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VCE Mathematical Methods ¾
Transformations [1.3]
Workbook

Outline:

Learning Objectives:

- [ ] **MM34 [1.3.1]** – Applying x' and y' notation to find transformed points, find the interpretation of transformations and altered order of transformations.
- [ ] **MM34 [1.3.2]** – Find transformed functions.
- [ ] **MM34 [1.3.3]** – Find transformations from transformed function (Reverse Engineering).

Section A: Introduction To Transformations

Sub-Section: Image And Pre-Image

Context: Transformations

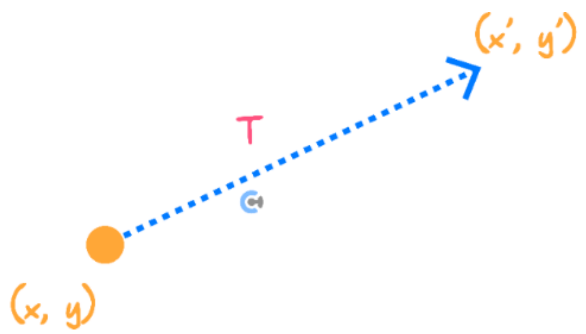
- Transformation is a super important topic in MM34.
- It sets the foundation for the hardest topic:

Topic: Family of Functions

- It is important to learn the foundations well for future questions!

What do we call an Original Coordinate and a Transformed Coordinate?

Image and Pre-Image



- The original coordinate is called the Pre Image.
- The transformed coordinate is called the Image.

Pre-Image: (x, y)

Image: (x', y')

Space for Personal Notes

Question 1

It is known that $(2,3)$ transformed into $(4,5)$. State the value of x' and y'.

$x' = 4$

$y' = 5$

NOTE: The x' and y' notation will be used quite heavily!

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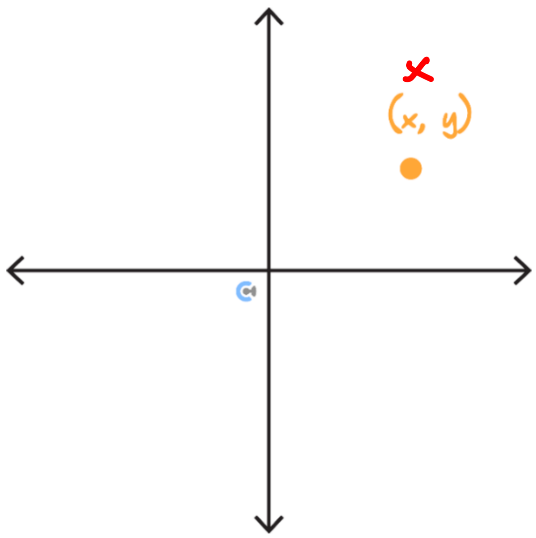
Sub-Section: Dilation

Let's do a quick revision of Dilation!

Exploration: Dilation

- Consider the point below:



- Let's plot the coordinates:
  - $P1$: Dilation by a factor 2 from the x-axis. $y' = 2y$
  - $P2$: Dilation by a factor $\frac{1}{2}$ from the x-axis. $y' = \frac{1}{2}y$
  - $P3$: Dilation by a factor 2 from the y-axis. $x' = 2x$
  - $P4$: Dilation by a factor $\frac{1}{2}$ from the y-axis. $x' = \frac{1}{2}x$

Dilation

Dilation by a factor a from the x-axis: $y' = ay$

Dilation by a factor b from the y-axis: $x' = bx$

Question 2

Find the image (x', y') after applying the following transformations to the point (x, y).

Dilation by a factor 3 from the x-axis.

Dilation by a factor $\frac{1}{5}$ from the y-axis.

$x' = \frac{1}{3}x$

$y' = 3y$

NOTE: We are applying the transformations on (x, y) not (x', y').

<u>**Misconception**</u>

"Shouldn't we do the opposite to x?"

TRUTH: Transformation applies the same for x and y.

How it is represented on the function is different, however. (More on this later!)

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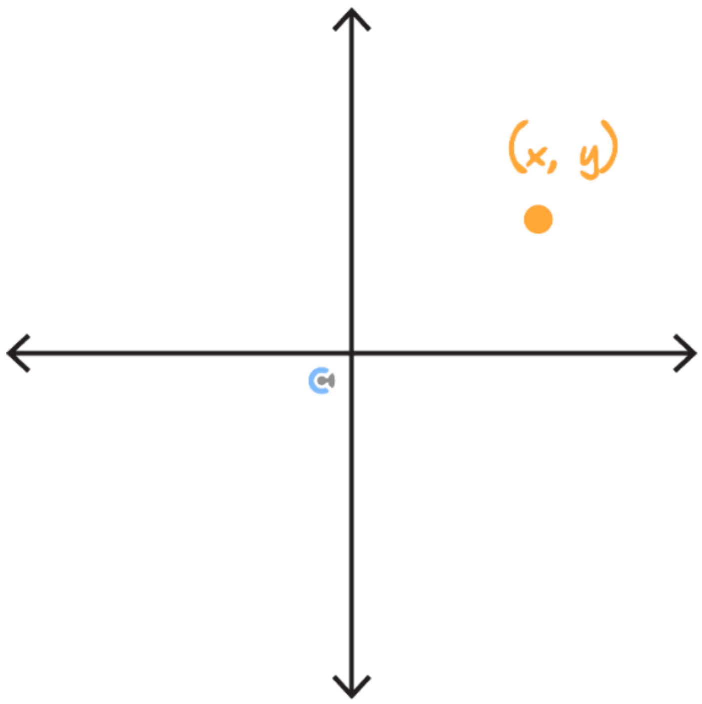
Sub-Section: Reflection

Let's do a quick revision of Reflection!

Exploration: Reflection

- Consider the point below:



- Let's plot the coordinates:
  - $P1$: Reflection in the x-axis. $y=0$ $y' = -y$
  - $P2$: Reflection in the y-axis. $x' = -x$

Reflection

Reflection in the x-axis: $y' = -y$

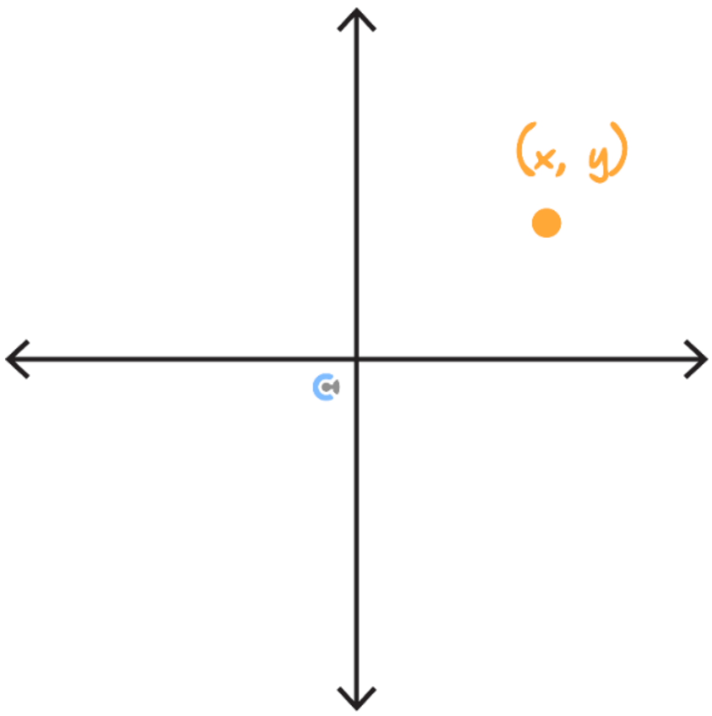
Reflection in the y-axis: $x' = -x$

Sub-Section: Translation

Let's do a quick revision of Translation!

Exploration: Translation

- Consider the point below:



- Let's plot the coordinates (ignore the scale):
  - $P1$: Translation by 2 units in the negative direction of the x-axis. $x' = x - 2$
  - $P2$: Translation by 3 units in the negative direction of the y-axis. $y' = y - 3$

Translation

Translation by c units in the positive direction of the x-axis: $x' = x + c$

Translation by d units in the positive direction of the y-axis: $y' = y + d$

Question 3

Find the image (x', y') after applying the following transformations to (x, y).

Translation by 2 units in the positive direction of the x-axis.

Translation by 5 units in the negative direction of the y-axis.

Key Takeaways

- ☑ The transformed point is called the image and is denoted by (x', y').
- ☑ The dilation factor is multiplied by the original coordinates.
- ☑ Reflection makes the original coordinates the negative of their original values.
- ☑ Translation adds a unit to the original coordinates.

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Section B: Transformation Of Points

Sub-Section: Basic Transformation Of Points

Let's try to apply all types of transformations to a point!

Question 4 Walkthrough.

Find the image (x', y') after applying the following transformations to (x, y).

Dilation by a factor 2 from the x-axis.

Dilation by a factor 3 from the y-axis.

Reflection in the x-axis.

Translation by 3 units in the negative direction of the x-axis.

Translation by 1 unit in the positive direction of the y-axis.

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$x' = 3x - 3.$

$y' = -2y + 1.$

Question 5

Find the image (x', y') after applying the following transformations to (x, y).

Translation by 2 units in the positive direction of the x-axis.

Translation by 1 unit in the negative direction of the y-axis.

Dilation by a factor 4 from the x-axis.

Dilation by a factor $\frac{1}{3}$ from the y-axis.

Reflection in the x-axis.

$x' = \frac{1}{3}(x+2)$

$y' = -4(y-1)$

Question 6 Extension.

Find the image (x', y') after applying the following transformations to (x, y).

Translation by a units in the negative direction of the x-axis.

Translation by b units in the positive direction of the y-axis.

Dilation by a factor c from the x-axis.

Dilation by a factor $\frac{2}{d}$ from the y-axis.

Reflection in the x-axis.

NOTE: Order is important!

- Apply the next transformation on top of everything that has already been done!

Sub-Section: The Order Of Transformations

Discussion: From the previous question, what happens when the translation is applied first?

What is the Order of Transformations the same as?

The Order of Transformation

Order = BODMAS Order

Question 7 Walkthrough.

Consider the point (x, y) was transformed into a point $(2x + 8, y)$ by the transformation T.

Jennifer thinks the transformation was:

> "~~A translation by 8 units in the positive direction~~ of the x-axis, followed by a dilation by a factor 2 from the y-axis."

$2(x+8)$ $2x+16$

Meanwhile, David thinks the transformation was:

> "A dilation by a factor 2 from the y-axis, followed by a translation by 8 units in the positive direction of the x-axis."

Who is correct? And why?

$x' = 2x+8$: Expanded : DRT

Question 8

Consider the point (x, y) was transformed into a point $(3(x-1), y)$ by the transformation T.

Mary thinks the transformation was:

"A translation by 1 unit in the negative direction of the x-axis, followed by a dilation by a factor 3 from the y-axis."

Meanwhile, Sam thinks the transformation was:

"A dilation by a factor 3 from the y-axis, followed by a translation of 1 unit in the negative direction of the x-axis."

Who is correct? And why?

$x' = 3(x-1)$

1 down

Dil 3 fm y

Question 9 Extension.

Consider the point (x, y) was transformed into a point $(2ax + 4a, y)$ by the transformation T.

Jennifer thinks the transformation was: $2a(x+2) = 2ax + 4a$

"A translation by 2 units in the positive direction of the x-axis, followed by a dilation by a factor $2a$ from the y-axis."

Meanwhile, David thinks the transformation was:

"A dilation by a factor $2a$ from the y-axis, followed by a translation by $2a$ units in the positive direction of the x-axis."

Who is correct? And why? Jen.

<u>Discussion:</u> If the order is the same as the BODMAS order, how do we change the order of transformations?

Expand = DRT

Factor = Translate First

Question 10 Walkthrough.

The series of transformations, "a dilation by a factor $\frac{1}{3}$ from the x-axis and a translation by 2 units up" yields the same result as the series of transformations, "a translation by a units up and a dilation by a factor b from the x-axis." Find the values of a and b.

+

$y' = \frac{1}{3}y + 2.$

$y' = \frac{1}{3}(y + 6).$

$6 = a.$

$b = \frac{1}{3}$

Question 11

The series of transformations, "a dilation by a factor 3 from the y-axis, a reflection in the y-axis and a translation by 6 units left" yields the same result as the series of transformations, "a translation by c units right, a reflection in the y-axis and a dilation by a factor d from the y-axis." Find the values of c and d.

$x' = -3x - 6$

$x' = -3(x + 2).$

$c = 2$
$d = 3.$

Question 12 Extension.

The series of transformations, "a dilation by a factor 2 from the y-axis, a reflection in the y-axis, a dilation by a factor 3 from the x-axis, a translation by 6 units left and a translation by 3 units down", yields the same result as the series of transformations, "a translation by c units right, a reflection in the y-axis, a dilation by a factor d from the y-axis, a translation k units down, and a dilation by a factor m from the x-axis." Find the values of c, d, k and m.

$x' = -2x - 6 = -2(x+3)$

$y' = 3y - 3 = 3(y-1)$

$c = 3$
$d = 2$

$k = 1$
$m = 3$

NOTE: Dilation factors don't change!

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Sub-Section: Interpreting The Transformation Of Points

Active Recall: Order of Transformation

Order = BODMAS Order

Question 13

Consider the transformation which maps:

$$x' = 3x - 6$$

$$y' = -2(y - 2)$$

a. State the transformations in DRT (Dilation, Reflection, Translation) order.

Expanded.

$x' = \cancel{3}x - 6$

$y' = \cancel{-2}y + 4.$

Dil 3 fr y

Dil 2 fr x

Reflect in x

6 left

4 up

b. State the transformations in the translation in first order.

$x' = 3(x - 2)$

$y' = -2(y - 2)$

2 left

2 down

Dil 3 fr y

Dil 2 fr x

Ref in x

NOTE: Expanding or factorising changes the order of transformation.

Discussion: Could the order of x and y transformations change?

Analogy: James' Weight

- James says,

"I was 2 kg lighter last year!"

- Did James gain or lose weight over the year?

Fatter.

- Similarly consider:

old x. New x

$$x = x' - 2$$

$x' = x + 2$

- Did x increase or decrease by 2?

Increased by 2 .= 2 Right.

- It is more evident if we make x' the subject.

$$x' = x + 2$$

Interpretation of Transformations

- When the new variables x' and y' are the subject, we can read the transformation directly.

$$x' = x + 5 \rightarrow \text{5 right}$$

- When the old variables x and y are the subjects instead, we must read the transformation in the opposite way.
- This includes the order of transformation!

$$x = x' - 5 \rightarrow \text{5 right}$$

NOTE: This includes the order of transformation!

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Discussion: Can you all see where this is going?

Question 14 Walkthrough.

Consider the transformation which maps:

$y - 4 = 2y'$

$\frac{1}{2}y - 2 = y'$

$$x = \frac{1}{3}x' + 2$$

$$y = 2y' + 4$$

$x - 2 = \frac{1}{3}x'$

$3x - 6 = x'$

a. State the transformations in DRT (Dilation, Reflection, Translation) order.

Dil 3 fm y
Dil $\frac{1}{2}$ fm x

6 left.
2 down

b. State the transformations in the translation in first order.

$y' = \frac{1}{2}(y-4)$, $x' = 3(x-2)$

2 left
4 down
Dil 3 fm y
Dil $\frac{1}{2}$ fm x

TIP: It is best to make x' and y' the subject before you interpret the transformations.

Question 15

Consider the transformation which maps:

$$x = -2x' - 5 \longrightarrow x + 5 = -2x'$$

$$-\frac{1}{2}x - \frac{5}{2} = x'$$

$$y = \frac{3}{2}y' + 1$$

$$y - 1 = \frac{3}{2}y'$$

$$\frac{2}{3}y - \frac{2}{3} = y'$$

a. State the transformations in DRT (Dilation, Reflection, Translation) order.

Dil $\frac{2}{3}$ fm x

Dil $\frac{1}{2}$ fm y

Ref in y

$\frac{2}{3}$ down

$\frac{5}{2}$ left

b. State the transformations in the translation in first order.

$y' = \frac{2}{3}(y-1).$ $\quad x' = -\frac{1}{2}(x+5)$

1 down

5 right

Dil $\frac{2}{3}$ fm x

Dil $\frac{1}{2}$ fm y

Reflect in y.

Space for Personal Notes

Question 16 Extension.

Consider the transformation which maps:

$$x = ax' + b$$

$$y = -c(y' - d)$$

a. State the transformations in DRT (Dilation, Reflection, Translation) order.

$\frac{1}{a}(x-b) = x'$ $\quad -\frac{1}{c}y = y' - d$

$\frac{1}{a}x - \frac{b}{a} = x'$. $\quad y' = -\frac{1}{c}y + d$

Dil $\frac{1}{a}$ from y

Dil $\frac{1}{c}$ from x

Reflection in x

$\frac{b}{a}$ left

d up

b. State the transformations in the translation in first order.

$\frac{1}{a}(x-b) = x'$. $\quad -\frac{1}{c}(y-cd) = y'$

b left

cd down

Reflection in x

Dil $\frac{1}{a}$ from y

Dil $\frac{1}{c}$ from x

Key Takeaways

- Transformations should be interpreted when x' and y' are isolated.
- The order of transformation follows the BODMAS order.
- To change the order of transformations, we either factorise or expand.

Section C: Transformation Of Functions

Sub-Section: Applying Transformations To Functions

Let's now work with Functions!

Transformation of Functions

- The aim is to get rid of the old variables, x and y, and have the new variables, x' and y', instead.

$$y = f(x) \rightarrow y' = f(x')$$

- Steps:
  1. Transform the points.
  2. Make x and y the subjects.
  3. Substitute them into the function.

Question 17 Walkthrough.

Apply the following transformations to $y = x^3$.

Reflection in the y-axis.

Translation by 3 units to the right.

Dilation by a factor 3 from the y-axis.

1) $x' = 3(-x+3)$

2) $\frac{x'}{3} = -x+3$

$\frac{x'}{3} - 3 = -x$

$-\frac{x'}{3}+3 = x$

3) $y = x^3$ ← old variables

$y = \left(-\frac{x'}{3}+3\right)^3$

$\boxed{y = \left(-\frac{x}{3}+3\right)^3}$

Your turn!

Question 18

Apply the following transformations to the functions below:

a. $f(x) = x^3$.

1) $x' = 3(-x+3)$

$y' = 2y$

9

Dilation by a factor 2 from the x-axis.

Reflection in the y-axis.

Translation by 3 units to the right.

Dilation by a factor 3 from the y-axis.

2) $\frac{x'}{3} = -x+3$, $\frac{y'}{2} = y$

$x = -\frac{x'}{3} + 3$,

3) $y = x^3$

$\frac{y'}{2} = \left(-\frac{x'}{3} + 3\right)^3$

$y = 2\left(-\frac{x'}{3} + 3\right)^3$

b. $f(x) = \log_e(x)$.

Dilation by a factor 2 from the y-axis.

Dilation by a factor $\frac{1}{3}$ from the x-axis.

Translation by 5 units to the left.

Translation by 3 units up.

Reflection in the y-axis.

1) $x' = -(2x-5)$

$y' = \frac{1}{3}y + 3$

2) $-x' = 2x-5$

$-x'+5 = 2x$

$\frac{1}{2}(-x'+5) = x$

$y'-3 = \frac{1}{3}y$

$3(y'-3) = y$

3) $y = \log_e(x)$

$3(y'-3) = \log_e\left(\frac{1}{2}(-x'+5)\right)$

$y'-3 = \frac{1}{3}\log_e\left(\frac{1}{2}(-x'+5)\right)$

$y = \frac{1}{3}\log_e\left(\frac{1}{2}(-x+5)\right) + 3$

Question 19 Extension.

Apply the following transformations to $y = 2^x$.

Translation by a units to the right.

Reflection in the y-axis.

Dilation by a factor 3 from the y-axis.

Translation by d units up.

A dilation by a factor 2 from the x-axis.

A reflection in the x-axis.

① $x' = -3(x+a)$

$y' = -2(y+d)$

② $-\frac{x'}{3} = x+a \qquad \frac{-y'}{2} = y+d$

$x = -\frac{x'}{3} - a \qquad y = -\frac{y'}{2} - d$

③ $y = 2^x$

$-\frac{y'}{2} - d = 2^{-\frac{x'}{3} - a}$

$-\frac{y'}{2} = 2^{-\frac{x'}{3} - a} + d$

$y = -2 \cdot 2^{-\frac{x}{3} - a} - 2d$

<u>Active Recall:</u> Interpretation of Transformations

- When the new variables x' and y' are the subject, we can read the transformation directly.

$$x' = x + 5 \rightarrow 5 \text{ right}$$

- When the original variables x and y are the subject instead, we must read the transformation in the opposite way.
- This includes the order of transformation!

$$x = x' - 5 \rightarrow 5 \text{ right}$$

<u>Discussion:</u> Which form is the transformation of x stuck in? $x = \cdots$ or $x' = \cdots$. Hence, would the transformation of x be represented as it is, or in the opposite way?

<u>Calculator Tip:</u> Finding Transformed Functions

- Save the function as $f(x)$.
- Substitute the x and y in terms of x' and y'.
- Solve for y!

Question 20 Tech-Active.

Apply the following transformations to $y = 2\sin(3x) + 10$.

Dilation by a factor 2 from the x-axis.

Dilation by a factor $\frac{1}{4}$ from the y-axis.

Reflection in the y-axis.

Translation of 2 units right.

Translation of 5 units down.

$x' = -\frac{1}{4}x + 2$

$x' - 2 = -\frac{1}{4}x$

$-4(x'-2) = x$

1) define $f(x) = 2\sin(3x) + 10$

2) Transform $y = f(x)$

3) Type this up

$y = \boxed{2f(-4(x-2)) - 5}$

$15 - 4\sin(12x - 24)$

Sub-Section: Finding The Applied Transformations

Now let's go backwards!

Reverse Engineering

- Steps:
  1. Add the dashes (') back to the transformed function.
  2. Make $f(\ )$ the subject.
  3. Equate the LHS of the original and transformed functions to the RHS of the original and transformed functions.
  4. Make x' and y' the subjects and interpret the transformations.

Question 21 Walkthrough.

Find the transformations required for $y = x^2$ to be transformed to $y = 2\left(\frac{x-3}{2}\right)^2 - 1$.

1) $y = x^2$

$y' = 2\left(\frac{x'-3}{2}\right)^2 - 1$

2) $y = x^2$

$\frac{y'+1}{2} = \left(\frac{x'-3}{2}\right)^2$

3) $y = \frac{y'+1}{2}$, $x = \frac{x'-3}{2}$.

4) $2y = y'+1$, $2x = x'-3$

$y' = 2y-1$. $2x+3 = x'$

Dil 2 from x

Dil 2 fm y

1 down

3 right

Your turn!

Question 22

State a series of transformations (in order) that allow $f(x)$ to be transformed into $g(x)$.

a. $f(x) = 3e^{2x-1} + 2$ and $g(x) = e^{\frac{1}{3}x+1} + 1$.

1) $y = 3e^{2x-1} + 2$

$y' = e^{\frac{1}{3}x'+1} + 1$

2) $\frac{y-2}{3} = e^{2x-1}$

$y'-1 = e^{\frac{1}{3}x'+1}$

3) $\frac{y-2}{3} = y'-1$. $\quad 2x-1 = \frac{1}{3}x'+1$

4) $\frac{y}{3} - \frac{2}{3} + 1 = y'$ $\quad 2x-2 = \frac{1}{3}x'$

$\underbrace{\frac{y}{3} + \frac{1}{3} = y'}$ $\quad 6x-6 = x'$

Dil $\frac{1}{3}$ fm x
Dil 6 fm y
$\frac{1}{3}$ up
6 left

b. $f(x) = (x-2)^3 + 1$ and $g(x) = 2(2x+4)^3 - 5$.

1) $y = (x-2)^3 + 1$

$y' = 2(2x'+4)^3 - 5$

2) $y-1 = (x-2)^3$

$\frac{y'+5}{2} = (2x'+4)^3$

3) $y-1 = \frac{y'+5}{2}$. $\quad x-2 = 2x'+4$

4) $2y-2 = y'+5$ $\quad x-6 = 2x'$

$2y-7 = y'$. $\quad \frac{1}{2}x-3 = x'$

Dil 2 fm x
Dil $\frac{1}{2}$ fm y
7 down
3 left

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Question 23 Extension.

Find a sequence of transformations required for $y = 3 - 4\sqrt{4(x+1)^2 + 3}$ to be transformed to $y = 2\sqrt{x^2 - 2x + 4}$.

1) ~~$y = 3 - 4\sqrt{4(x+1)^2 + 3}$~~

$y = 3 - 4\sqrt{(2x+2)^2 + 3}$

$y' = 2\sqrt{(x'-1)^2 + 3}$

2/3)

$\frac{y-3}{-4} = \frac{y'}{2}$, $2x+2 = x'-1$

4) $-\frac{1}{2}(y-3) = y'$ $2x+3 = x'$

$-\frac{1}{2}y + \frac{3}{2} = y'$

Dil $\frac{1}{2}$ fm x

Dil 2 fm y

Reflect in x

$\frac{3}{2}$ up

3 right

Key Takeaways

- We transform the coordinates first, then transform the function.
- To transform the function, replace its old variables with the new ones.
- To find the transformations, simply equate LHS with RHS after separating the transformations of x and y.

Contour Check

Learning Objective: [1.3.1] – Applying x' and y' notation to find transformed points, find the interpretation of transformations and altered order of transformations.

Key Takeaways

- The transformed point is called the ____________ and is denoted by ______________.
- The dilation factor is ________________ to the original coordinate.
- Reflection makes the original coordinates the _______________ of their original values.
- Translation ___________ a unit to the original coordinate.
- Transformations should be interpreted when _______________ are isolated.
- The order of transformation follows the ______________ order.
- To change the order of transformations, we either ____________________________.

Learning Objective: [1.3.2] – Find transformed functions.

Key Takeaways

- To transform the function, replace its _____________________ with the new one.

Learning Objective: [1.3.3] – Find transformations from transformed function (Reverse Engineering).

Key Takeaways

- To find the transformations, simply equate the ____________________ after separating the transformations of x and y.

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