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# VCE Mathematical Methods ¾
## Functions & Relations Exam Skills [1.2]
## Homework Solutions

### Homework Outline:

| | |
|---|---|
| Compulsory Questions | Pg 2 – Pg 20 |
| Supplementary Questions | Pg 21 – Pg 42 |

## Section A: Compulsory Questions

### Sub-Section: [1.2.1] - Finding a new domain to fix composite functions

**Question 1**

Consider the following functions defined over their maximal domains,

$$f(x) = x^2 - 1 \text{ and } g(x) = \sqrt{x}$$

**a.** Show that $g(f(x))$ does not exist.

$f(x)$ has a global minimum at $(0, -1)$.
ran $f = [-1, \infty)$ and dom $g = [0, \infty)$.
Therefore, $g(f(x))$ does not exist since ran $f \not\subseteq$ dom $g$.

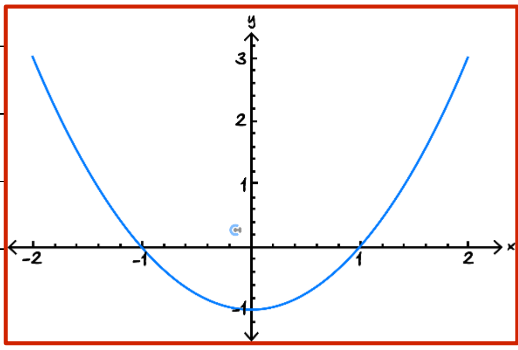
**b.** Find the maximal domain of $f$ such that $g(f(x))$ exists.

We require that ran $f \geq 0$. Therefore we solve

$$x^2 - 1 \geq 0$$
$$x^2 \geq 1$$
$$x \leq -1 \text{ or } x \geq 1$$

Therefore, domain $f = \mathbb{R} \setminus (-1, 1)$.

```
In[1]:= Plot[x^2 - 1, {x, -2, 2}]
```

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**Question 2**

Consider the functions,

$$f: [0, \infty) \to \mathbb{R}, f(x) = -3\sqrt{x} \text{ and } g: (-\infty, -3) \to \mathbb{R}, g(x) = \log_e(x^2 - 9)$$

**a.** Show that $g(f(x))$ does not exist.

ran $f = (-\infty, 0]$ and dom $g = (-\infty, -3)$
Therefore, $g(f(x))$ does not exist since ran $f \not\subseteq$ dom $g$

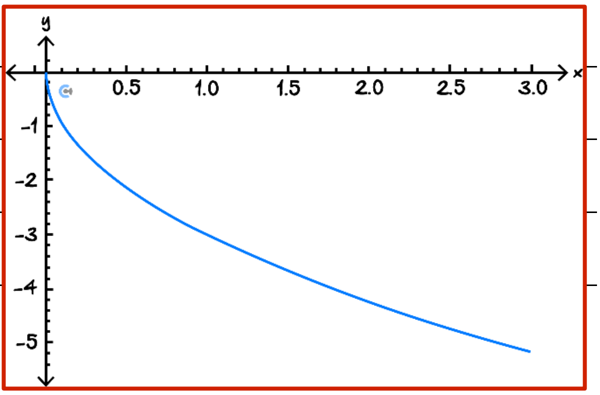
**b.** Find the maximal domain of $f$ such that $g(f(x))$ exists.

```
In[22]:= Plot[-3 √x, {x, 0, 3}]
```

We require that ran $f$ = dom $g = (-\infty, -3)$. Therefore we solve

$$-3\sqrt{x} = -3$$
$$\sqrt{x} = 1$$
$$x = 1$$

By considering the graph of $f$ we conclude that domain $f = (1, \infty)$.

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**Question 3**

Consider the following functions defined over their maximal domains,

$$f(x) = \frac{1}{x-1} \text{ and } g(x) = \sqrt{x^2 - 1}$$

**a.** Show that $g(f(x))$ does not exist.

$f$ is a hyperbola with no vertical shift.
dom $g \geq 0 \implies x^2 - 1 \geq 0 \implies x \leq -1$ or $x \geq 1$.
Therefore, ran $f = \mathbb{R} \setminus \{0\}$ and dom $g = (-\infty, -1] \cup [1, \infty)$.
Therefore, $g(f(x))$ does not exist since ran $f \not\subseteq$ dom $g$.

**b.** Find the maximal domain of $f$ such that $g(f(x))$ exists.

We require that ran $f$ = dom $g = (-\infty, -1] \cup [1, \infty)$. Therefore we solve

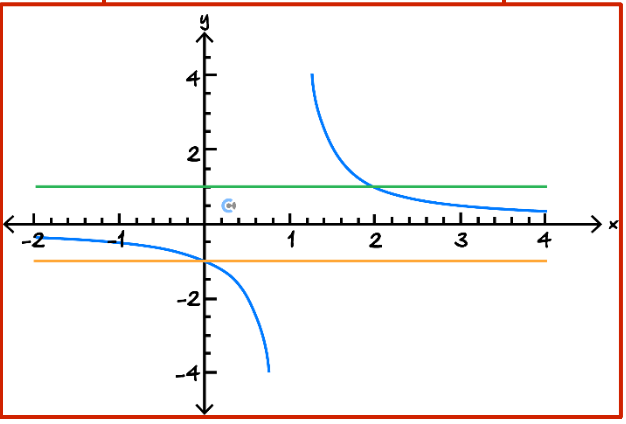
$$\frac{1}{x-1} = -1$$
$$1 = 1 - x$$
$$x = 0$$

and

$$\frac{1}{x-1} = 1$$
$$1 = x - 1$$
$$x = 2$$

By considering the graph of $f$ we conclude that domain $f = [0, 1) \cup (1, 2]$.

```
In[16]:= Plot[{1/(x - 1), -1, 1}, {x, -2, 4}]
```

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## Sub-Section: [1.2.2] - Finding the range of complex composite functions

**Question 4**

Find the range of $f(x) = \log_3(x^2 - 1)$, where $f$ is defined on its maximal domain.

ran $\log_3(x)$ $= \mathbb{R}$ for $x > 0$
Now $x^2 - 1$ has range $[-1, \infty)$.
Therefore we must have ran $f(x) = \mathbb{R}$ since $(0, \infty) \subseteq [-1, \infty)$

**Question 5**

Find the range of $f(x) = \log_2(x^2 + 16)$.

$x^2 + 16$ has a minimum value of 16 when $x = 0$. $\log_2(16) = 4$
As $x \to \infty$ $f(x) \to \infty$
Therefore, ran $f(x) = [4, \infty)$

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**Question 6**

The functions $f$ and $g$ are defined over their maximal domains. Let,

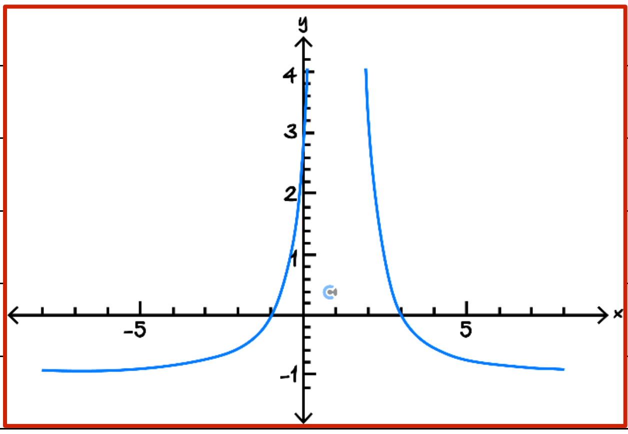
$$f(x) = \frac{x+1}{x-1} \text{ and } g(x) = x^2 - 2x$$

Find the range of $h(x) = g(f(x))$.

$$\begin{aligned} h(x) &= \left(\frac{x+1}{x-1}\right)^2 - \frac{2(x+1)}{x-1} \\ &= \frac{(x+1)^2 - 2(x+1)(x-1)}{(x-1)^2} \\ &= \frac{x^2 + 2x + 1 - 2(x^2 - 1)}{(x-1)^2} \\ &= \frac{-x^2 + 2x + 3}{(x-1)^2} \\ &= -\frac{x^2 - 2x - 3}{(x-1)^2} \\ &= -\frac{(x-1)^2 - 4}{(x-1)^2} \\ &= -1 + \frac{4}{(x-1)^2} \end{aligned}$$

Therefore, $h(x)$ is a truncus and so ran $h = (-1, \infty)$

```
In[39]:= Plot[((x + 1)/(x - 1))^2 - (2 (x + 1))/(x - 1), {x, -8, 8}]
```

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## Sub-Section: [1.2.3] - Finding the gradient of inverse functions

**Question 7**

Consider the one-to-one function $f$ with the following properties:

$$f(3) = 4, f(2) = 3, f'(3) = 1 \text{ and } f'(2) = 6$$

Let $g$ be the inverse function of $f$. Find the gradient of $g$ when $x = 3$.

We have that $g(3) = f^{-1}(3) = 2$ and that $f'(2) = 6$. Therefore

$$g'(3) = \frac{1}{6}$$

Remember the formula:

$$g'(x) = \frac{1}{f'(f^{-1}(x))}, \quad \text{where } g(x) = f^{-1}(x).$$

**Question 8**

Consider the one-to-one function $f$ with the following properties:

$$f(a) = 3, f(1) = a, f'(1) = c \text{ and } f'(a) = d$$

Let $g$ be the inverse function of $f$. Find the gradient of $g$ when $x = a$.

We have that $g(a) = 1$ and that $f'(1) = c$. Therefore

$$g'(a) = \frac{1}{c}$$

Remember the formula:

$$g'(x) = \frac{1}{f'(f^{-1}(x))}, \quad \text{where } g(x) = f^{-1}(x).$$

**Question 9**

Let $g$ be the inverse function of $f$. It is known that:

$$g'(a) = b \text{ and } f'(c) = \frac{1}{b}$$

where $f'(x)$ and $g'(x)$ are one-to-one functions.

Find $g(a)$.

We have that

$$g'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{1/b} = b$$

Therefore,

$$f^{-1}(a) = c$$

and so $g(a) = c$.

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## Sub-Section: Exam 1 Questions

**Question 10**

Find the maximal domain of the following functions:

**a.** $f(x) = \sqrt{4-x} + \log_e(x^2 + 4x + 3)$.

For $\sqrt{4-x}$ we require $4 - x \geq 0 \implies x \leq 4$
For the log We require that $x^2 + 4x + 3 > 0$.

$$\begin{aligned} x^2 + 4x + 3 &= 0 \\ (x+3)(x+1) &= 0 \\ x &= -3, -1 \end{aligned}$$

Positive quadratic shape so $x^2 + 4x + 3 > 0 \implies x < -3$ or $x > -1$.
Therefore, dom $f = (-\infty, -3) \cup (-1, 4]$

**b.** $g(x) = 2x + \sqrt{\frac{1}{-x^2+x+12}}$.

We require that $-x^2 + x + 12 > 0$.

$$\begin{aligned} -x^2 + x + 12 &= 0 \\ (4-x)(x+3) &= 0 \\ x &= -3, 4 \end{aligned}$$

Negative quadratic shape so $-x^2 + x + 12 > 0 \implies x \in (-3, 4)$
Therefore dom $g = (-3, 4)$

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**Question 11**

Let $f: (0, \infty) \to \mathbb{R}$, where $f(x) = \log_2(x)$ and $g: \mathbb{R} \to \mathbb{R}$, where $g(x) = x^2 + 4$.

**a.**

**i.** Find the rule for $h$, where $h(x) = f(g(x))$.

$$h(x) = \log_2(g(x)) = \log_2(x^2 + 4).$$

**ii.** State the value of $x$ for which $h$ is minimised.

$\log_2(x)$ is an increasing function.
$x^2 + 4$ is minimised when $x = 0$.
Therefore $x = 0$.

**iii.** State the domain and range of $h$.

dom $h$ = dom $g = \mathbb{R}$ since $g(x) > 0$ for all $x \in \mathbb{R}$.
$h$ is minimal when $x = 0 \implies h(0) = \log_2(4) = 2$.
Therefore ran $h = [2, \infty)$

**b.** Let $k: (-\infty, a] \to \mathbb{R}$, where $k(x) = \log_2(x^2 + 4)$.

**i.** Find the largest value of $a$ such that $k^{-1}$, the inverse function of $k$, exists.

We require $k$ to be one-to-one.
Therefore $a = 0$.

**ii.** Find the rule for $k^{-1}$.

Let $x = \log_2(y^2 + 4)$ and re-arrange to find $y$.

$$2^x = y^2 + 4$$
$$y^2 = 2^x - 4$$
$$y = \pm\sqrt{2^x - 4}$$

Now ran $k^{-1}$ = dom $k = (-\infty, 0]$ so

$$k^{-1}(x) = -\sqrt{2^x - 4}$$

**iii.** State the domain and range of $k^{-1}$.

dom $k^{-1}$ = ran $k = [2, \infty)$
ran $k^{-1}$ = dom $k = (-\infty, 0]$

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**Question 12**

Let $f: (-\infty, 1] \to \mathbb{R}, f(x) = \sqrt{1-x}$.

**a.** State the range of $f$.

$$\text{ran } f = [0, \infty)$$

**b.** Define the inverse function, $f^{-1}$, of $f$. Use functional notation.

Let $x = \sqrt{1-y}$ and re-arrange to find $y$

$$x^2 = 1 - y$$
$$y = 1 - x^2$$

Now dom $f^{-1}$ = ran $f = [0, \infty)$. Therefore,

$$f^{-1}: [0, \infty) \to \mathbb{R},\ f^{-1}(x) = 1 - x^2.$$

**c.** Find all points of intersection of $f$ and $f^{-1}$.

NOTE: $f$ is a decreasing function so we cannot just equate $f(x) = x$.
We must solve $f(x) = f^{-1}(x)$

$$\sqrt{1-x} = 1 - x^2$$
$$1 - x = \left((1-x)(1+x)\right)^2$$
$$1 - x = (1-x)^2(1+x)^2$$
$$(1-x)(1-(1-x)(1+x)^2) = 0$$

Therefore $x = 1$ or

$$1 - (1-x)(x^2+2x+1) = 0$$
$$1 = x^2 + 2x + 1 - x^3 - 2x^2 - x$$
$$1 = -x^3 - x^2 + x + 1$$
$$x^3 + x^2 - x = 0$$
$$x(x^2 + x - 1) = 0$$

Therefore $x = 0$ or

$$x^2 + x - 1 = 0$$
$$\left(x + \frac{1}{2}\right)^2 = \frac{5}{4}$$
$$x + \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$
$$x = \frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})$$

Now by considering the domains of $f$ and $f^{-1}$ the points of intersection are

$$(0,1) \quad \text{and} \quad \left(\frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}-1)\right) \quad \text{and} \quad (1,0).$$

NOTE: Just solving $\sqrt{1-x} = x \implies x = \frac{1}{2}(\sqrt{5}-1)$ only, so we miss two other points of intersection.

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## Sub-Section: Exam 2 Questions

**Question 13**

The function $f$ defined by $f : A \to \mathbb{R}, f(x) = (x - 3)^2 + 2$ will have an inverse function if its domain $A$ is:

**A.** $\mathbb{R}$

**B.** $\mathbb{R}^+ \cup \{0\}$

**C.** $x \geq 2$

**D.** $x \leq 3$

**Question 14**

The function $f(x) = \frac{\sqrt{x^2-9}}{2x}$ has a maximal domain and range.

**A.** Domain $= \mathbb{R}\backslash(-3, 3)$ and Range $= [0, \infty)$.

**B.** Domain $=$ R$\backslash(-3,3)$ and Range $= (-\frac{1}{2}, \frac{1}{2})$

**C.** Domain $= [-3, 0] \cup [3, \infty)$ and Range $= [1, \infty)$.

**D.** Domain $= [-\infty, 0) \cup [3, \infty)$ and Range $= [1, \infty)$.

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**Question 15**

Let $f$ be a one-to-one differentiable function, and the following values are known,

$$f(2) = 5, f(3) = 9, f'(2) = 3 \text{ and } f'(3) = 8$$

Let $g(x) = f^{-1}(x)$, the value of $g'(5)$ is:

**A.** $\frac{1}{8}$

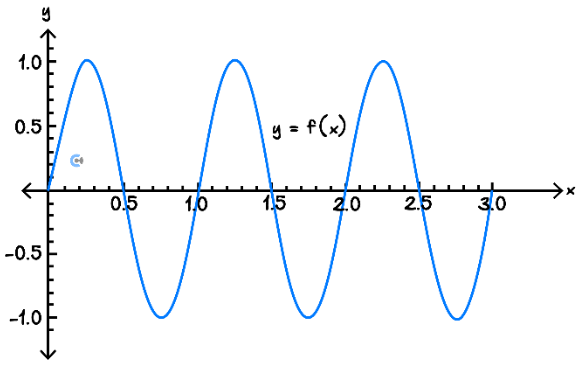
**B.** $\frac{1}{9}$

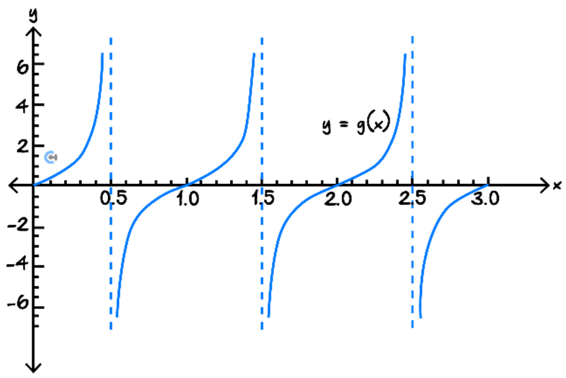
**C.** $\frac{1}{3}$

**D.** $\frac{1}{5}$

**Question 16**

Consider the functions $f$ and $g$ graphed below.




How many $x$-intercepts do the function $h(x) = f(x)g(x)$ have for $x \in [0, 3]$?

**A.** 4

**B.** 5

**C.** 6

**D.** 7

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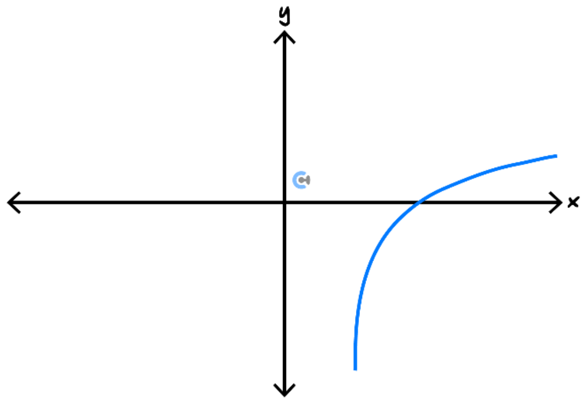
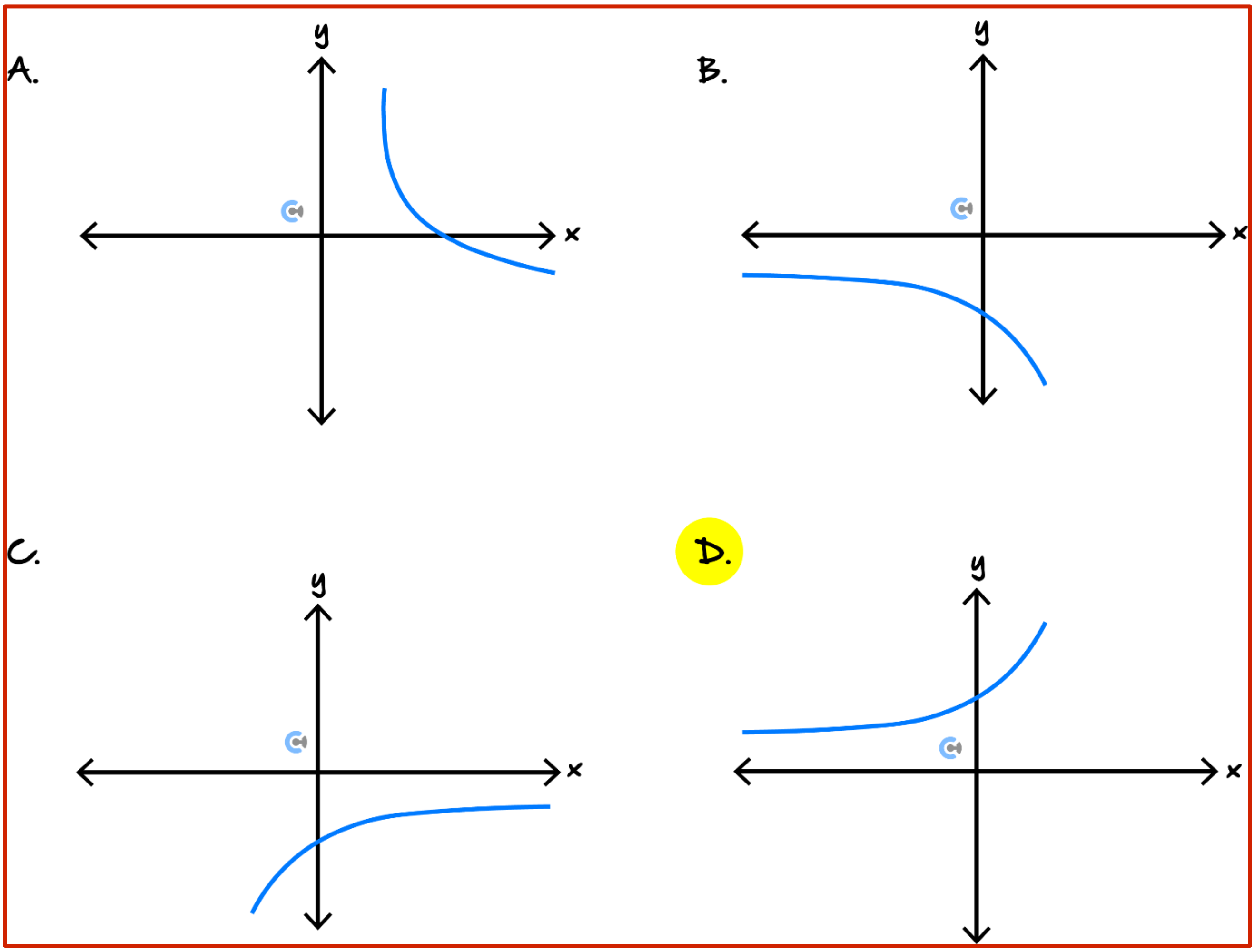
**Question 17**

Part of the graph of $y = f(x)$ is shown below.




The inverse function $f^{-1}$ is best represented by:

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**Question 18**

Consider the following functions defined on their maximal domains.

$$f(x) = \sqrt{3 - x}$$

$$g(x) = \log_e\left(\frac{1}{x}\right)$$

**a.** Find the maximal domain and range of $\frac{1}{f(x)} - g(x)$.

$$\text{Domain} = (0, 3) \text{ and Range} = \mathbb{R}$$

**b.** Show that $f(g(x))$ does not exist.

dom $f = (-\infty, 3]$ and ran $g = \mathbb{R}$
$g(f(x))$ is not defined because ran $f \not\subseteq$ dom $g$

**c.** Show that $g(f(x))$ does not exist.

dom $g = (0, \infty)$ and ran $f = [0, \infty)$
$g(f(x))$ is not defined because ran $f \not\subseteq$ dom $g$

**d.** Restrict the domains of $f$ and $g$ to be as large as possible so that both $f(g(x))$ and $g(f(x))$ are defined.

If dom $f = (-\infty, 3)$ then $g(f(x))$ is defined.
Then we must restrict dom $g$ so that ran $g = (-\infty, 3) \implies$ dom $g = (e^{-3}, \infty)$.
Conclude that dom $f = (-\infty, 3)$ and dom $g = (e^{-3}, \infty)$

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**Question 19**

Consider the function:

$$f\colon(-\infty, a] \to \mathbb{R}, f(x) = -\frac{1}{2}x^2 + 6x + \frac{3}{2}$$

**a.** Find the smallest value of $a$ such that the inverse function $f^{-1}$ exists.

$f(x) = \frac{39}{2} - \frac{1}{2}(x-6)^2$.
Therefore, $a = 6$.

**b.** Define the inverse function, $f^{-1}$.

$x = \frac{39}{2} - \frac{1}{2}(y-6)^2 \implies y = 6 \pm \sqrt{39 - 2x}$

ran $f^{-1}$ = dom $f = (-\infty, 6]$ and dom $f^{-1}$ = ran $f = \left(-\infty, \frac{39}{2}\right]$. Therefore,

$$f^{-1}: \left(-\infty, \frac{39}{2}\right] \to \mathbb{R},\ f^{-1}(x) = 6 - \sqrt{39 - 2x}$$

**c.** Find all points of intersection between $f$ and $f^{-1}$.

Solve $f(x) = x \implies x = 5 \pm 2\sqrt{7}$, only one solution valid for domains of both $f$ and $f^{-1}$.
Point of intersection $(5 - 2\sqrt{7}, 5 - 2\sqrt{7})$

**d.** Find the rule and domain for $f(f^{-1}(x))$.

$$f(f^{-1}(x)) = x \text{ for } x \leq \frac{39}{2}.$$

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## Section B: Supplementary Questions

### Sub-Section: [1.2.1] - Finding a new domain to fix composite functions

**Question 20**

Consider the functions the following functions defined over their maximal domains,

$$f(x) = \log_e(x) \text{ and } g(x) = e^x - 1$$

**a.** Show that $f(g(x))$ does not exist.

ran $g = (-1, \infty)$ and dom $g = (0, \infty)$
Therefore, $f(g(x))$ does not exist since ran $g \not\subseteq$ dom $f$.

**b.** Find the maximal domain of $g$ such that $f(g(x))$ exists.

We require that for all $x$ in the maximal domain of $g$, $g(x) > 0$. Hence

$$e^x - 1 > 0 \implies e^x > 1 \implies x > 0$$

Therefore, dom $g = (0, \infty)$

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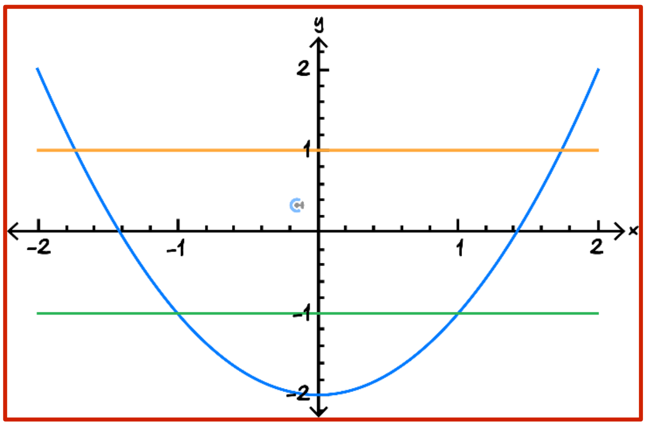
**Question 21**

Consider the following functions defined over their maximal domains,

$$f(x) = (x^2 - 2)^2 \text{ and } g(x) = \sqrt{x - 1}$$

Find the maximal domain of $f$ such that $g(f(x))$ exists.

Observe that the domain of $g(x)$ is $[1, \infty)$.
Thus for $g(f(x))$ to exist, we require $f(x) = (x^2 - 2)^2 \geq 1 \implies x^2 - 2 \geq 1$ or $x^2 - 2 \leq -1$.
We can solve for $x^2 - 2 = 1 \implies x = \pm\sqrt{3}$ and $x^2 - 2 = -1 \implies x = \pm 1$, and then sketch $y = x^2 - 2$ to solve our inequalities.




From this graph we see that $x^2 - 2 \leq -1$ if $x \in [-1, 1]$.
Similarly we see that $x^2 - 2 \geq 1$ if $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$
Hence the maximal domain of $f$ for $g(f(x))$ to exist $x \in (-\infty, -\sqrt{3}] \cup [-1, 1] \cup [\sqrt{3}, \infty)$

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**Question 22**

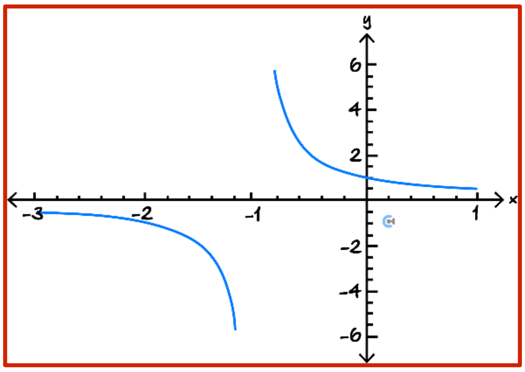
Consider the following functions defined over their maximal domains,

$$f(x) = \frac{1}{1+x} \text{ and } g(x) = \sqrt{16 - (x-1)^2}$$

Find the maximal domain of $f$ such that $g(f(x))$ exists.

We require that ran $f \subseteq$ dom $g$.
Observe that if $x \in$ dom $g$, then $(x-1)^2 \leq 16$, which leaves us with $x \in [-3, 5]$.
Hence $f(x) = \frac{1}{1+x} \in [-3, 5]$. We proceed by solving $f(x) = -3, 5$ and sketching our graph to get our inequality.

$$\frac{1}{1+x} = -3 \implies -3x = 4 \implies x = -\frac{4}{3}$$
$$\frac{1}{1+x} = 5 \implies 5x = -4 \implies x = -\frac{4}{5}$$




From these three pieces of information, we see that the maximal domain of $f$ such that $g(f(x))$ exists is $x \in \left(-\infty, -\frac{4}{3}\right] \cup \left[-\frac{4}{5}, \infty\right)$.

## Space for Personal Notes

**Question 23**

Consider the following functions,

$f: [0, 2) \to \mathbb{R}, f(x) = \log_2(4 - x^2)$ and $g: (-\infty, 2) \to \mathbb{R}, g(x) = 3(x - 1)^2 - 1$.

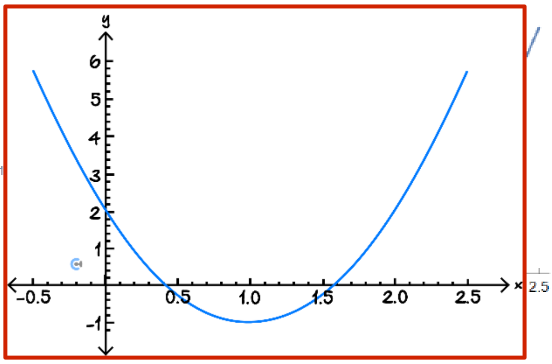
Find the largest interval of $x$ values for which $f(g(x))$ and $g(f(x))$ both exist.

We need $f(x) = \log_2(4 - x^2) < 2 \to 4 - x^2 < 4 \to x^2 > 0$

As $x \in dom\ f, x \in (0,2)$

```
In[141]:= Plot[3 (x - 1)^2 - 1, {x, -0.5, 2.5}]
```

Out[141]=




As $g\left(1 \pm \frac{1}{\sqrt{3}}\right) = 0$ and $g(0) = g(2) = 2$, from our graph we see that $3x^2 - 1 \in [0, 2)$ when

$$x \in \left(0, 1 - \frac{1}{\sqrt{3}}\right] \cup \left[1 + \frac{1}{\sqrt{3}}, 2\right).$$

Combining this restriction with our restriction for ran $f \subseteq$ dom $g$, we get that,

$$x \in \left[1 + \frac{1}{\sqrt{3}}, 2\right).$$

**Space for Personal Notes**

## Sub-Section: [1.2.2] - Finding the range of complex composite functions

**Question 24**

Find the range of $f(x) = e^{x^2+1}$.

The range of $g(x) = x^2 + 1$ is $[1, \infty)$.
The range of $h : [1, \infty) \to \mathbb{R}, h(x) = e^x$ is $[e, \infty)$.
As $f(x) = h(g(x))$, the range of $f$ is $[e, \infty)$.

**Question 25**

Find the range of $f: [0, \infty) \to \mathbb{R}, f(x) = \log_3(3^x + 8)$.

The range of $g : [0, \infty) \to \mathbb{R}, g(x) = 3^x + 8$ is $[9, \infty)$.
The range of $h : [9, \infty) \to \mathbb{R}, h(x) = \log_3(x)$ is $[2, \infty)$.
As $f(x) = h(g(x))$, the range of $f$ is $[2, \infty)$.

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**Question 26**

Find the range of $f(x) = \sqrt{\frac{x}{x+1}}$ where $f$ is defined on its maximal domain.

The range of $g(x) = \dfrac{x}{x+1} = 1 - \dfrac{1}{x+1}$ is $\mathbb{R}\backslash\{1\}$.
Thus the range of $f$ is the range of $\sqrt{x}$ on the intersection of it's maximal domain, $[0, \infty)$ and the range of $g$. Specifically,

$$\mathbb{R}\backslash\{1\} \cap [0, \infty) = [0, \infty)\backslash\{1\}$$

The range of $\sqrt{x}$ on this domain is $[0, \infty)\backslash\{1\}$, hence the range of $f$ is, $[0, \infty)\backslash\{1\}$.

Space for Personal Notes

**Question 27**

Consider the following functions defined on all real numbers,

$$f(x) = \sin(x) \text{ and } g(x) = \log_3(4x^2 - 4x + 2)$$

Find the range of $g(f(x))$.

We observe that the range of $f(x)$ is $[-1, 1]$. Hence the range of $g(f(x))$ is the range of $g$ restricted to $[-1, 1]$.
Now observing $g(x)$, we note that it is the composition of $\log_3(x)$ and $4x^2 - 4x + 2 = (2x - 1)^2 + 1$.
The range of $h(x) = (2x - 1)^2 + 1$ on the interval $[-1, 1]$ can be found by evaluating $h(-1), h(1)$ and the $y$-value of the turning point of $h$, which is 1.
As $h(-1) = (-3)^2 + 1 = 10$, and $h(-1) = 1^2 + 1 = 2$, we see that the range of $h$ on the interval $[-1, 1]$ is $[1, 10]$.
Since $\log_3(x)$ is an increasing function, the range of $\log_3(x)$ on the interval $[1, 10]$ is $[0, \log_3(10)]$.
Hence the range of $g(f(x))$ is $[0, \log_3(10)]$

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## Sub-Section: [1.2.3] - Finding the gradient of inverse functions

**Question 28**

Consider the function $f: [0, \infty) \to \mathbb{R}, f(x) = x^2$.

The gradient of $f$ at $x = a$ is $2a$.

Let $g$ be the inverse function of $f$. Find the gradient of $g$ when $x = 2$.

We observe that $g(x) = \sqrt{x}$, hence $g(2)$ is $\sqrt{2}$.
When $x = \sqrt{2}$, the gradient of $f$ is $2\sqrt{2}$.
Hence the gradient of $g(x)$ when $x = 2$ is $\frac{1}{2\sqrt{2}}$.

**Question 29**

Consider the one-to-one function $f$ with the following properties.

$$f(2) = 5, f(5) = 7, f'(2) = 3 \text{ and } f'(5) = 1$$

Let $g$ be the inverse function of $f$. Find the gradient of $g$ when $x = 5$.

We have that $g(5) = 2$ and that $f'(5) = 1$. Therefore

$$g'(5) = \frac{1}{1} = 1$$

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**Question 30**

Consider the function $f(x)$, the gradient of $f$ at $x = a$ is $2f(a) + 2a$, and $f(0) = 1$.

From this information, we can tell that the gradient of $f^{-1}$ at $x = b$ is $c$. Find $b$ and $c$.

If $f(a) = b$ and $f'(a) = \frac{1}{c}$, we know that the gradient of $f^{-1}$ at $x = b$ is $c$.
The only $a$ for which we know $f(a)$ and $f'(a)$ is $a = 0$.
Hence $b = f(0) = 1$, and $c = \frac{1}{f'(0)} = \frac{1}{2(1) + 2(0)} = \frac{1}{2}$.

**Question 31**

Consider the differentiable, one-to-one, function $f\colon (0, 1) \to \mathbb{R}$. It is known that:

1. $f'(x) = -[f(x)]^2$, for all $x \in (0, 1)$.
2. $\text{ran } f = (1, \infty)$.

If $g$ is the inverse function of $f$, find the domain and range of $g'(x)$.

**Hint:** $g'(a)$ denotes the gradient of $g$ at $x = a$.

The domain of $g'(x)$ is the domain of $g$ which is the range of $f = (1, \infty)$.
The range of $g'(x)$ is the reciprocal of the range of $f'(x)$.
As $f'(x) = -[f(x)]^2$, the range of $f'(x)$ is $(-\infty, -1)$.
Hence the range of $g'(x)$ is $(-1, 0)$.

## Sub-Section: Exam 1 Questions

**Question 32**

Let $f: [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x+4}$.

**a.** State the range of $f$.

$[2, \infty)$

**b.** Let $g: (-\infty, c] \to \mathbb{R}$, $g(x) = x^2 + 6x + 7$, where $c < 0$.

Find the largest possible value of $c$ such that the range of $g$ is a subset of the domain of $f$.

We require $g(x) = x^2 + 6x + 7 \geq 0$.
We solve $g(x) = 0$ by completing the square, thus,

$$x^2 + 6x + 9 - 2 = (x+3)^2 - 2 = 0 \implies x + 3 = \pm\sqrt{2} \implies x = -3 \pm \sqrt{2}$$

As $g(x)$ is a positive parabola, for $g(x) \geq 0$ either, $x \leq -3 - \sqrt{2}$ or $x \geq -3 + \sqrt{2}$.
Hence $c = -3 - \sqrt{2}$.

**c.** For the value of $c$ found in **part b.**, state the range of $f(g(x))$.

For the value of $c$ found in part b, the range of $g$ is $[0, \infty)$.
Hence the range of $f(g(x))$ is simply the range of $f = [2, \infty)$

**d.** Let $h: \mathbb{R} \to \mathbb{R}, h(x) = x^2 + 5$.

State the range of $f(h(x))$.

$f(h(x)) = \sqrt{x^2+5+4} = \sqrt{x^2+9}$. Hence the range of $f(h(x)) = [3, \infty)$.

**Question 33**

Let $f: (-2, \infty) \to \mathbb{R}, f(x) = 3 - \frac{4}{(x+2)^2}$.

State the rule and domain of $f^{-1}$.

We solve $x = f(y)$ for $y$, thus,

$$x = 3 - \frac{4}{(y+2)^2}$$
$$\implies 3 - x = \frac{4}{(y+2)^2}$$
$$\implies \frac{4}{3-x} = (y+2)^2$$
$$\implies \frac{2}{\sqrt{3-x}} = y + 2 \qquad \text{Since dom } f = (-2, \infty)$$
$$\implies y = \frac{2}{\sqrt{3-x}} - 2$$

The domain of $f^{-1}$ is simply the range of $f$ which is $(-\infty, 3)$. Hence the function $f^{-1}$ is,

$$f^{-1}: (-\infty, 3) \to \mathbb{R}, f^{-1}(x) = \frac{2}{\sqrt{3-x}} - 2$$

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**Question 34**

**a.** Let $f: \mathbb{R}\backslash\{3\} \to \mathbb{R}$, $f(x) = \frac{1}{x-3}$. Find the rule for $f^{-1}$.

We solve $f(y) = x$ for $y$. Thus,

$$x = \frac{1}{y-3} \implies \frac{1}{x} = y - 3 \implies y = \frac{1}{x} + 3$$

Hence the rule for $f^{-1}$ is, $f^{-1}(x) = \frac{1}{x} + 3$

**b.** State the domain of $f^{-1}$.

The domain of $f^{-1}$ is the range of $f$ which is $\mathbb{R}\backslash\{0\}$.

**c.** Let $g(x) = f(x - c) + d$ for $c, d \in \mathbb{R}$.

Find the values of $c$ and $d$, given that $g = f^{-1}$.

$g(x) = \dfrac{1}{x - c - 3} + d = \dfrac{1}{x} + 3$. Hence $d = 3$ and $c = -3$.

**d.** Given that $f'(1) = -\frac{1}{4}$ and $f'(4) = -1$, find the value of $g'(1)$.

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(1+3)} = \frac{1}{f'(4)} = -1$$

**Space for Personal Notes**

**Question 35**

Find the maximal domain of $f$, where $f(x) = \frac{1}{\sqrt{x^2-6x+5}}$.

For $x$ to be in the maximal domain of $f$, we require that $x^2 - 6x + 5 > 0$.
We can factorise $x^2 - 6x + 5$ as $(x-5)(x-1)$ to see that it is equal to 0 when $x = 1, 5$.
As $x^2 - 6x + 5$ is an upwards parabola, we see that it is greater than 0 when $x < 1$ or $x > 5$.
Hence the maximal domain of $f$ is $(-\infty, 1) \cup (5, \infty)$

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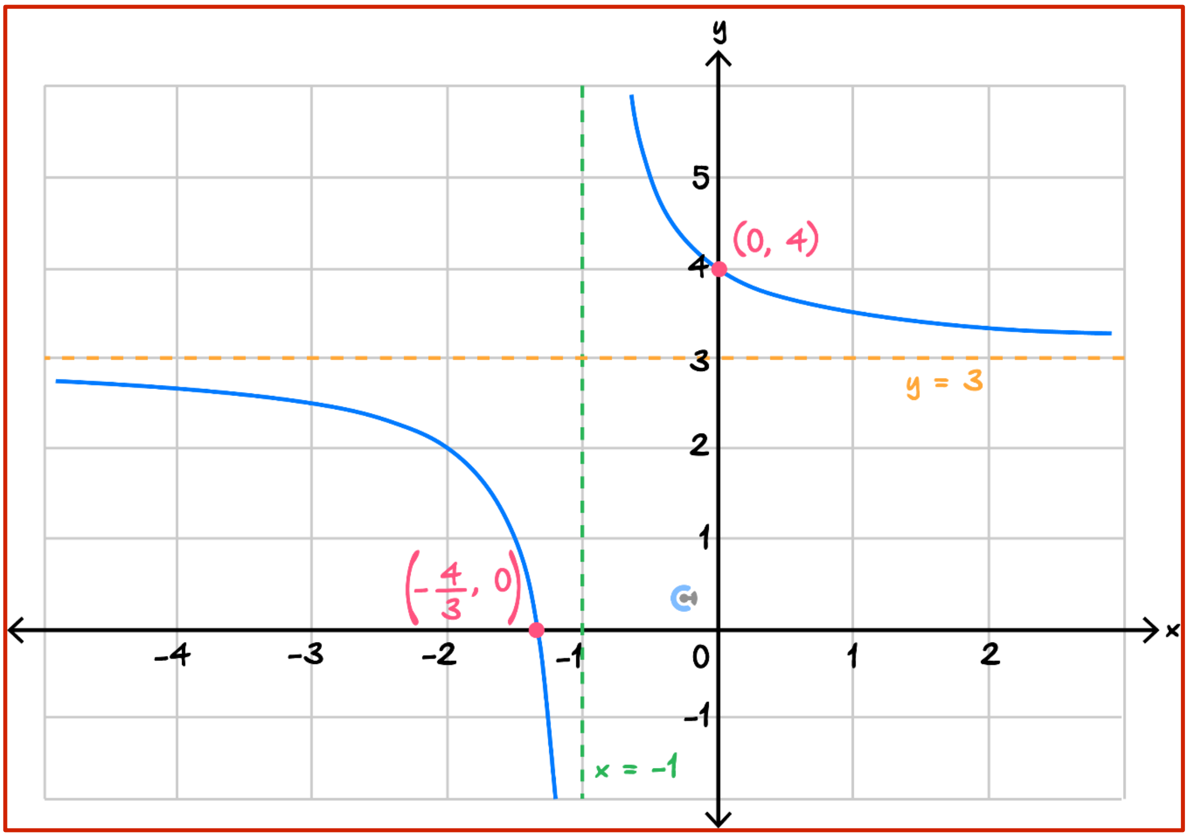
**Question 36**

**a.** Sketch the graph of $f(x) = 3 + \frac{1}{x+1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates.




**b.** Find the values of $x$ for which $f(x) \in (2, 4)$.

We see that $f(x) = 4$ when $x = 0$ and $f(x) = 2$ when $x = -2$.
From the above graph we can see that $f(x) \in (2, 4)$ when $x \in (-\infty, -2) \cup (0, \infty)$.

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## Sub-Section: Exam 2 Questions

**Question 37**

Which one of the following is the inverse function of $g: (-\infty, 2] \to \mathbb{R}, g(x) = 4(x - 2)^2 + 3$?

**A.** $f: [3, \infty) \to \mathbb{R}, f(x) = 2 + \frac{\sqrt{x-3}}{2}$

**B.** $f: [3, \infty) \to \mathbb{R}, f(x) = 2 - \frac{\sqrt{x-3}}{2}$

**C.** $f: [3, \infty) \to \mathbb{R}, f(x) = 4 + \frac{\sqrt{x-3}}{4}$

**D.** $f: [3, \infty) \to \mathbb{R}, f(x) = 4 - \frac{\sqrt{x-3}}{4}$

**Question 38**

The maximal domain of the function $f$ is $\left(-\infty, 1 - \sqrt{5}\right] \cup [1 + \sqrt{5}, \infty)$.

A possible rule of $f$ is:

**A.** $f(x) = \sqrt{5 - (x-1)^2}$

**B.** $f(x) = \log_e(5 - (x-1)^2)$

**C.** $f(x) = \frac{1}{\sqrt{5-(x-1)^2}}$

**D.** $f(x) = \frac{1}{\log_e(5-(x-1)^2)}$

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**Question 39**

Let $f$ be a one-to-one differentiable function and the following values are known,

$$f(-1) = 3, f(3) = 7, f'(-1) = 5 \text{ and } f'(3) = 2$$

Let $g(x) = f^{-1}(x)$, the value of $g'(3)$ is:

**A.** 5

**B.** 2

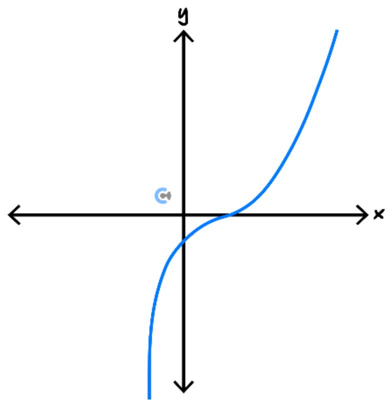
**C.** $\frac{1}{5}$

**D.** $\frac{1}{2}$

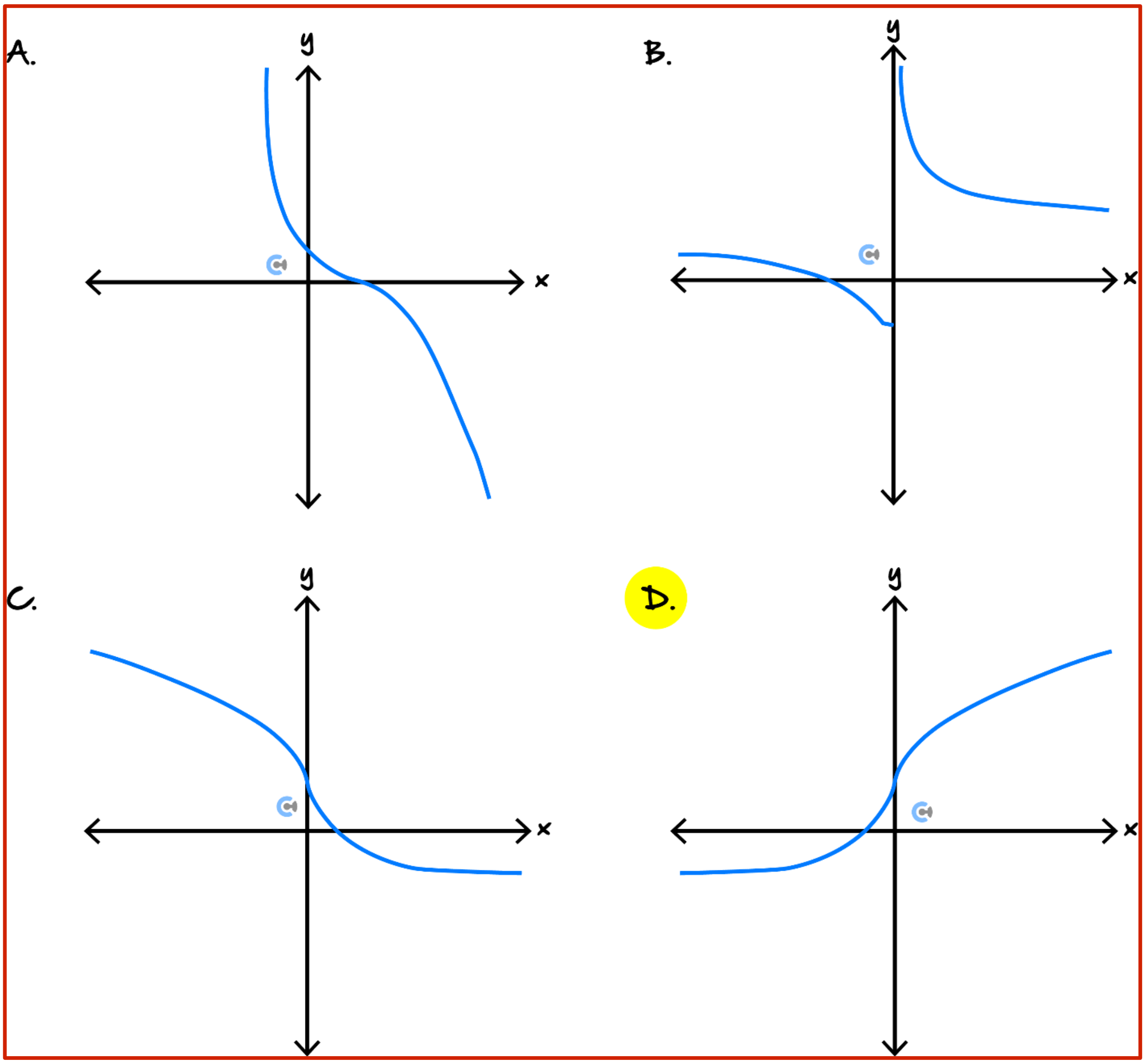
**Space for Personal Notes**

**Question 40**

Part of the graph of the function $f$ is shown below. The same scale has been used on both axes.




The corresponding part of the graph of the inverse function $f^{-1}$ is best represented by:



**Question 41**

Consider the following functions,

$$f : \left(-\frac{\sqrt{3}}{2}, \infty\right) \to \mathbb{R}, f(x) = \log_e\left(x + \frac{\sqrt{3}}{2}\right)$$

$$g : (-\infty, 3) \to \mathbb{R}, g(x) = \cos(x)$$

The largest interval of $x$ values for which $f(g(x))$ and $g(f(x))$ both exist is:

**A.** $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

**B.** $\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{6}\right)$

**C.** $\left(-\frac{5\pi}{6}, \frac{5\pi}{6}\right)$

**D.** $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

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**Question 42**

**a.** Express $\frac{3x+2}{x+3}$ in the form of $a+\frac{b}{x+2}$, where $a$ and $b$ are non-zero integers.

$$\frac{3x+2}{x+3}=\frac{3(x+3)-7}{x+3}=3+\frac{-7}{x+2}$$

Hence $a=3$ and $b=-7$.

**b.** Let $f:\mathbb{R}\backslash\{-3\}\to\mathbb{R}, f(x)=\frac{3x+2}{x+3}$.

**i.** Find the rule and domain of $f^{-1}$ and the inverse function of $f$.

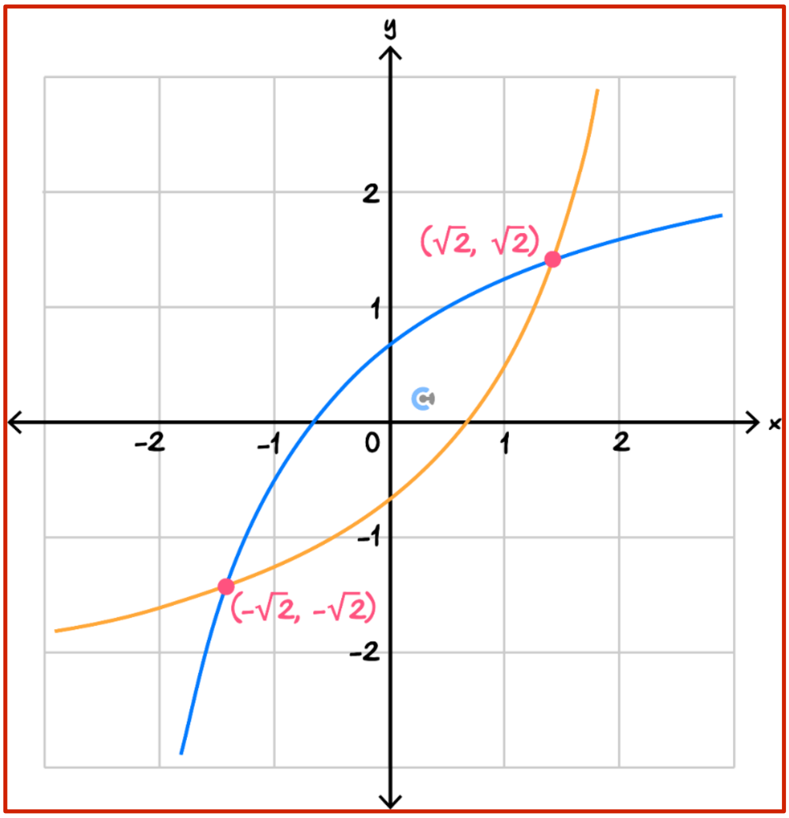
We solve $f(y)=x$ for $y$ to get the rule for $f^{-1}$.

Hence $f^{-1}(x)=\frac{2-3x}{x-3}$.

The domain of $f^{-1}$ is the range of $f$ which is $\mathbb{R}\backslash\{3\}$

**ii.** Part of the graph of $f$ is shown in the diagram below.

Sketch the graph of $y=f^{-1}$, labelling all points of intersection with their coordinates.



**c.** Let $g(x) = -\sqrt{16 - x^2}$.

**i.** Show that both $f(g(x))$ and $g(f(x))$ do not exist.

The domain of $f$ is $\mathbb{R}\backslash\{-3\}$.
The range of $g$ is $[-4, 0]$.
Thus ran $g \not\subseteq$ dom $f$ hence $f(g(x))$ does not exist.
The domain of $g$ is $[-4, 4]$
The range of $f$ is $\mathbb{R}\backslash\{3\}$.
Thus ran $f \not\subseteq$ dom $g$ hence $g(f(x))$ does not exist.

**ii.** Find the largest interval on which both $f(g(x))$ and $g(f(x))$ are defined on.

We solve $f(x) = 4 \implies x = -2$.
We solve $f(x) = -4 \implies x = -10$.
From the graph of $f$ we see that $f(x) \in [-4, 4]$ if $x \in (-\infty, -10] \cup [-2, \infty)$
Thus $g(f(x))$ is defined for $x \in (-\infty, -10] \cup [-2, \infty)$
We solve $g(x) = -3 \implies x = \pm\sqrt{7}$.
Thus $f(g(x))$ is defined for $x \in [-4, 4]\backslash\{\pm\sqrt{7}\}$.
Hence $f(g(x))$ and $g(f(x))$ are both defined on the intersection of the two sets, specifically $x \in [-2, 4]\backslash\{\sqrt{7}\}$.
This set contains two intervals, the bigger one being, $[-2, \sqrt{7})$.

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**Question 43**

Let $f(x) = 2^{-x}$ and $g(x) = 4x^2 - 4x + 3$.

**a.**

**i.** State the rule of $f(g(x))$.

$f(g(x)) = 2^{-4x^2-4x+3}$

**ii.** State the range of $f(g(x))$.

The range of $g(x) = (2x-1)^2 + 2$ is $[2, \infty)$.
As $f$ is a decreasing function, which tends towards 0 as $x \to \infty$, the range of $f$ is,

$$\left(0, \frac{1}{4}\right].$$

**b.** Let $h: [a, \infty) \to \mathbb{R}, h(x) = g(f(x))$. Find the smallest value of $a$ such that $h$ is a one-to-one function.

As $g(x) = (2x-1)^2 + 2$, the largest intervals for which it is a one-to-one function on are, $\left[\frac{1}{2}, \infty\right)$, or, $\left(\infty, \frac{1}{2}\right]$.
As $f(x)$ is a decreasing function, the range of $f$ when restricted to $[a, \infty)$ is $(0, 2^a]$.
Since $(0, 2^{-a}] \not\subseteq \left[\frac{1}{2}, \infty\right)$ for all $a$, we must consider values of $a$ for which

$$(0, 2^{-a}] \subseteq \left(\infty, \frac{1}{2}\right].$$

The smallest such value of $a$ is $a = 1$.

**c.** For the value of $a$ found in **part b.**, state the rule and domain for $h^{-1}$.

We can solve $h(y) = x$ for $y$. This implies that $g(f(y)) = x$, thus we will first solve $g(z) = x$ for $z$, restricting our attention to $z < \frac{1}{2}$ because of our work from part b. This yields,

$$z = \frac{1}{2}(1 - \sqrt{x-2})$$

Now we solve $f(y) = z$, we see that $y = -\log_2\left(\frac{1}{2}(1-\sqrt{x-2})\right) = 1 - \log_2(1 - \sqrt{x-2})$.

Hence the rule for $h^{-1}$ is $h^{-1}(x) = 1 - \log_2(1 - \sqrt{x-2})$.

The domain for $h^{-1}$ is the range of $h$. As the range of $f$ restricted to $[1, \infty)$ is $\left(0, \frac{1}{2}\right]$, the range of $h$ is simply the range of $g$ restricted to $\left(0, \frac{1}{2}\right]$.

As $g$ is one-to-one in this interval, the range of $g$ restricted to $\left(0, \frac{1}{2}\right]$ is $[2, 3)$.

Hence the domain of $h^{-1}$ is $[2, 3)$.

**d.** How many solutions does the equation $f(g(x)) + g(f(x)) = 0$ have?

We sketch the graph of $f(g(x)) + g(f(x))$.

As it is entirely above the $x$-axis, our equation has 0 solutions.

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