}$ has a range of $[0,5]$.

The following sequence of transformations maps the graph of f onto the graph of g:

- A dilation by a factor of 2 from the x-axis, followed by,
- A reflection in the y-axis, followed by,
- A translation of 3 units left and 1 unit up.

Find the domain and range of g.

We observe that under our transformation,

$$(x,y)\mapsto(x,2y)\mapsto(-x,2y)\mapsto(-x-3,2y+1)=(x',y')$$

We observe that x is in the domain of f if and only if $-x-3$ is in the domain of g, hence the domain of g is $[-4,-3]$.
Similarly, y is in the range of f if and only if $2y+1$ is in the range of g, hence the range of g is $[1,11]$

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Question 6

Consider the function, $f : [-2, 4) \to \mathbb{R}, f(x) = 3 - x^2$.

The function $g(x) = af\big(b(x + c)\big) + d$ has a domain of $(-1, 1]$ and a range of $[-1, 3)$.

Find the values of a, b, c and d.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation that maps the graph of f onto the graph of g. We observe that,

$$T(x, y) = \left(\frac{x}{b} - c, ay + d\right)$$

We use the fact that a transformation must map domain / range constraints with ">" or " < " onto constraints with " > " or " < ", and must map constraints with "≥" or "≤" onto constraints with "≥" or "≤".
Thus by plugging in the end points of the domain intervals that correspond to the same brackets in f and g we get,

$$\frac{-2}{b} - c = 1 \quad \text{and} \quad \frac{4}{b} - c = -1$$

Subtracting one equation from the other yields, $-\frac{6}{b} = 2 \implies b = -3$. Substituting this into any other equation yields, $c = -\frac{1}{3}$.
Now focusing on y, since the graph of f has a turning point at $x = 0$, we see that the range of f is $(-13, 3]$. Applying a similar logic to before we get,

$$-13a + d = 3 \quad 3a + d = -1$$

Subtracting one from the other yields $16a = -4 \implies a = -\frac{1}{4}$, and substituting this into any equation yields $d = -\frac{1}{4}$.

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Sub-Section [1.4.3]: Apply Transformations of Functions to Find Transformed Points and Tangents

Question 7

Find the image of the point $A(2,3)$ under the transformation:

$$T: \mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = \left(2x-3, -\frac{y}{2}+1\right)$$

The image of A under T is,

$$\left(2(2)-3, -\frac{3}{2}+1\right) = \left(1, -\frac{1}{2}\right)$$

Question 8

The equation of the tangent to the graph of $f(x)$ at the point $(-3,5)$ is $y = \frac{1}{3}x + 6$.

The transformation, $T(x,y) = (2x+1, -y)$ maps the graph of f onto the graph of g.

Find the equation of the tangent to the graph of g when x is equal to -5.

Observe that the image of $(-3,5)$ under T is $(-5,-5)$.

Thus the tangent to the graph of g at $x=-5$ is simply the image of the line $y = \frac{1}{3}x + 6$ under T.

Thus the equation of our tangent is,

$$y = -\left(\frac{1}{3}\left(\frac{x-1}{2}\right)+6\right) = \frac{-35-x}{6}$$

Question 9

The equation of the tangent to the graph of $f(x)$ when $x = 2$ is $y = -5x - 2$.

The following sequence of transformations maps the graph of f to the graph of g:

- A dilation by a factor of 2 from the y-axis, followed by,
- A dilation by a factor of 3 from the x-axis, followed by,
- A reflection in the y-axis, followed by,
- A translation of 3 units in the positive direction of the x-axis, followed by,
- A translation of 2 units in the negative direction of the y-axis.

a. Find a point, A on the graph of g.

We observe that under our transformation,

$$(x, y) \mapsto (-2x, 3y) \mapsto (-2x + 3, 3y - 2)$$

We know that the point $(2, -12)$ is on the graph of f, hence a point A on the graph of g is,

$$(-2(2) + 3, 3(-12) - 2) = (-1, -38)$$

b. Find the tangent to g at the point A.

We simply apply the transformation to our tangent to get,

$$y = 3\left(-5\left(\frac{x-3}{-2}\right) - 2\right) - 2 = \frac{15x - 61}{2}$$

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Sub-Section [1.4.4]: Find Transformations with Constraints

Question 10

Consider the transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the following sequence of transformations:

- A translation by a factor of a in the positive direction of the x-axis, followed by,
- A dilation by a factor of b from the y-axis.

T maps the graph of $f(x) = x^2 + 1$ onto the graph of $g(x) = 4(x-1)^2 + 1$. Find the values of a and b.

We observe that under T, $(x, y) \mapsto (b(x+a), y)$, thus,

$$g(x) = f\left(\frac{x}{b} - a\right) = \left(\frac{x}{b} - a\right)^2$$

Since we do not have a coefficient in front of our $\left(\frac{x}{b} - a\right)^2$ term, we will bring the 4 into the quadratic to get, $g(x) = (2x-2)^2 + 1$.

Thus we can now compare coefficients to see that $b = \frac{1}{2}$ and $a = 2$.

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Question 11

Describe a sequence of two translations, followed by a dilation and a reflection that maps the graph of $y = x^2$ onto the graph of $y = 3(10 - 5x)^2 + 4$.

Since we are first applying translations before dilations and reflections, our transformation looks like $(x', y') = (a(x+b), c(y+d))$, where exactly one of a or c is negative and exactly one of a or c has a magnitude of 1.
Under this transformation we see that the graph of $y = x^2$ gets mapped to the graph of

$$y = c\left(\left(\frac{x}{a} - b\right)^2 + d\right).$$

If we were to compare coefficients right now, the magnitude of both a and c would not be 1, hence we will take a factor of 5 out of the quadratic to get,

$$y = 75(2 - x)^2 + 4$$

We can now compare coefficients to get $a = -1$, $b = -2$, $c = 75$ and $d = \frac{4}{75}$.
From here we see that our transformations are,

- A translation of 2 units left and $\frac{4}{75}$ units up, followed by,
- A dilation by a factor of 75 from the x-axis, followed by,
- A reflection in the y-axis.

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Question 12

Let $f(x) = 2(x+1)^2$ and $g(x) = 2 - 3x$.

A transformation, $T(x, y) = (x + a, by + c)$ maps the graph of $f \circ g$ onto the graph of $g \circ f$.

Find the values of a, b and c.

Note that $(f \circ g)(x) = 2(3-3x)^2$ and $(g \circ f)(x) = 2 - 6(x+1)^2$.
Since we cannot apply a horizontal dilations / reflections we need to get $(f \circ g)(x)$ into the form,

$$(f \circ g)(x) = 2(3-3x)^2 = 18(x-1)^2$$

Now we observe that the rule for the image of the graph of $(f \circ g)$ under T is,

$$y = 18b(x-a-1)^2 + c$$

By comparing coefficients with the graph of $g \circ f$, $y = 2 - 6(x+1)^2$ we see that $a = -2$, $b = -\frac{1}{3}$ and $c = 2$.

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Sub-Section [1.4.5]: Find Transformations of the Inverse Functions

Question 13

Let $f : [0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$.
Describe a sequence of transformations that maps the graph of f onto the graph of g, where the inverse function of g is defined as such:

$$g^{-1}: [-1, \infty) \to \mathbb{R}, g(x) = 2(x+1)^2$$

By inverting g^{-1} we see that the function g is,

$$g : [0, \infty) \to \mathbb{R}, g(x) = \sqrt{\frac{x}{2}} - 1$$

Thus a sequence of transformations that map the graph of f onto the graph of g are,

- A dilation by a factor of 2 from the y-axis, followed by,
- A translation of one unit down.

Question 14

Consider the one-to-one functions, $f(x)$ and $g(x)$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (2x + 3, y + 5)$ maps the graph of f onto the graph of g.

Describe a sequence of transformations that maps the graph of f^{-1} onto the graph of g^{-1}.

To get a transformation that maps the graph of f^{-1} to the graph of g^{-1}, we simply swap x and y in the definition of T. Thus such a transformation could be,

$$S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (x + 5, 2y + 3)$$

This transformation can be described as,

- A dilation by a factor of 2 from the x-axis followed by,
- A translation of 5 units right and 3 units up.

Question 15

Consider the functions, $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x + 1$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 2^{2-6x} - 3$.

The transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, cy)$$

maps the graph of f^{-1} onto the graph of g^{-1}. Find the values of a, b, and c.

We observe that the transformation $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (cx, ay + b)$ will map the graph of f onto the graph of g.
Hence $g(x) = a(2^{\frac{x}{c}} + 1) + b$.
We can rearrange the rule of g to $g(x) = 4(2^{-6x}) - 3$, to be able to compare coefficients.
Hence $c = -\frac{1}{6}$, $a = 4$ and $b = -7$.

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Sub-Section [1.4.6]: Find Opposite Transformations

Question 16

Describe a sequence of transformations that maps the graph of $y = 2(x-3)^2 + 4$ onto the graph of $y = x^2$.

Like with transforming simple to complex functions, to transform complex to simple functions, we will first apply some algebra to the complex function to get the simple function.

Observe that $\frac{1}{2}[2((x+3)-3)^2+4]-2 = x^2$.

From here we can read off the transformations using the quick method.

- A dilation by a factor of $\frac{1}{2}$ from the x-axis, followed by,
- A translation of 2 units down and 3 units left.

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Question 17

The following sequence of transformations maps the graph of $y = f(x)$ onto the graph of $y = 2\log_e(3 - x) + 4$:

- A reflection in the y-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the x-axis, followed by,
- A translation of 3 units right and 4 units up.

Find the rule of f.

Under our transformation we see that,

$$(x, y) \mapsto \left(-x, \frac{1}{2}y\right) \mapsto \left(3 - x, \frac{1}{2}y + 4\right) = (x', y')$$

Thus for any pair (x', y') satisfying $y' = 2\log_e(3 - x') + 4$, we know that (x, y) satisfy $y = f(x)$.
Thus we substitute (x, y) in for (x', y') in the above equation to get,

$$\frac{1}{2}y + 4 = 2\log_e(3 - (3 - x)) + 4 \implies y = 4\log_e(x)$$

Hence the rule for f is $f(x) = 4\log_e(x)$.

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Question 18

Let $f : [2, \infty) \to \mathbb{R}, f(x) = 4x^2 - 16x - 1$ and $g : (-\infty, 0] \to \mathbb{R}, g(x) = x^2$.

The transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$$

maps the graph of f onto the graph of g. Find the values of a, b, and c.

We first complete the square for f to get an easier expression to work with. We will also bring in the 4 coefficient into the $(x-2)^2$ term as we do not have any vertical dilations to work with. Thus,

$$f(x) = 4(x-2)^2 - 17 = (2x-4)^2 - 17$$

From here we see that the rule for the image of the graph of f under T will be,

$$y = \left(2\left(\frac{x-b}{a}\right) - 4\right)^2 - 17 + c$$

Since elements in the domain of f go to infinity, and elements in the domain of g go to negative infinity, we require a reflection in the y-axis, hence a is negative. By comparing coefficients we see that $a = -2$ and $c = 17$.

Lastly as $2\frac{x-b}{-2} - 4 = -x$ we see that $b - 4 = 0 \implies b = 4$.

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Sub-Section: Exam 1 Questions

Question 19

a. A translation T maps the graph of $y = x\cos(x)$ onto the graph of $y = (\pi - x)\cos(x)$, where,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + a, y)$$

And a is a real constant.

State the value of a.

Observe that the image of $y = x\cos(x)$ under T is $y = (x - a)\cos(x - a)$.
Since we want to change $(x - a)$ to $(a - x)$ we use the fact that $\cos(x + \pi) = -\cos(x)$, thus,

$$(x - a)\cos(x - a) = (a - x)\cos(x + \pi - a)$$

From here we see that $a = \pi$

b. The equation of the tangent to the graph of $y = x\cos(x)$ when $x = 0$ is $y = x$.

Find the equation of the tangent to the graph of $y = (x - \pi)\cos(x)$ when $x = \pi$.

To transform the graph of $y = x\cos(x)$ onto the graph of $y = (x - \pi)\cos(x)$ we can apply the transformation, $(x, y) \mapsto (x + \pi, -y)$.
This transformation also maps the point $(0, 0)$ onto the point $(\pi, 0)$.
Hence to get our tangent we simply apply the transformation $(x, y) \mapsto (x + \pi, -y)$ onto the line $y = x$.
This yields the equation,

$$y = -(x - \pi) = \pi - x$$

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Question 20

Let $f : (-\infty, -2) \to \mathbb{R}, f(x) = \frac{1}{2x+4}$ and $g : (-\infty, 0) \to \mathbb{R}, g(x) = \frac{1-4x}{2x}$.

a. Show that $f(g(x)) = x$.

$$f(g(x)) = \frac{1}{2\frac{1-4x}{2x} + 4} = \frac{1}{\frac{1}{x} - 4 + 4} = \frac{1}{\frac{1}{x}} = x$$

b. Describe a sequence of **translations** that maps the graph of f onto the graph of g.

We observe that $f(x) = \frac{1}{2(x+2)}$ and $g(x) = \frac{1}{2x} - 2$.
As $g(x) = f(x-2) - 2$ our sequence of translations is,

- A translation of 2 units right, followed by,
- A translation of 2 units down.

c. Let $k : (-\infty, -1) \to \mathbb{R}, k(x) = f(2x)$.

Describe a transformation that maps the graph of g onto the graph of k^{-1}, the inverse function of k.

We observe that f is the inverse function of g. Thus we apply the same transformations to map f onto k as we do to map g onto k^{-1}, but swap x and y.
Thus our required transformation is a dilation by a factor of $\frac{1}{2}$ from the x-axis.

d. Let $h: (-2, \infty) \to \mathbb{R}, f(x) = \frac{1}{2x+4}$ have the same rule as f but with a different domain.

Describe a sequence of transformations that maps the graph of g onto the graph of h.

We observe that $-g(-x) = \frac{1}{2x} + 2$.
However since we applied a reflection in the y-axis, the domain of $-g(-x)$ is now $(0, \infty)$. We want the domain to become $(-2, \infty)$. Thus our transformations to map $g(x)$ to $h(x)$ are,

- A reflection in the x and y-axis, followed by,
- A translation of 2 units left and 2 units down.

Question 21

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 9$.

a. Find the co-ordinates of the axis intercepts of f.

$(0, -9), (3, 0)$ and $(-3, 0)$.

b. Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

- Dilation by a factor of $\frac{1}{3}$ from the y-axis.
- Dilation by a factor of 2 from the x-axis.
- Translation by two units to the right.

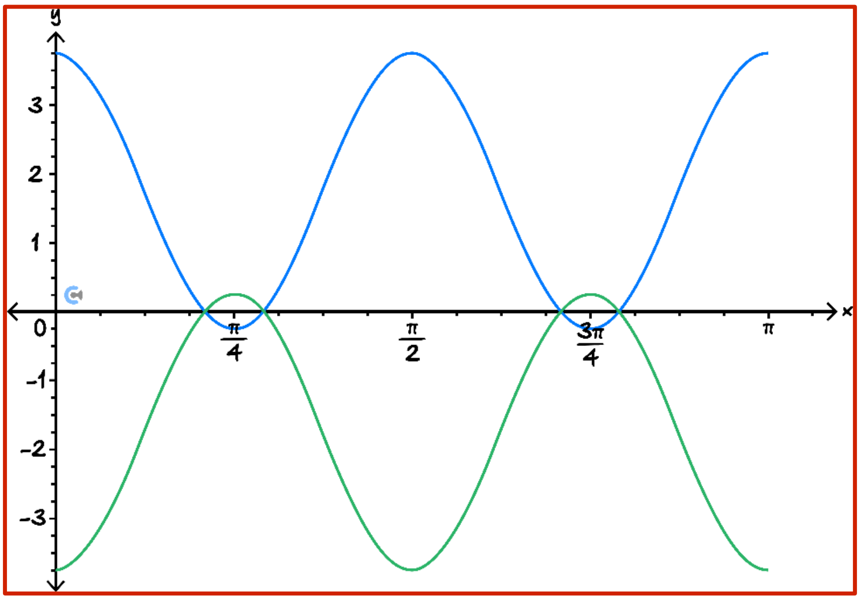
State the co-ordinates of the axis intercepts of h.

Observe that under our transformations, $(x, y) \mapsto \left(\frac{x}{3} + 2, 2y\right)$.
Since we have no vertical translations, the x-axis intercepts for h are simply the image of the x-axis intercepts of f.
Thus the co-ordinates of the x-axis intercepts are $(3, 0)$ and $(1, 0)$.
For the y-axis intercept we note that, $h(x) = 2f(3(x-2))$, thus $h(0) = 2f(-6) = 2(36-9) = 54$. Hence the co-ordinates of the y-axis intercept is $(0, 54)$.

Question 22

The graph of $y = f(x)$, where $f : [0, \pi] \to \mathbb{R}, f(x) = 2\cos(4x) + \sqrt{3}$ is shown below.

a. On the axes below, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of f in the horizontal axis.



b. Let $h : D \to \mathbb{R}, h(x) = 2\cos(4x) + \sqrt{3}$, where $h(x)$ has the same rule as $f(x)$ with a different domain.

The graph of $y = h(x)$ is translated a units in the positive horizontal direction and b units in the negative vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.

i. Find the value for b.

Observe that

$$g(x) = -f(x) = -2\cos(4x) - \sqrt{3} = 2\cos(4x - \pi) + \sqrt{3} - 2\sqrt{3} = h\left(x - \frac{\pi}{4}\right) - 2\sqrt{3}.$$

From here we read that $b = 2\sqrt{3}$

ii. Find the smallest positive value for a.

From the working above we see that $a = \frac{\pi}{4}$.

iii. Hence, or otherwise, state the domain, D, of $h(x)$.

$$D = \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

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Question 23

Describe a sequence of transformations that maps the graph of $y = 3\sin(2x)$ onto the graph of $y = \cos(x)$.

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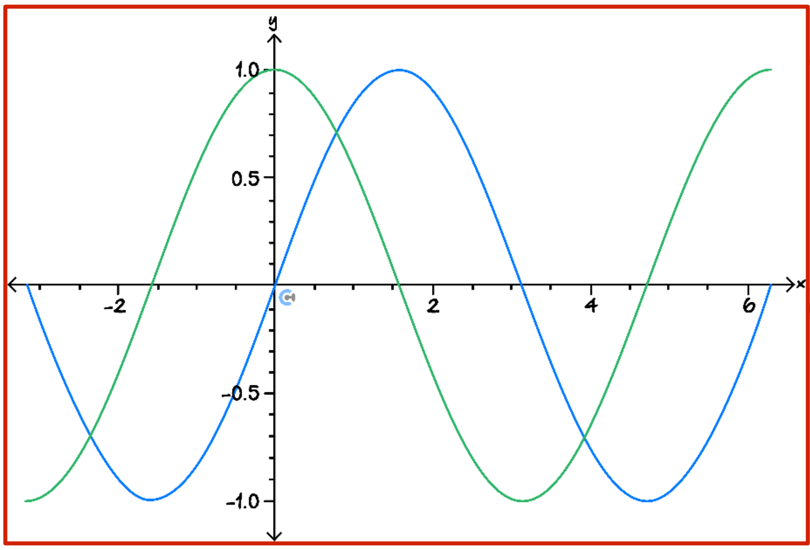
We observe from the graphs of $\sin(x)$ (in blue) and $\cos(x)$ (in orange), that $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$.



A dilation by factor $\frac{1}{3}$ from the x-axis followed by a dilation by factor 2 from the y-axis maps $y = 3\sin(2x)$ to $y = \sin(x)$, then we just need to translate $\frac{\pi}{2}$ units left. Thus, the sequence of transformations is:

- A dilation by a factor of $\frac{1}{3}$ from the x-axis, followed by,
- A dilation by a factor of 2 from the y-axis, followed by,
- A translation of $\frac{\pi}{2}$ units to the left.

Sub-Section: Exam 2 Questions

Question 24

The point $A(1, 5)$ lies on the graph of the function f. A transformation maps the graph of f to the graph of g, where $g(x) = 2f(3 - x) + 2$. The same transformation maps the point A to the point B.

The coordinates of the point P are:

A. $f(x) = (2, 12)$

B. $f(x) = (4, 12)$

C. $f(x) = (2, 8)$

D. $f(x) = (4, 8)$

Question 25

The point $A(u, v)$ is transformed by $T(x, y) = \left(3x - 1, -\frac{1}{5}y + 2\right)$.

If the image of A is $(1, 1)$, then A is:

A. $\left(2, \frac{9}{5}\right)$

B. $(2, 5)$

C. $\left(\frac{2}{3}, 5\right)$

D. $\left(\frac{2}{3}, \frac{9}{5}\right)$

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Question 26

The sequence of transformations that maps the graph of $y = e^x$ onto the graph of $y = e^{3x+6}$ is:

A. A translation of 6 units right followed by a dilation by a factor of 3 from the y-axis.

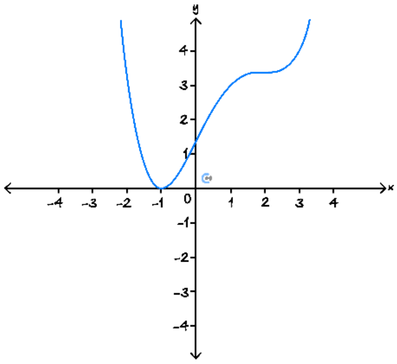
B. A translation of 6 units left followed by a dilation by a factor of $\frac{1}{3}$ from the y-axis.

C. A translation of 2 units left followed by a dilation by a factor of 3 from the y-axis.

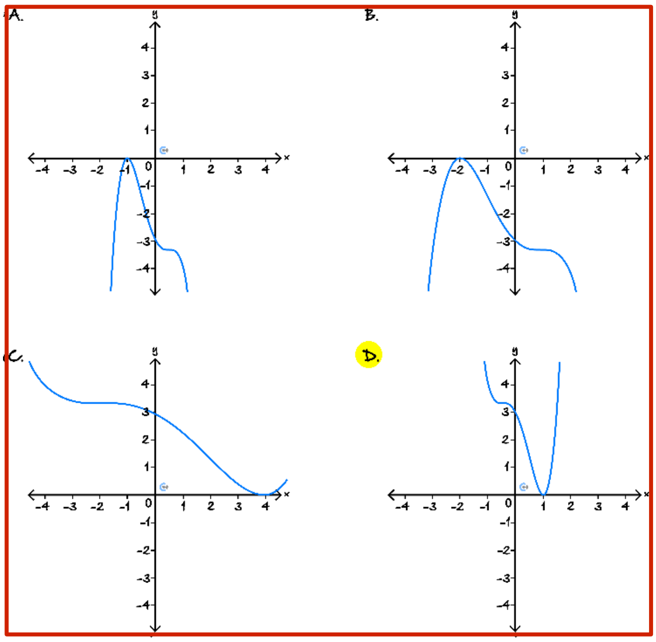
D. A dilation by a factor of $\frac{1}{3}$ from the y-axis followed by a translation of 6 units right.

Question 27

The graph of $y = f(x)$ is shown below.



The corresponding part of the graph of the inverse function $f(1 - 2x)$ is best represented by:



Question 28

The line $y = -\frac{1}{3}x + 5$ is tangent to the graph of f when $x = 3$.

The following sequences of transformations map the graph of f onto the graph of g:

1. A dilation by a factor of 2 from the y-axis, followed by,

2. A translation of 3 units in the negative direction of the y-axis.

Which of the following statements is true?

A. The line $y = -\frac{1}{6}x + 2$ is tangent to g at the point $(3, 4)$.

B. The line $y = -\frac{2}{3}x + 2$ is tangent to g at the point $(6, 1)$.

C. The line $y = -\frac{1}{6}x + 2$ is tangent to g at the point $(6, 1)$.

D. The line $y = -\frac{2}{3}x + 2$ is tangent to g at the point $\left(\frac{3}{2}, 1\right)$.

Question 29

Consider the functions,

$$f : \mathbb{R} \to \mathbb{R}, f(x) = 4x^3 - 3x^2 - 6x + 5$$

$$g : \mathbb{R} \to \mathbb{R}, g(x) = (4x - 5)(x + 1)^2$$

a. Find the co-ordinates of the axial intercepts of f.

We solve $f(x) = 0$ to get the co-ordinates of the x-axis intercepts of $\left(-\frac{5}{4}, 0\right)$ and $(1, 0)$.
We evaluate $f(0)$ to get the do-ordinates of the y-axis intercepts of $(0, 5)$.

b.

i. Hence, or otherwise, describe a sequence of **reflections and dilations**, T that maps the graph of f onto the graph of g.

We realise that both the x-axis and y- axis intercepts of f are equal to the negative of the x and y-axis intercepts of g. Hence our transformation is,

- A reflection in the x-axis, followed by.
- A reflection in the y-axis.

ii. Describe a sequence of **translations**, S that maps the graph of f onto the graph of g.

We observe that the concave turning point of f is at the point $\left(-\frac{1}{2}, \frac{27}{4}\right)$, whilst the concave turning point of g is at $(-1, 0)$. Since these functions have the "same scale" it is sufficient to translate that the turning point of f onto the turning point of g. Thus our sequence of translations is,

- A translation of $\frac{1}{2}$ units left, followed by,
- A translation of $\frac{27}{4}$ units down.

c. The image of a point $P(u, v)$ under both S and T is the same.

Find the values of u and v.

We note that $T(u, v) = (-u, -v)$ and $S(u, v) = \left(u - \frac{1}{2}, v - \frac{27}{4}\right)$.

Thus we solve $-u = u - \frac{1}{2}$ and $-v = v - \frac{27}{4}$ simultaneously to get $P(u, v) = \left(\frac{1}{4}, \frac{27}{8}\right)$

d. Show that $h(x) = f(x) + g(x)$ has the property that $h(-x) = -h(x)$.

We observe that $g(-x) = -f(x)$ and $f(-x) = -g(x)$, thus,

$$h(-x) = f(-x) + g(-x) = -g(x) - f(x) = -h(x)$$

Question 30

Consider the function, $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x$.

a. The transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + 3, 2y)$$

maps the graph of f onto the graph of $g : \mathbb{R} \to \mathbb{R}, g(x) = af(x)$. Find the value of a.

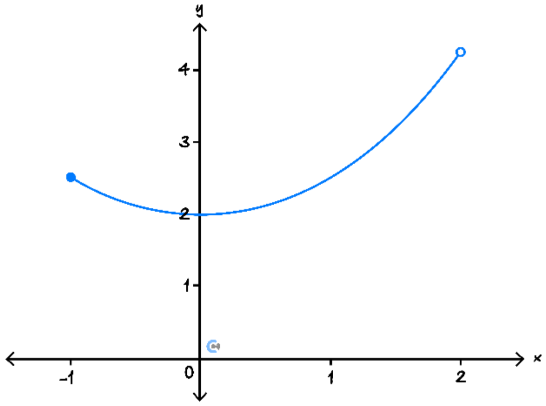
$$g(x) = 2 \times 2^{x-3} = \frac{1}{4} \times 2^x.$$

Hence $a = \frac{1}{4}$.

b. Hence, describe another transformation that maps the graph of f onto the graph of g.

A dilation by a factor of $\frac{1}{4}$ from the x-axis.

c. Let $h : [-1, 2) \to \mathbb{R}$, $h(x) = f(x) + f(-x)$. The graph of h is drawn below.



i. State the range of h.

From the graph of h we see that it's range is $[2, 2^2 + 2^{-2}) = \left[2, \frac{17}{4}\right)$

ii. Hence, or otherwise, state the domain and range of the image of the graph of h under T.

Domain $= [-1 + 3, 2 + 3) = [2, 5)$.

Range $= \left[2 \times 2, 2 \times \frac{17}{4}\right) = \left[4, \frac{17}{2}\right)$

iii. Describe a sequence of possible transformations that maps the graph h onto a graph with a domain of $[-1, 2)$ and a range of $\left(2, \frac{17}{4}\right]$.

We do not need to touch the domain. Since the brackets on the range are inverted we will first reflect and then translate up.

- A reflection in the x-axis, followed by,
- A translation of $\frac{25}{4}$ units upwards.

d. The equation of the tangent to the graph of f when $x = a$ is $y = 2^a\left(1 + \log_e(2)(x - a)\right)$.

i. Find the equation of the tangent to the graph of h when $x = a$.

The tangent to the graph of $f(-x)$ at $x = a$, is the reflection of the line $y = 2^{-a}(1 + \log_e(2)(x + a))$ in the y-axis.
Hence the tangent to the graph of $f(-x)$ at $x = a$ is

$$y = 2^{-a}(1 + \log_e(2)(a - x))$$

Now to get the tangent to the graph of h at $x = a$ we simply add the two tangents of f and $f(-x)$ at $x = a$ to get,

$$y = 2^{-a}(1 + \log_e(2)(a - x)) + 2^a(1 + \log_e(2)(x - a))$$

ii. Let $k(x) = h(4 - x)$.

Find the equation of the tangent to the graph of k when $x = a$.

$$\begin{aligned} y &= 2^{-4+a}(1 + \log_e(2)(4 - a - 4 + x)) + 2^{4-a}(1 + \log_e(2)(4 - x - 4 + a)) \\ &= 2^{-4+a}(1 + \log_e(2)(x - a)) + 2^{4-a}(1 + \log_e(2)(a - x)) \end{aligned}$$

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Section B: Supplementary

Sub-Section [1.4.1]: Apply Quick Method to Find Transformations

Question 31

Find the image of the graph of $y = x^2$ under the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (1 - 2x, y + 5)$.

Apply the transformation $x \mapsto 1 - 2x$ in an opposite manner, so we replace x with $\frac{x-1}{2}$ Thus (applying the y-axis transformations as well) we get,

$$y = \left(\frac{x-1}{2}\right)^2 + 5$$

Question 32

Describe a sequence of transformations that maps the graph of $y = x^3$ onto the graph of $y = 2(3x + 2)^3 - 3$.

In our equation we replace x with $3x + 2$, thus we apply those transformations in reverse including the order.

- A translation of 2 units left, followed by,
- A dilation by a factor of $\frac{1}{3}$ from the y-axis, followed by,

Then we apply the y-axis transformations as normal.

- A dilation by a factor of 2 from the x-axis, followed by,
- A translation of 3 units down.

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Question 33

Find the image of the graph of $y = 2\log_2(x) - 3$ under the following sequence of transformations:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units left and 3 units up, followed by,
- A reflection in the y-axis, followed by,
- A dilation by a factor of 5 from the y-axis.

We observe that the last 3 transformations apply to x, thus applying them in reverse (including the order) yields,

$$x \to \frac{1}{5}x \to -\frac{1}{5}x \to -\frac{1}{5}x + 2$$

Applying the y-axis transformations in order yields,

$$y \to 3y + 3$$

Thus, the rule for the image of our graph under the transformations is,

$$y = 3(2\log_2\left(-\frac{1}{5}x + 2\right) - 3) + 3 = 6\log_2\left(-\frac{1}{5}x + 2\right) - 6$$

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Question 34

Consider four linear functions, $p_1(x)$, $p_2(x)$, $q_1(x)$ and $q_2(x)$.

A transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x', y')$$

maps the graph of $y = f(x)$ onto the graph of $y = (p_1 \circ p_2 \circ f \circ q_2 \circ q_1)(x)$. Express x' in terms of x and y' in terms of y.

By the quick method we apply the reverse of the x-axis transformations in the reverse order, thus $x' = (q_1^{-1} \circ q_2^{-1})(x)$.
We apply the y-axis transformations in the correct order, this yields $y' = (p_1 \circ p_2)(y)$.

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Sub-Section [1.4.2]: Apply Transformations of Functions to Find its Domain and Range

Question 35

The function $f : \mathbb{R} \to \mathbb{R}$ has a range of $[2, \infty)$.

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (5 - 2x, 3 + y)$ maps the graph of f onto the graph of g. State the domain and range of g.

We simply apply T to both our domain and range.
As x is a real number $5 - 2x$ can be any real number.
As $y \geq 2$, we know that $y + 3 \geq 5$.
From here we see that the domain of g is R and the range of g is $[5, \infty)$.

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Question 36

The function $f : (-\infty, -1] \to \mathbb{R}$ has a range of $[-2, \infty)$.

Describe a sequence of transformations that maps the graph of f onto a graph of a function with a domain of $[0, \infty)$ and a range of $(-\infty, 2]$.

Since our domain and ranges both swap the signs of the ∞, we require reflections about both axes.

- A reflection about the x-axis, followed by,
- A reflection about the y-axis.

After applying these transformations, we have a domain of $[1, \infty)$ and a range of $(-\infty, 2]$.

- We just need a translation to fix the domain.
- A translation of 1 unit to the left

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Question 37

Consider the function, $f : \mathbb{R}\backslash\{-2\} \to \mathbb{R}, f(x) = \frac{3}{(x+2)^2} - 5$.

The following sequence of transformations maps the graph of f onto the graph of g:

- A reflection in the x-axis, followed by,
- A dilation by a factor of 3 from the x-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the y-axis, followed by,
- A translation of 3 units up and 2 units left.

State the domain and range of g.

Recall that the domain of f is $\mathbb{R}\backslash\{-2\}$ and the range of f is $(-5, \infty)$.
Under our transformations,

$$(x, y) \mapsto (x, -y) \mapsto \left(\frac{1}{2}x, -3y\right) \mapsto \left(\frac{1}{2}x - 2, 3 - 3y\right)$$

Now we just apply these transformations to our domain and range.
If $x \neq -2$, then $\frac{1}{2}x - 2 \neq -3$ and if $y > -5$, then $3 - 3y < 18$.
Hence the domain of g is $\mathbb{R}\backslash\{-3\}$ and the range of g is $(-\infty, 18)$.

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Question 38

Let $f : (-2, 1] \to \mathbb{R}, f(x) = 2(x + 1)^2 - 3$.

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, cy + d)$ where a and c are both non-zero.

The transformation T maps the graph of f onto the graph of g.

a. Explain why the range of g will always be of the form $[p, q]$ for some real $p < q$.

The range of f is $[-3, 5]$.
Let $y' = cy + d$. We note that y' is in the range of g if and only if y is in the range of f.
As we know that $-3 \leq y \leq 5$, we see that $-3c + d \leq y' \leq 5c + d$ if $c > 0$ or, $-3c + d \geq y' \geq 5c + d$ if $c < 0$.
As $c \neq 0$, in both cases these restrictions create an interval with square brackets.

b. Explain why the domain of g will always be of the form $(p, q]$ or $[p, q)$ for some real $p < q$.

The domain of f is $(-2,1]$.
Let $x' = ax + b$. We note that x' is in the domain of g if and only if x is in the domain of f.
As we know that $-2 < x \leq 1$, we see that, $-2a + b < x \leq a + b$ if $a > 0$ or, $-2a + b > x \geq a + b$ if $a < 0$.
The first restriction produces a range of the form $(p, q]$ whilst the second produces a range of the form $(q, p]$

c. For what values of a is the domain of g of the form $(p, q]$.

$a > 0$.

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Sub-Section [1.4.3]: Apply Transformations of Functions to Find Transformed Points and Tangents

Question 39

The equation of the tangent to the graph of $f(x)$ at the point $(1, 3)$ is $y = 2x + 1$.

The transformation, $T(x, y) = \left(x, \frac{y}{3} + 1\right)$ maps the graph of f onto the graph of g.

Find the equation of the tangent to the graph of g at the point $(1, 2)$.

As the image of the point (1,3) under T is (1,2), we simply apply T to our tangent line. Thus our tangent to the graph of g at (1,2) is,

$$y = \frac{1}{3}(2x + 1) + 1 = \frac{2x + 4}{3}$$

Question 40

The points $(2, 4)$ and $(4, 7)$ lie on the graph of $f(x)$.

Evaluate $g(2)$, where $g(x) = 3f(6 - x) + 5$.

$$g(2) = 3f(6 - 2) + 5 = 3f(4) + 5.$$

As the point (4,7) lies on the graph of $y = f(x)$, we see that $f(4) = 7$, hence, $g(2) = 21 + 5 = 26$.

Question 41

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ described by the following sequence of transformations:

- A dilation by a factor of 2 from the x-axis, followed by,
- A translation by a factor of 4 in the negative direction of the x-axis, followed by,
- A dilation by a factor of $\frac{1}{3}$ from the y-axis, followed by,
- A translation by a factor of 5 in the positive direction of the y-axis.

The image of $A(u, v)$ under T is $(3, 7)$. Find the values of u and v.

Under T we observe that,
$(x, y) \to (x, 2y) \to (x - 4, 2y) \to x - 4$
$3, 2y \to x - 4$
Applying this transformation to A yields,

$$T(A) = \left(\frac{u-4}{3}, 2v + 5\right) = (3,7).$$

Thus $u = 13$ and $v = 1$

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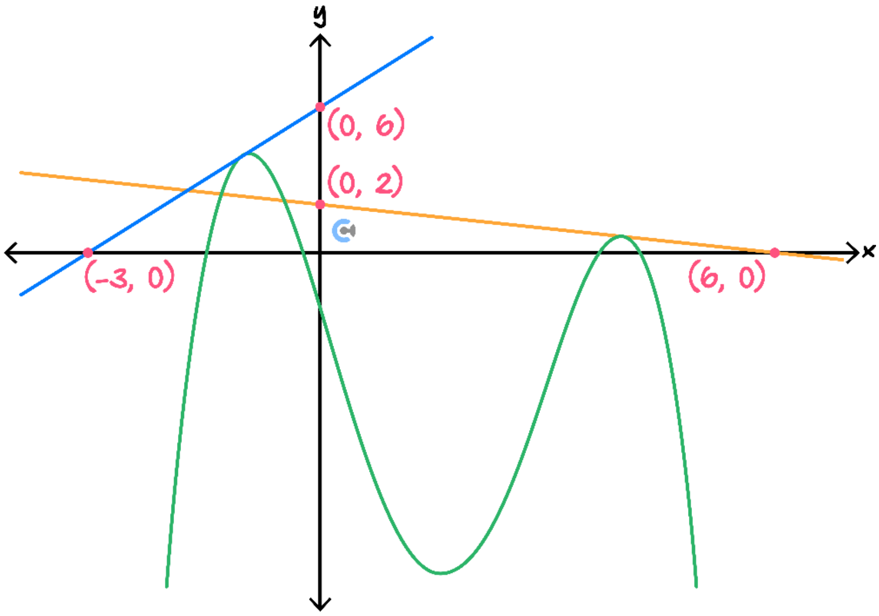
Question 42

The graph of $y = f(x)$ is drawn below along with two tangents at $x = 4$ and at $x = -1$.



Find the equation of the tangent to the graph of $g(x) = 1 - 3f(2 - 2x)$ when $x = -1$.

A possible transformation that maps the graph of f onto the graph of g is,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(1 - \frac{1}{2}x, 1 - 3y\right)$$

Thus, the pre-image of $(-1, g(-1))$ under T is $(4, f(4))$, thus T maps the tangent to f at $x = 4$ onto the tangent to g at $x = -1$.

This tangent has the equation, $y = 2 - \frac{1}{3}x$.

Applying T to this tangent yields the equation, $y = 1 - 3\left(2 - \frac{2-2x}{3}\right) = -3 - 2x$

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Sub-Section [1.4.4]: Find Transformations with Constraints

Question 43

Consider the transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the following sequence of transformations:

- A dilation by a factor of a from the x-axis.
- A translation by a factor of b in the positive direction of the y-axis.

T maps the graph of $f(x) = \sqrt{x}$ onto the graph of $g(x) = \sqrt{9x} + 6$.

Find the values of a and b.

Under T we see that $(x, y) \to (x, ay + b)$.
Thus, the image of the graph of f under T has a rule of, $y = a\sqrt{x} + b = g(x)$.
We take the 9 out of the square root in the rule of g to get $g(x) = 3\sqrt{x} + 6$.
Now we can compare coefficients to get $a = 3$ and $b = 6$

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Question 44

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$ maps the graph of $y = 2^x$ onto the graph of $y = 8 \times 2^{3x-1} - 5$.

Find the values of a, b and c.

The rule for the image of the graph of $y = 2x$ under T is,

$$y = 2^{\frac{x-b}{a}} + c = 8 \times 23x - 1 - 5$$

As we do not have a dilation from the x-axis, we take the 8 into the exponential to get

$$y = 2^{3x+2} - 5.$$

From here we can compare coefficients to get $c = -5$, $a = \frac{1}{3}$ and $b = -6$

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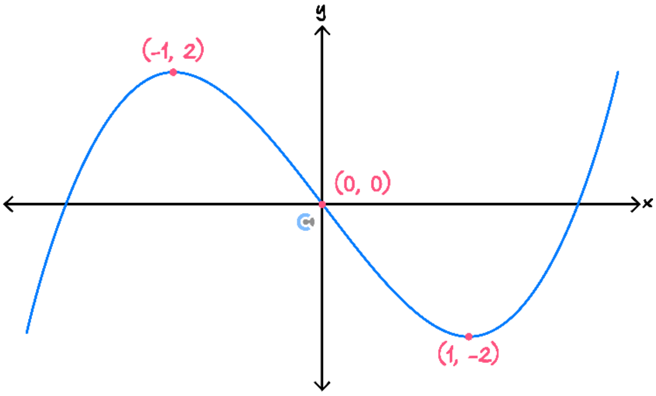
Question 45

The graph of $y = x^3 - 3x$ is drawn below.



The transformation,

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(x, y) = (a - x, b - y)$$

maps the graph of $y = x^3 - 3x$ onto the graph of $y = (x - 1)^3 - 3x + 5$.

Find the values of a and b.

The image of the graph of $y = x^3 - 3x$ under T is,
$y = b - (a - x)^3 + 3(a - x) = b + (x - a)^3 - 3x + 3a$
Thus $a = 1$ and $b + 3a = 5 \Longrightarrow b = 2$.

Question 46

Consider the functions,

$$f : [-1, \infty) \to \mathbb{R}, f(x) = x^2 + 2x + 2$$

$$g : (-\infty, 1] \to \mathbb{R}, g(x) = 4(2x - 1)^2 + 3$$

Describe a sequence of a dilation followed by two translations and lastly a reflection that maps the graph of f onto the graph of g.

Looking at the domain of f and g, our reflection must be in the y-axis.
We now complete the square for the rule of f to get $f(x) = (x+1)^2 + 1$.
Since we have 1 dilation, we can bring the 4 into the square in the rule of g to get $g(x) = (4x - 2)2 + 3$, and from here we see that we need to apply,

- A dilation by a factor of $\frac{1}{4}$ from the y-axis.

This maps the rule of f onto the rule of $y = (4x + 1)^2 + 1$. As our last transformation will be a reflection in the y-axis, we need to use our two translations to map the graph of $y = (4x+1)^2 + 1$ onto the graph of $y = (4x+2)^2 + 3$.
Hence our two translations are,

- A translation of $\frac{1}{4}$ units left, followed by,
- A translation of 2 units up.

Lastly, we apply our reflection in the y-axis.

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Sub-Section [1.4.5]: Find Transformations of the Inverse Functions

Question 47

Consider the function, $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}, f(x) = \frac{2}{x-1} + 4$.

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + a, y + b)$ maps the graph of f onto the graph of its inverse function. Find the values of a and b.

The horizontal asymptote of f is $y = 4$, whilst the horizontal asymptote of f^{-1} is $y = 1$.
Thus, we need to translate the graph of f 3 units down, i.e. $b = -3$.
The vertical asymptote of f is $x = 1$, whilst the vertical asymptote of f^{-1} is $x = 4$.
Thus we need to translate the graph of f 3 units to the right, i.e. $a = 3$.

Question 48

Consider the one-to-one functions, $f(x)$ and $g(x)$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (3 - x, 2y + 7)$ maps the graph of f onto the graph of g.

Describe a sequence of transformations that maps the graph of f^{-1} onto the graph of g^{-1}.

We can swap x and y in the equation of T to get a transformation $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (2x + 7, 3 - y)$ that maps the graph of f^{-1} onto the graph of g^{-1}.
From here we can read off a sequence of transformations.

- A dilation by a factor of 2 from the y-axis, followed by,
- A reflection in the x-axis, followed by,
- A translation of 7 units to the right and 3 units up

Question 49

Let $f : (-\infty, 2] \to \mathbb{R}, f(x) = 3x^2 - 12x + 11$ and $g : [-3, \infty) \to \mathbb{R}, g(x) = 2\sqrt{x+3} + 4$.

a. Describe a sequence of transformations that maps the graph of f onto the graph of g^{-1}.

We first find the rule for g^{-1} by solving $g(y) = x$ for y. Thus,

$$2\sqrt{y+3} + 4 = x \implies \frac{x-4}{2} = \sqrt{y+3} \implies y = \frac{(x-4)^2}{4} - 3$$

Furthermore the domain of g^{-1} is the range of g which is $[4, \infty)$. Similarly, the range of g^{-1} is $[-3, \infty)$.
From here we see that our transformation needs,

- A reflection in the y-axis

to align our domains (Now the image of f after this reflection has a domain of $[-2, \infty)$, which a simple translation can map to the domain of g^{-1}).
The rule for the image of the graph of f after applying that reflection is

$$y = 3x^2 + 12x + 11 = 3(x+2)^2 - 1.$$

Now we apply the following transformations to map the graph of f onto the graph of g^{-1}.

- A dilation by a factor of $\frac{1}{12}$ from the x-axis, followed by,
- A translation of 6 units to the right and 2 units up

b. Hence, or otherwise, describe a sequence of transformations that maps the graph of g onto the graph of f^{-1}.

To map the graph of f^{-1} onto the graph of g we would just swap x and y in the transformation in **part a.** However, we are mapping the graph of g onto the graph of f^{-1}, thus we also need to swap our transformations and their order. Thus, our transformations are,

- *A* translation of 6 units down and 2 units to the left, followed by,
- *A* dilation by a factor of 12 from the y-axis, followed by,
- *A* reflection in the x-axis.

Question 50

Consider the function f which has the property that $f(x-3)-3 = f^{-1}(x)$.

The transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = (4x+1, 2-y)$ maps the graph of f onto the graph of g.

Describe a sequence of basic transformations (translations, dilations and reflections in the x and y-axis only) that maps the graph of g onto the graph of g^{-1}.

We will approach this problem by mapping the graph of g onto the graph of f, then onto the graph of f^{-1}, and finally onto the graph of g^{-1}.

Observe that the transformation $S: \mathbb{R}^2 \to \mathbb{R}^2, S(x,y) = \left(\frac{x-1}{4}, 2-y\right)$ undoes T, hence maps the graph of g onto the graph of f.

Then we apply the transformation $R: \mathbb{R}^2 \to \mathbb{R}^2, R(x,y) = (x+3, y-3)$ to map the graph of f onto the graph of f^{-1}.

We can swap x and y in the rule for T to create a transformation $Q: \mathbb{R}^2 \to \mathbb{R}^2, Q(x,y) = (2-x, 4y+1)$ that maps the graph of f^{-1} onto the graph of g^{-1}.

We compose these 3 transformations to create a transformation

$$\begin{aligned} U: \mathbb{R}^2 \to \mathbb{R}^2, U(x,y) &= Q(R(S(x,y))) \\ &= Q\left(R\left(\frac{x-1}{4}, 2-y\right)\right) \\ &= Q\left(\frac{x-1}{4}+3, 2-y-3\right) \\ &= \left(2-\frac{x-1}{4}-3, 4(-y-1)+1\right) = \left(-\frac{x}{4}-\frac{3}{4}, -4y-3\right) \end{aligned}$$

Hence our transformation Q can be described with the following sequence of transformations,

- A reflection in both the x-axis and the y-axis, followed by,
- A dilation by a factor of $\frac{1}{4}$ from the y-axis and a dilation by a factor of 4 from the x-axis, followed by,
- A translation of $\frac{3}{4}$ units left and 3 units down.

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Sub-Section [1.4.6]: Find Opposite Transformations

Question 51

Describe a sequence of transformations that maps the graph of $y = 3e^{2x+1} - 4$ onto the graph of $y = e^x$.

Observe that $\frac{1}{3}(3e^{2x+1} - 4) + \frac{4}{3} = e^{2x+1}$.
Thus we can undo the "y" transformations with,

- A dilation by a factor of $\frac{1}{3}$ from the x-axis, followed by,
- A translation of $\frac{4}{3}$ units up.

Since $2x + 1 = x'$ we can undo the "x" transformations with,

- A dilation by a factor of 2 from the y-axis, followed by,
- A translation of 1 unit right.

Question 52

The transformation, $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(2x + 3, \frac{1}{3}y - 4\right)$ maps the graph of $y = f(x)$ onto the graph of $y = x^3$.

Find the rule of f.

Choose a point, (x, y) on the graph of $y = f(x)$. Let (x', y') be the image of that point under T.
We can substitute $x' = 2x + 3$ and $y' = \frac{1}{3}y - 4$ into the equation $y' = (x')^3$ to get,

$$\frac{1}{3}y - 4 = (2x + 3)^3 \implies y = f(x) = 3(2x + 3)^3 + 12$$

Question 53

The following sequence of transformations maps the graph of f onto the graph of $y = \sqrt{x}$, for $x \in (2, \infty)$:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units left and 4 units up, followed by,
- A reflection in both the x-axis and the y-axis.

State the rule and domain of f.

Under our transformation we see that,

$$(x, y) \mapsto (x, 3y) \mapsto (x-2, 3y+4) \mapsto (2-x, -3y-4)$$

Choose a point (x, y) on the graph of x, and let $(x', y') = (2-x, -3y-4)$.
We see that $y' = \sqrt{x'}$, thus substituting the above values into this equation yields the rule for $f(x)$, specifically,

$$-3y - 4 = \sqrt{2-x} \implies y = f(x) = -\frac{\sqrt{2-x}}{3} - \frac{4}{3}.$$

Now choose some x in the domain of f. Then $2 - x = x'$ is in the domain of the image of f under T, hence $2 - x > 2 \implies x < 0$.
Hence the domain of f is $(-\infty, 0)$

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Question 54

Describe a transformation different from $(x, y) \mapsto (x, y)$, that maps the graph of $y = a(x-k)^5 + b(x-k)^3 + h$ onto itself.

We first map our graph onto the graph of $y = ax^5 + bx^3$. Then the transformation $(x, y) \mapsto (-x, -y)$ will map the graph of $y = ax^5 + bx^3$ onto itself, after which we can undo our first transformation to get back to our original graph.
The transformation to map the graph of $y = a(x-k)^5 + b(x-k)^3 + h$ onto the graph of $y = ax^5 + bx^3$ is $(x, y) \mapsto (x-k, y-h)$, and we can undo that transformation with the transformation, $(x, y) \mapsto (x+k, y+h)$. Now we combine these 3 transformations to get,

$$(x, y) \mapsto (x-k, y-h) \mapsto (k-x, h-y) \mapsto (2k-x, 2h-y)$$

Hence $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (2k-x, 2h-y)$ is our desired transformation.

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Sub-Section: Exam 1 Questions

Question 55

The following sequence of transformations maps the graph of $y = f(x)$ onto the graph of $y = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x\right)$:

- A translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ in from the y-axis, followed by,
- A dilation by a factor of 2 from the x-axis.

Find the rule of f.

Under our transformations,

$$(x, y) \mapsto \left(x + \frac{\pi}{6}, y\right) \mapsto \left(\frac{1}{2}x + \frac{\pi}{12}, 2y\right) = (x', y')$$

If a point (x, y) sits on the graph of $y = f(x)$, then (x', y') sits on the graph of $y' = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x'\right)$.
We simply substitute x and y into this equation to get

$$2y = \frac{1}{2}\cos\left(\frac{\pi}{3} - x - \frac{\pi}{6}\right) \implies y = \frac{1}{4}\cos\left(\frac{\pi}{6} - x\right)$$

Hence $f(x) = \frac{1}{4}\cos\left(\frac{\pi}{6} - x\right)$.

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Question 56

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 2 - \frac{1}{2}x^3$, and let $g : \mathbb{R} \to \mathbb{R}, g(x) = 6 - 2x$.

a.

i. Find $(g \circ f)(x)$.

$$(g \circ f)(x) = 6 - (4 - x^3) = 2 + x^3.$$

ii. Find $(f \circ g)(x)$ and express it in the form $k + m(x - h)^3$, where m, k and h are integers.

$$(f \circ g)(x) = 2 - \frac{1}{2}(6 - 2x)^3 = 2 - \frac{1}{2} \times (-2)^3 \times (x - 3)^3 = 2 + 4(x - 3)^3$$

b. The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + b, ay + c)$, where a, b and c are integers, maps the graph of $y = (f \circ g)(x)$ onto the graph of $y = (g \circ f)(x)$.
Find the values of a, b and c.

Looking at the x-transformations we need to turn $(x-3)^3$ into x^3, hence we will map $x \mapsto x + 3$.
Looking at the y-transformations, we observe that $\frac{1}{4}(2 + 4x^3) + \frac{3}{2} = 2 + x^3$, thus we must map $y \mapsto \frac{1}{4}y + \frac{3}{2}$.
Hence $b = 3, a = \frac{1}{4}$ and $c = \frac{3}{2}$.

Question 57

Let $f : [1, \infty) \to \mathbb{R}, f(x) = 4(x - 1)^2 - 3$ and let $g : [2, \infty) \to \mathbb{R}, g(x) = 1 - \sqrt{x - 2}$.

a. Let g^{-1} be the inverse function of g.

i. State the domain and range of g^{-1}.

Dom g^{-1} = Ran $g = (-\infty, 1]$.
Ran g^{-1} = Dom $g = (-\infty, 2]$.

ii. Find the rule of g^{-1}.

We solve $g(y) = x$ for y, thus,

$$\begin{aligned} x &= 1 - \sqrt{y - 2} \\ \implies (1 - x)^2 &= y - 2 \\ \implies y &= 2 + (x - 1)^2 \end{aligned}$$

b. The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$ maps the graph of f onto the graph of g^{-1}.

Find the values of a, b and c.

Due to the domain of g we know that we need a reflection in the y-axis, hence $a < 0$. The rule for the image of the graph of f under T is,

$$y = 4\left(\frac{x-b}{a} - 1\right)^2 - 3 + c$$

Since $-3 + c = 2$ we see that $c = 5$.
After bringing the 4 into the quadratic we see that $a = -2$, thus we require that,

$$2\left(\frac{x-b}{-2} - 1\right) = b - x - 2 = 1 - x \implies b = 3$$

Question 58

Let $f : \mathbb{R}\backslash\{a\} \to \mathbb{R}, f(x) = \frac{1}{x-a} + b$.

a. Find the rule and domain for the graph of f^{-1} in terms of a and b.

We solve $f(y) = x$ for y, thus

$$x = \frac{1}{y-a} + b \implies y - a = \frac{1}{x-b} \implies y = \frac{1}{x-b} + a$$

Hence the rule for f^{-1} is $f^{-1}(x) = \frac{1}{x-b} + a$, and the domain for f^{-1} is the range of f which is $\mathbb{R}\backslash\{b\}$.

b. The following sequence of transformations maps the graph of f to the graph of f^{-1}:

- A translation of 4 units in the positive direction of the x-axis, followed by,
- A translation of 4 units in the negative direction of the y-axis.

Find the value of a in terms of b.

Under those transformations we know that $(x, y) \mapsto (x+4, y-4)$, hence the image of the graph of f under that transformation is,

$$y = \frac{1}{x-4-a} + b - 4$$

Since this is equal to $\frac{1}{x-b} + a$ we see that $4 + a = b$ and $b - 4 = a$.
Both of these conditions imply that $a = b - 4$.

c. Let $g(x) = \frac{1}{x-c} + d$. A transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + h, y + k)$$

maps the graph of g onto the graph of g^{-1}.

What restrictions are there on the values of h and k?

Under T the rule for the image of the graph of g is,

$$y = \frac{1}{x-h-c} + d + k$$

Since this is equal to $g^{-1}(x) = \frac{1}{x-d} + c$, we see that $h + c = d$ and $d + k = c$.
As $h = d - c$ and $k = c - d$ we see that $h = -k$.

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Sub-Section: Exam 2 Questions

Question 59

The graph of the function f passes through the point $(2, -3)$.

If $h(x) = 3f(x - 2)$, then the graph of the function h must pass through the point:

A. $(0, -1)$

B. $(4, -9)$

C. $(0, -9)$

D. $(4, -1)$

Question 60

The graph of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x - 1$, is reflected in the y-axis and then translated 2 units to the left and then 3 units up.

Which one of the following is the rule of the transformed graph?

A. $y = 2^{2-x} + 2$

B. $y = 2^{2+x} + 2$

C. $y = \left(\frac{1}{2}\right)^{-2-x} + 2$

D. $y = \frac{1}{4}\left(\frac{1}{2}\right)^{x} + 2$

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Question 61

The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, which maps the graph of $y = 4 - \log_e\left(\frac{x-1}{2}\right)$ onto the graph of $y = \log_e(x)$, has the rule:

A. $T(x, y) = \left(\frac{x-1}{2}, 4 - y\right)$

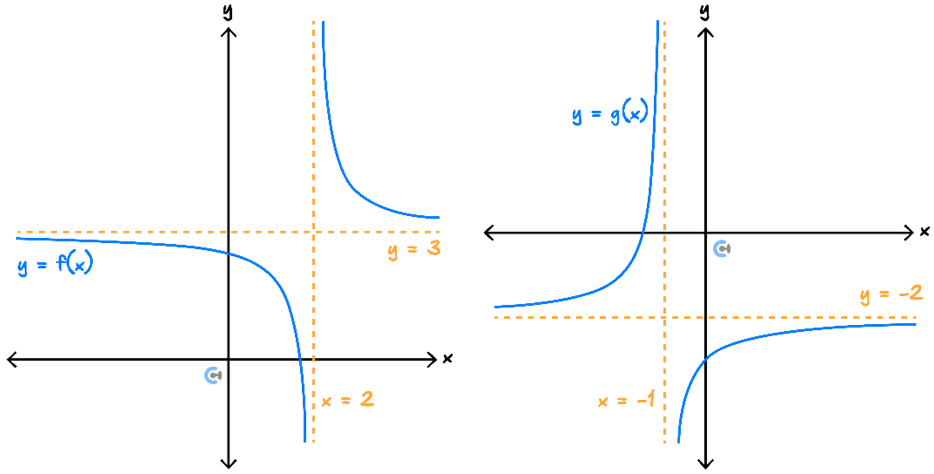
B. $T(x, y) = (2x + 1, -y - 4)$

C. $T(x, y) = (2x + 1, 4 - y)$

D. $T(x, y) = \left(\frac{x-1}{2}, -y - 4\right)$

Question 62

Consider the graph of f and g below, which have the same scale,



If T transforms the graph of f onto the graph of g, then:

A. $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (1 - x, y - 5)$

B. $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x - 3, y - 5)$

C. $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x - 3, 5 - y)$

D. $T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (1 - x, 2 - y)$

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Question 63

The graph of the function g is obtained from the graph of the function $f : [-2, 3] \to \mathbb{R}, f(x) = 2x^2 - 4x + 5$, by a dilation of factor 2 from the y-axis, followed by a dilation of factor $\frac{1}{3}$, from the x-axis, followed by a reflection in the y-axis, and finally, followed by a translation of 1 unit in the negative direction of the y-axis.

The domain and range of g are respectively:

A. $[-6,4]$ and $\left[\frac{8}{3}, 6\right]$

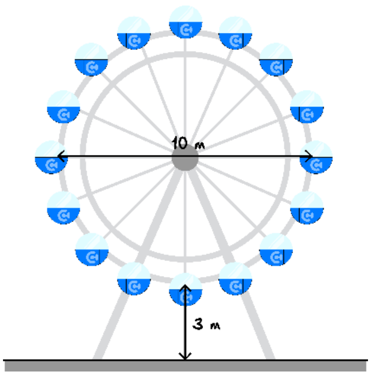
B. $\left[-1, \frac{2}{3}\right]$ and $[21,41]$

C. $[-6,4]$ and $\left[\frac{2}{3}, \frac{17}{3}\right]$

D. $[-6,4]$ and $[0,6]$

Question 64

The Contour Ferris Wheel pictured below takes 30 minutes to complete a trip.



Thus, the height of the bottom of a carriage t minutes after the start of a trip is given by,

$$h(t) = 8 - 5\cos\left(\frac{\pi t}{15}\right)$$

a. Describe a sequence of transformations that maps the graph of $\sin(t)$ onto the graph of h.

Observe that $\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$.

Thus we first translate our graph $\frac{\pi}{2}$ units left and dilate by a factor of $\frac{15}{\pi}$ from the y-axis.

This gives us $y = \cos\left(\frac{\pi t}{15}\right)$.

To get this into our desired form we now, simply reflect our graph in the t-axis, then dilate it by a factor of 5 from the t-axis and translate it 8 units up.

b. The horizontal displacement, d from the bottom of the carriage to the centre of the roller coaster t minutes after the start of a trip is,

$$d(t) = 5\sin\left(\frac{\pi t}{15}\right)$$

The transformation, $T(t, y) = (t + a, y + b)$ maps the graph of h onto the graph of d.

i. Find b.

$b = 8$

ii. Find a possible value of a.

We require $5\sin\left(\frac{\pi(t-a)}{15}\right) = -5\cos\left(\frac{\pi t}{15}\right)$

Since $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$ we simply need $-\frac{\pi a}{15} = -\frac{\pi}{2} \implies a = \frac{15}{2}$

c. 15 minutes into a trip on the Ferris Wheel, Caitlin crashes her car into the Ferris Wheel. This causes the Ferris Wheel to stop for 5 minutes before starting again at half speed.

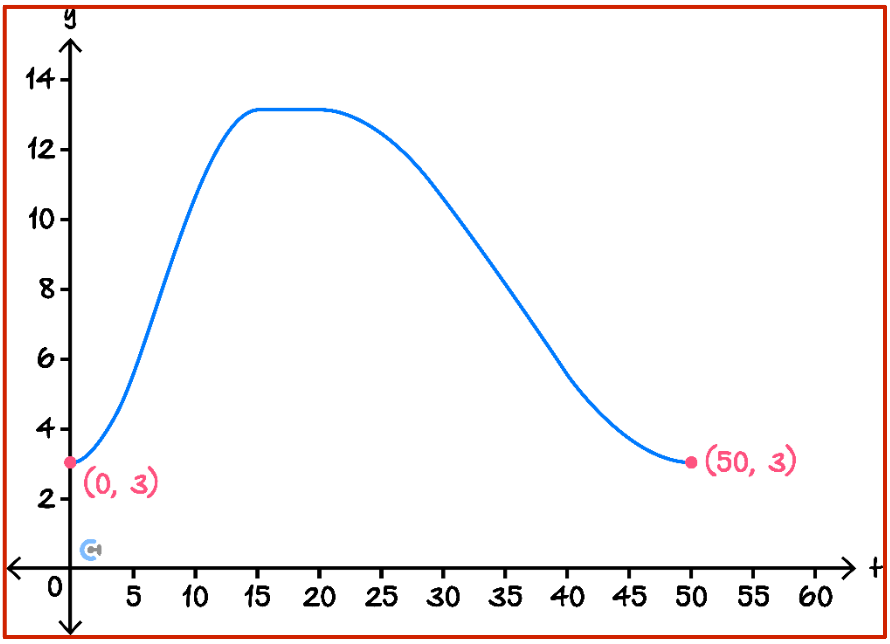
The height of the Ferris wheel in this trip, $h_1 : [0, r] \to \mathbb{R}$ is given by the following function:

$$h_1(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(pt + q) & 20 \leq t \leq r \end{cases}$$

Find a set of possible values of p, q, k and r.

We know that $k = h(15) = 8 - 5\cos(\pi) = 13$.
Since it would take 15 minutes to finish the trip before Caitlin crashed her car into the Ferris wheel, it will now take 30 minutes in double time.
Hence $r - 20 = 30 \implies r = 50$.
Since we are going at half speed, after the crash we see that $p = \frac{1}{2}$. Now we simply require that $h\left(\frac{1}{2} \times 20 + q\right) = 13 \implies 10 + q = 15 \implies q = 5$

d. Part of the graph of h_1 is drawn on the axis below. Draw the rest of the graph of h_1 labelling endpoints with their co-ordinates.



Question 65

Consider the function, $f:(-1,1) \to \mathbb{R}, f(x) = (2x-1)^2(x+1)$.

a. State the range of f.

From the graph of f we see that the range is $[0,2]$.

b. The following sequence of transformations, T, maps the graph of f onto the graph of g:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units down and 5 units left, followed by,
- A reflection in the y-axis.

i. State the rule of g.

Under T we see that $(x,y) \mapsto (x,3y) \mapsto (x-5,3y-2) \mapsto (5-x,3y-2) = (x',y')$. From the quick method, as $x = 5-x'$ we see that $g(x) = 3f(5-x)-2 = 3(x-6)(2x-9)^2-2$

ii. State the domain of g.

We apply the transformation $x \mapsto 5-x$ onto the interval $(-1,1)$ to get the domain of g.
Thus the domain of g is $(4,6)$.

iii. State the range of g.

We apply the transformation $y \mapsto 3y-2$ onto the interval $[0,2]$ to get the range of g.
Thus the range of g is $[-2,4]$

c. The tangent to the graph of f at the point $A\left(-\frac{1}{4}, \frac{27}{16}\right)$ is given by the equation,

$$y = \frac{9}{8} - \frac{9x}{4}.$$

i. Find B, the image of A under T.

$$B = \left(5 - \left(-\frac{1}{4}\right), 3\left(\frac{27}{16}\right) - 2\right) = \left(\frac{21}{4}, \frac{49}{16}\right)$$

ii. Find the equation of the tangent to the graph of g at point B.

We simply apply out transformation to the line to get,

$$y = 3\left(\frac{9}{8} - \frac{9(5-x)}{4}\right) - 2 = \frac{27x}{4} - \frac{259}{8}$$

d. A transformation, $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (-x, a - y)$ maps the graph of f onto itself.

i. State the value of a.

The rule for the image of the graph of f under S is $y = a - f(-x)$.
As this is meant to equal $f(x)$, we see that $a - f(0) = f(0) \implies a = 2f(0) = 2$

ii. Hence, or otherwise, describe a sequence of transformations in terms of S and T as required, that maps the graph of g to itself, but does not map A to itself.

Let $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ undo the transformation T. Specifically,

$$T^{-1}(x, y) = \left(5 - x, \frac{y+2}{3}\right)$$

To map the graph of g onto itself, we can first apply T^{-1} to map the graph of g onto the graph of f, then apply S to map the graph of f onto itself, and then apply T to map the graph of f onto the graph of g.

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