CONTOUREDUCATION

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VCE Mathematical Methods ¾
Functions & Relations [1.1]
Workbook

Outline:

Section A: Domain of Functions

Sub-Section: Maximal Domains

Functions and Relations

- Our topics today:

  1. Domain

  2. Composite Functions

  3. Inverse Functions

- None of these can be understood without being able to find a domain of a function.

- Today's class will get progressively harder so be sure to ask questions when you have.

Starting with domain!

Maximal Domain

- **Definition**: The largest possible set of input values (elements of the domain) for which the function is well-defined.

- **Three Important Rules:**

Functions	Maximal Domain
$\sqrt{z}$	$z \geq 0$
$\log(z)$	$z > 0$
$\frac{1}{z}$	$z \neq 0$

Steps

1. Find the restriction of the inside.
2. Sketch the graph if needed.
3. Solve for domain.

Let's have a look at a question together!

Question 1 Walkthrough.

Find the maximal domain of each of the following functions.

a. $f(x) = 3\sqrt{4x+3} - 2$

$4x+3 \geq 0$

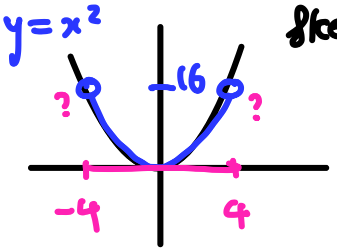
$x \geq -\frac{3}{4}$

b. $h(x) = \log_2(-x^2 + 16)$

$-x^2+16 > 0$

$-x^2 > -16$

M:1 $x^2 < 16$



Sketch

$x^2 = 16$

$x = \pm 4$

$x \in (-4, 4)$

Active Recall: Steps to Find Maximal Domain

1. Find the restriction of the inside.
2. Sketch the graph if needed.
3. Solve for domain

Your turn!

Question 2

Find the maximal domain of the following functions.

a. $f(x) = -\sqrt{-2x-4}+1$

$-2x-4 \geq 0$

$-2x \geq 4$

$x \leq -2$

b. $\frac{1}{x^2-9}$

$x^2-9 \neq 0$

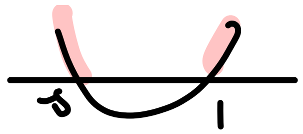
$x^2 \neq 9$

$x \in \mathbb{R} \setminus \{-3, 3\}$

c. $h(x) = -\log_2(x^2 + 4x - 5)$

$x^2+4x-5 > 0$

$(x+5)(x-1) > 0$



$x \in (-\infty,-5) \cup (1,\infty)$

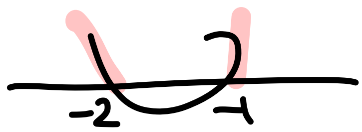
Question 3 Extension.

State the maximal domain of the following function.

$$y = \frac{1}{\sqrt{x^2 + 3x + 2}}$$

$x^2+3x+2 > 0$

$(x+2)(x+1) > 0$



$x \in (-\infty,-2) \cup (-1,\infty)$

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Sub-Section: Domain of Sum, Difference, and Product of Functions

What about a domain of the sum of two functions?

Analogy: Students

- Consider two students with different availabilities:

Student	Function	Availability
Raj	f	10 A.M. − 2 P.M.
Rithvik	g	11 A.M. − 5 P.M.

- When can these two meet? $f+g$: $[11, 2]$

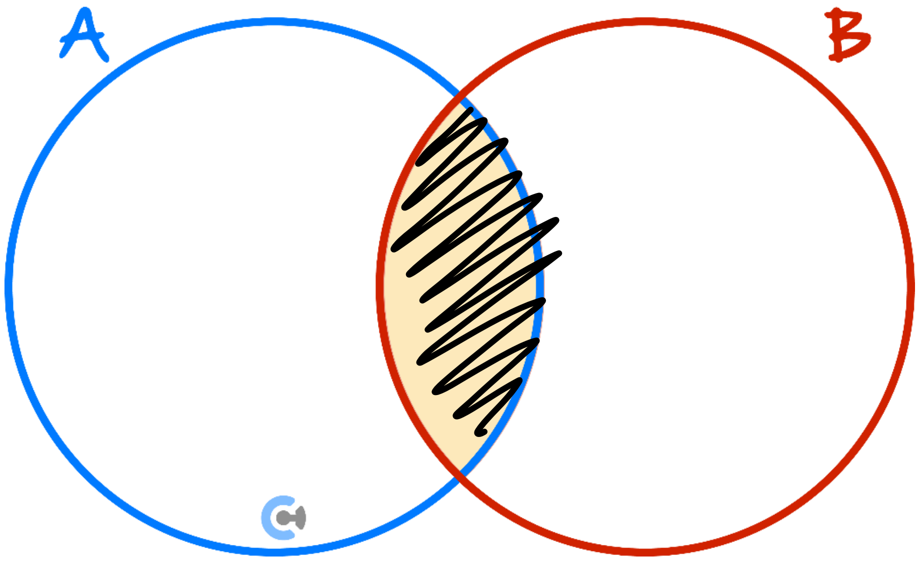
Intersection

Meeting	$f+g$	

This is the same as finding the domain of the sum of two functions!

Exploration: Domain of Sum, Differen

- If the domain of f is A and the domain of g is B, what would be the **domain of $f + g$**?



- For $f + g$ to be defined, do both f and g be defined? [Yes] / [No]

- How can both f and g defined? (Hint: Look at the diagram above.)

$$\text{Dom } f + g = \text{Dom } f \cap \text{Dom } g$$

- Will this work for $f - g$? [Yes] / [No]

- Will this work for $f \times g$? [Yes] / [No]

- Will this work for $\frac{f}{g}$? [Yes] / [No]

Discussion: Why not for $\frac{f}{g}$?

$\text{Dom } f \cap \text{Dom } g \cap \{x : g(x) \neq 0\}.$

Definition

Sums, Differences, and Products of Functions

- **Rules:**

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(f \times g)(x) = f(x) \times g(x)$$

- **Idea:**

$$\textit{Domain of sum or product of two functions} = \textit{Intersection of the two domains}$$

- **Steps:**

1. Find the domain of each function
2. Find the intersection (draw number line if needed)

Let's look at some questions together!

Question 4 Walkthrough.

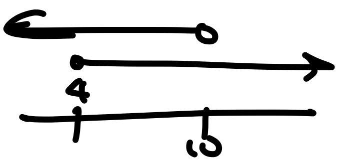
Find the maximal domain of the following function:

$$g(x) = \sqrt{x-4} + \log_3(10-x)$$

$x - 4 \geq 0$ $\qquad$ $10 - x > 0$

$x \geq 4$ $\qquad$ $10 > x$

$[4, \infty) \cap (-\infty, 10)$



$x \in [4, 10)$

TIP: Read the inequalities out loud to avoid making mistakes!

Recall!

Active Recall: To find the maximal domain we

- Find the domain of each function
- Find the intersection of the function domains

Your turn!

Question 5

Find the maximal domain of each of the following functions.

a. $\sqrt{10-x} + \frac{-1}{x-4}$

$10 - x \geq 0$

$10 \geq x$

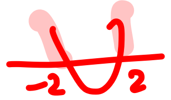
$x - 4 \neq 0$

$x \neq 4$

$x \in (-\infty, 10] \setminus \{4\}$. $= \underbrace{(-\infty, 4) \cup (4, 10]}$

b. $\log_3(x^2-4) + \frac{3}{x^2-1}$

$x^2 - 4 > 0$



$x^2 - 1 \neq 0$

$x \neq \pm 1$

$x \in (-\infty, -2) \cup (2, \infty)$

Question 6 Extension.

State the maximal domain of the following function.

$$y = \underbrace{\sqrt{4 - x}} - \log_3\left(\frac{1}{x + 4}\right)$$

$4 - x \geq 0$

$4 \geq x$

$x \in (-\infty, 4]$

$\frac{1}{x+4} > 0.$

$: x \in (-4, \infty)$

$x \in (-4, 4]$

Key Takeaways

- Inside of a log > 0.
- Inside of a root ≥ 0.
- Denominator $\neq 0$.
- Domain of sum or product of two functions = Intersection of the two domains.

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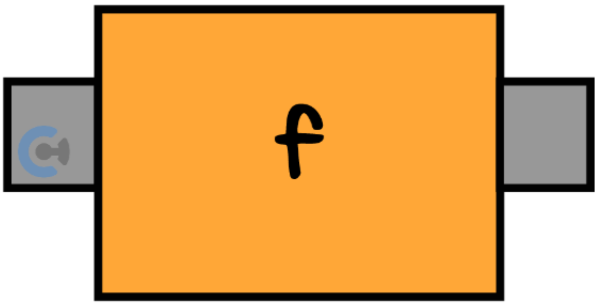
Section B: Composite Functions

Sub-Section: Basics of Composition

Analogy: Function and Machines

- Functions can be thought of as a simple machine.
  - Takes an input.
  - Performs some operation on that input.
  - Returns an output.



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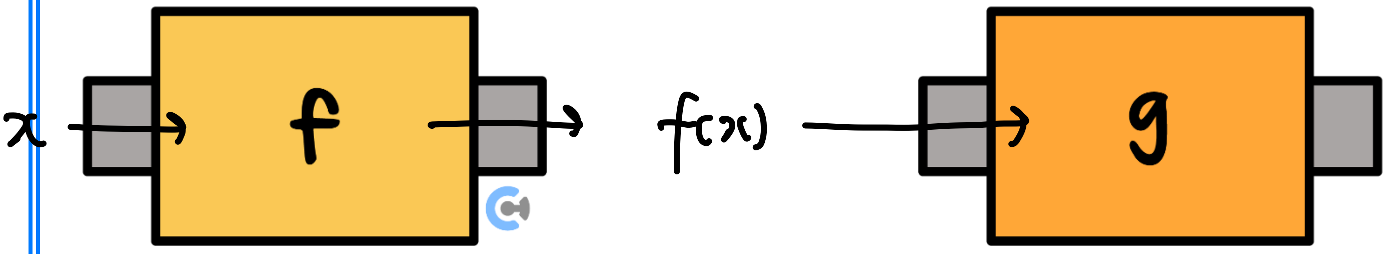
What would happen if we stacked two functions one after another?

Composite Functions



- Definition: A series of functions.
- Representation of the Above:

$$y = g(f(x)) = g \circ f(x)$$

NOTE: Inside Function = 1^{st} function in the series.

Try this question!

Question 7

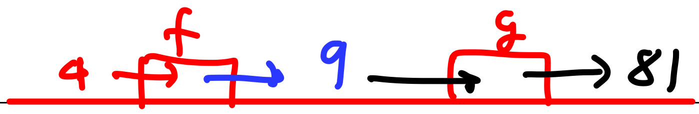
Consider two functions $f(x) = 2x + 1$ and $g(x) = x^2$ performed in order. That is, the **output from f becomes the input of g**.

a. $g(f(x))$:

b) What would be the output of the combined function if the initial input is $x = 4$?

a) $g(f(x)) = g(2x+1) = (2x+1)^2$

b) $g(f(4)) = (8+1)^2 = 81$



Sub-Section: Validity of Composite Functions

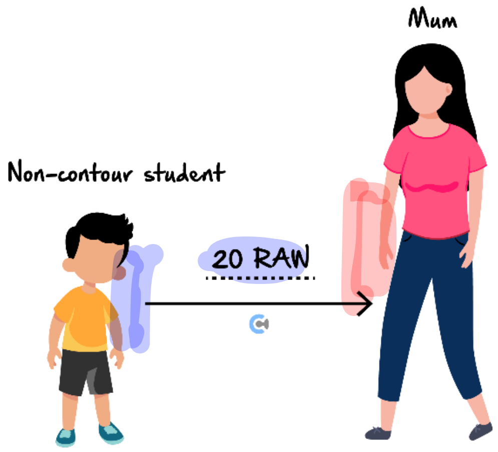
Do composite functions work all the time?

Analogy: Non-Contour Student Getting a 20 Raw.

- **Let's consider a Non-Contour student giving their study score to their mum.**
- Their mum is only willing to accept $[40\ Raw, 50\ Raw]$

$$\textit{Mum}: \text{"}\textit{Anything below is outside my domain}!\text{"}$$

- What would happen if the Non-Contour student gave their 20 Raw to their mum?



- Would this composition work? [Yes] / [No]

RIP Non-Contour Student

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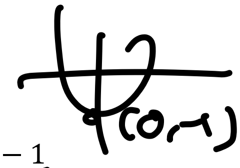
Let's turn the above analogy into a mathematical one!

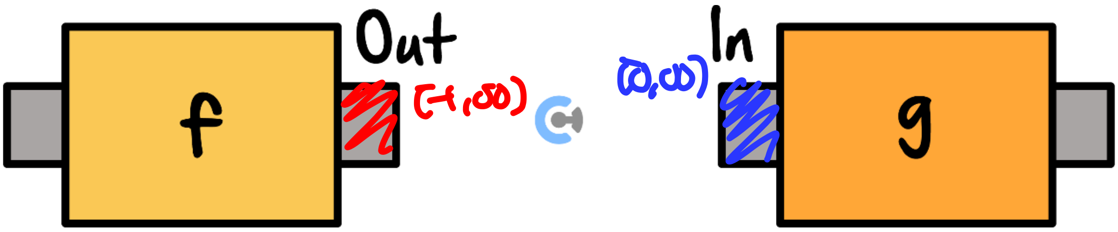
Exploration: Validity of Composition Function.

- Consider $g(f(x))$ for the following functions:



$$f: R \to R, f(x) = x^2 - 1$$

$$g: [0, \infty) \to R, g(x) = \sqrt{x}$$



- What range of values does $f(x)$ **produce**? [-1,∞)
- What range of values can $g(x)$ **accept**? [0,∞)
- So, can $g(x)$ take in **everything** that is outputted by $f(x)$? [Yes] / [No]
- Hence, can this composite function exist? [Yes] / [No]

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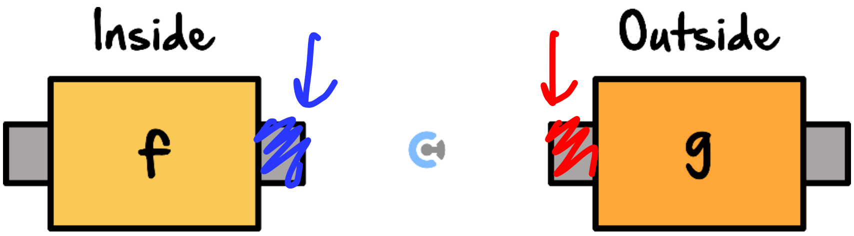
Validity of Composite Functions



- Output of $f(x)$: Range of Inside *(Label Above)*
- Input of $g(x)$: Dom of Outer *(Label Above)*
- Composite Function is only valid if:

Range of Inside $\subseteq$ Dom of Out

Subset = "part of".

- Acronym:

RIDO

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Let's look at some questions together!

Question 8 Walkthrough.

$x > 0$

Consider the functions $f(x) = x^2 - 4$ and $g(x) = \log_e(x)$ defined over their maximal domain.

a. Does $f(g(x))$ work?

Range g $\subseteq$ Dom f RIDO.

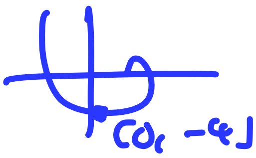
$\mathbb{R} \subseteq \mathbb{R}$

$\therefore f(g(x))$ exists

b. Does $g(f(x))$ work?

Range f $\not\subseteq$ Dom g

$[-4, \infty) \not\subseteq (0, \infty)$



$\therefore g(f(x))$ doesn't work

$g \circ f$

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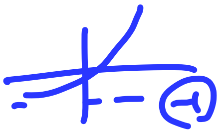
Your turn!

Question 9

Consider the functions $f(x) = x^3$ and $g(x) = 3^x - 1$ defined over their maximal domain.

a. Does $f(g(x))$ work?

Range $g \subseteq$ Dom f



$(-1, \infty) \subseteq \mathbb{R}$

$\therefore f(g(x))$ works

b. Does $g(f(x))$ work?

Range $f \subseteq$ Dom g

$\mathbb{R} \subseteq \mathbb{R}$

$\therefore g(f(x))$ works

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Question 10 Extension.

Consider the functions $f(x) = \sqrt{x+4}$ and $g(x) = x^2 - a$ defined over their maximal domain.

Given that $f(g(x))$ is defined, state the largest value of a.

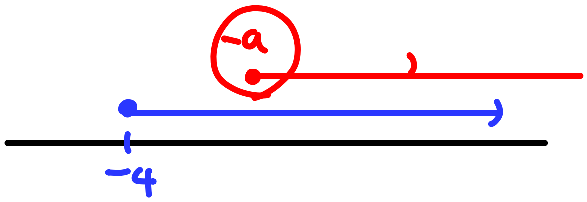
Range $g \subseteq$ Dom f

$[-a, \infty) \subseteq [-4, \infty)$

$-4 \leq -a$

$4 \geq a$

$\boxed{a = 4}$



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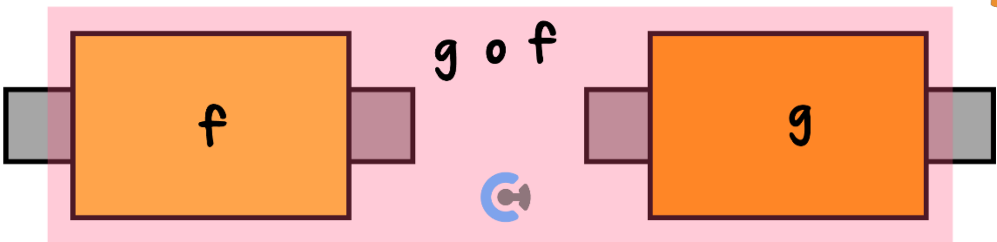
Sub-Section: Domain of Composite Functions

How do we find the domain of a composite function?

Exploration: Domain of the Composite Function

Input of the composition

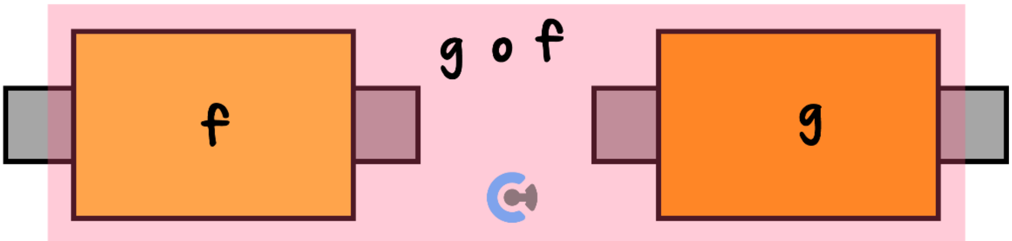


- Composite function input based on: [First] / [Second] Function
- Composite function domain based on: [First] / [Second] Function

$$\textit{Domain of Composite} = \textit{Domain of } [\textit{Inside}] \,/\, [\textit{Outside}]$$

Domain of Composite Functions

Input of the composition



$$\textit{Domain of Composite} = \textit{Domain of Inside}$$

* If comp exists ($R \subseteq D$ ✓)

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Try the following question!

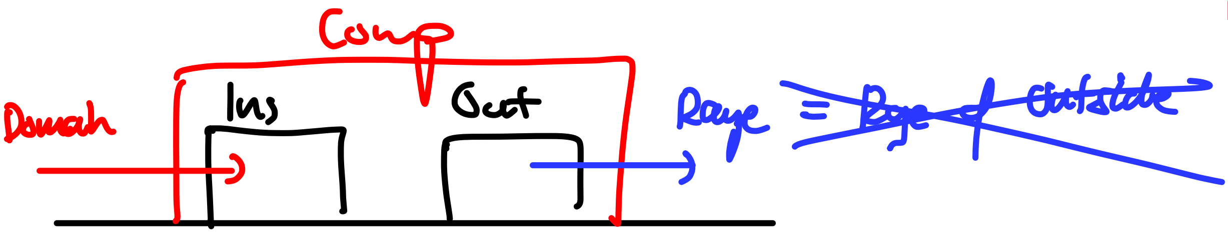
Question 11

Consider the functions $f(x) = \sqrt{x+4}$ and $g(x) = x^2 + 4$ defined over their maximal domain.

State the domain of the composite function $g(f(x))$.

$$\text{Dom } g(f(x)) = \text{Dom } f$$
$$= [-4, \infty)$$

<u>Discussion:</u> What would the range of the composite function be then?



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Sub-Section: Range of Composite Functions

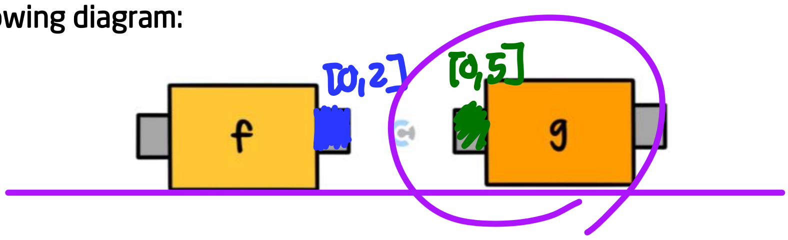
Misconception

"The range of a composite function must be the range of the outside function."

TRUTH: The range of the composite function is a subset of the range of the outside function.

Exploration: Range of Composite Functions

- Consider the following diagram:



- Consider that: Range of f: $[0, 2]$, Domain of g: $[0, 5]$

- Does the composite function work? [Yes] / [No]
- Does $f(x)$ give $g(x)$ every possible value $g(x)$ can take? [Yes] / [No]
- Does the function g in the composite function produce the entire range of g? [Yes] / [No]
- Why? 1st func only gives "some of" wht 2nd can take.
- Would the range of the composite function equal to the range of $g(x)$? [Yes] / [No]

$$\textit{Range Comp} \ \underline{\ \subseteq\ } \ \textit{Range Outside}$$

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Range of the Composite Functions

Range of Composite ⊆ Range of the Outside

- Finding the range of composition function: Use the domain and the rule, just like another function.

Your turn!

Question 12

Consider the functions:

$$f: R \to R, f(x) = x^2 + 4$$
$$g: R \to R, g(x) = x + 6$$

a. For the composite function $g(f(x))$, state the rule and domain.

$g(x^2+4) = x^2+4+6$

$= x^2+10, \quad x \in \mathbb{R}.$

b. State the range of $g(f(x))$.

$y = x^2+10, \quad x \in \mathbb{R}$

$\therefore \quad y \in [10, \infty)$

c. State the range of $g(x)$.

$y = x+6, \quad x \in \mathbb{R}$

$y \in \mathbb{R}$

d. Explain why the range of g is not the same as the range of $g \circ f$.

$g \circ f$ Range f is not the entire domain of g

<u>Discussion:</u> To make $\boldsymbol{Range\ Comp} = \boldsymbol{Range\ Outside}$, what must be the range of inside equal to?

Range Inside = Dom Out

Max'ed out!

<u>Key Takeaways</u>

- $f(g(x)) = f \circ g(x)$.
- Range (output) of Inside $\subseteq$ Domain (Input) of Outside.
- Domain of Composite = Domain of Inside (1^{st}) Function.
- Range of Composite $\subseteq$ Range of the Outside.

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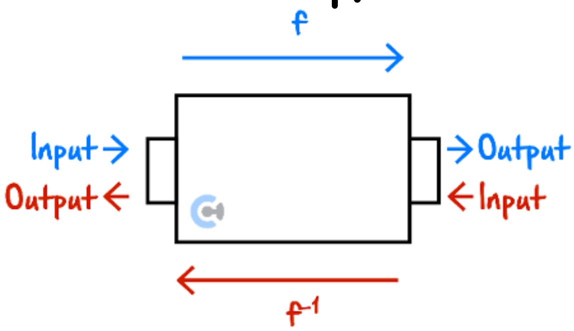
Section C: Inverse Functions

Sub-Section: Basics of Inverses

What does "Inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the opposite



Discussion: What would be the inverse of $f(x) = x + 2$?

$f^{-1}(x) = x - 2$

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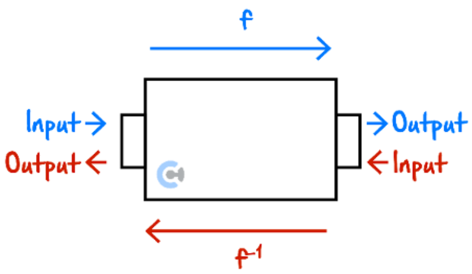
Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

Solving for an Inverse Relation

- Swap x and y.



Question 13

Find the inverse of $f(x) = 3x - 1$ by swapping x and y.

Swap x & y for inv.

$x = 3y - 1$

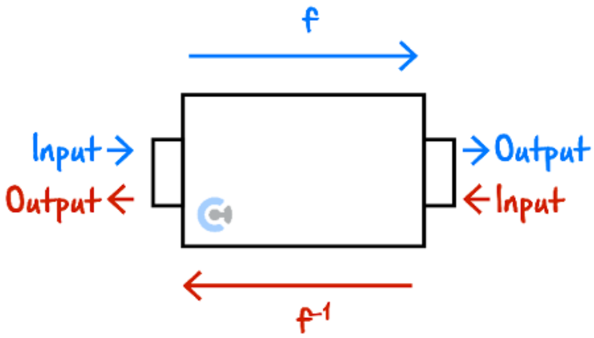
$y = \frac{1}{3}(x+1)$

$f^{-1}(x) = \frac{1}{3}(x+1)$

NOTE: $f(x) = y$.

Discussion: Hence, what would happen to the domain and range of the function when we find its inverse?

Domain and Range of Inverse Functions



$Dom\ f^{-1} = $ Range f

$Ran\ f^{-1} = $ Dom f

Question 14 Walkthrough.

Consider the function $f(x) = \sqrt{x+2} - 1$ defined for its maximal domain.

a. Find the rule for the inverse function.

b. State the domain and range of the inverse function.

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Question 15

Consider the function $f: [0, 4] \to R, f(x) = 2x + 1$.

a. Find the rule for the inverse function.

Swap x & y for inverse

$x = 2y + 1$

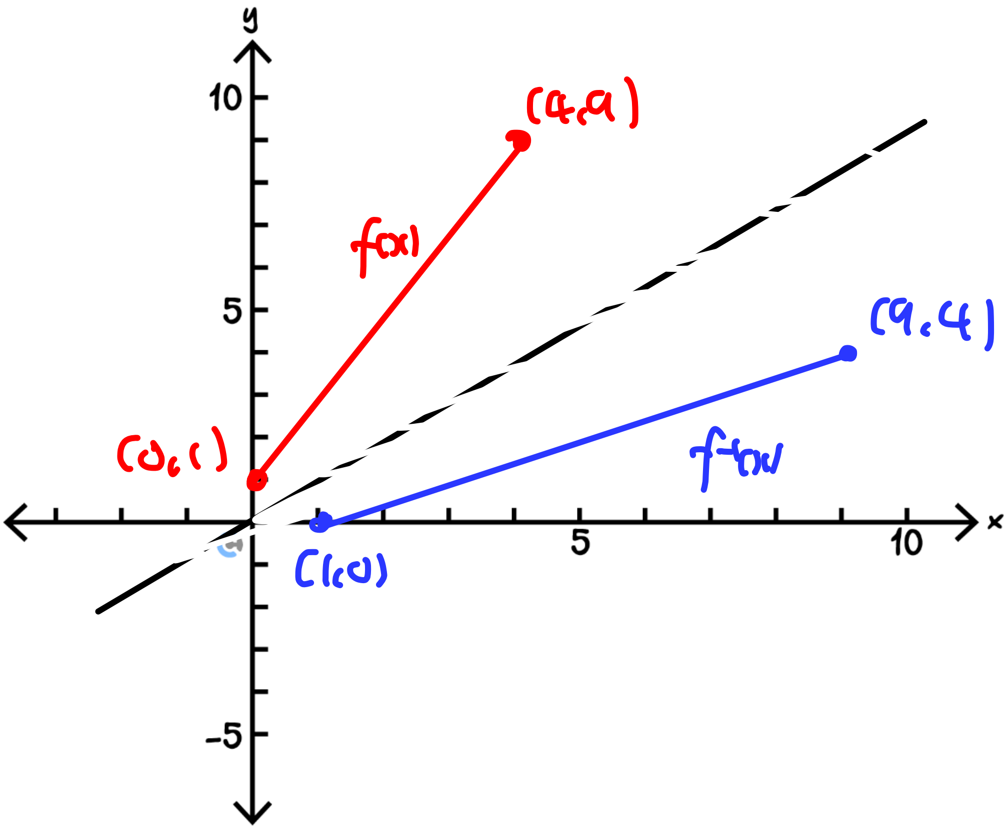
$\frac{x-1}{2} = y$. $\therefore f^{-1}(x) = \frac{x-1}{2}$

b. State the domain and range of the inverse function.

Dom = Range f = $[f(0), f(4)] = [1, 9]$

Range = Dom f = $[0, 4]$

c. Sketch the $f(x)$ and $f^{-1}(x)$ on the axis below.



Discussion: In the previous question, which line were the two inverses symmetrical along?

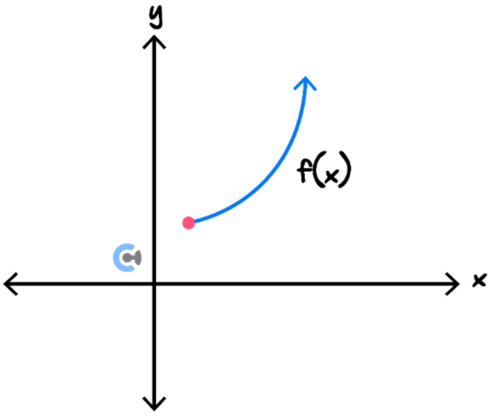
$y = x$

Sub-Section: Symmetry Around $y = x$

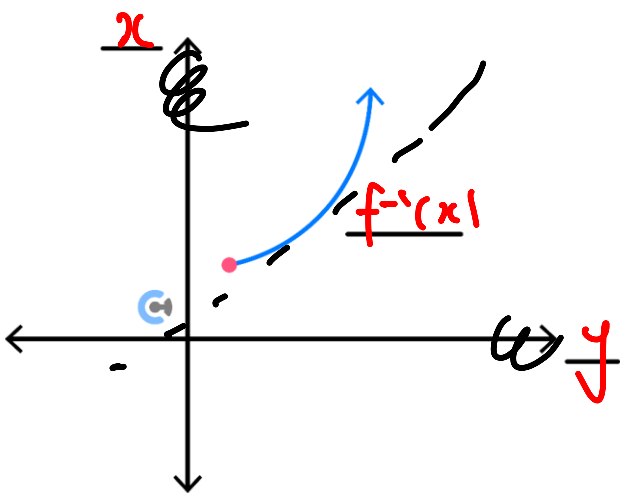
Why does this happen?

Exploration: Symmetry Around $y = x$

- Consider the following function:

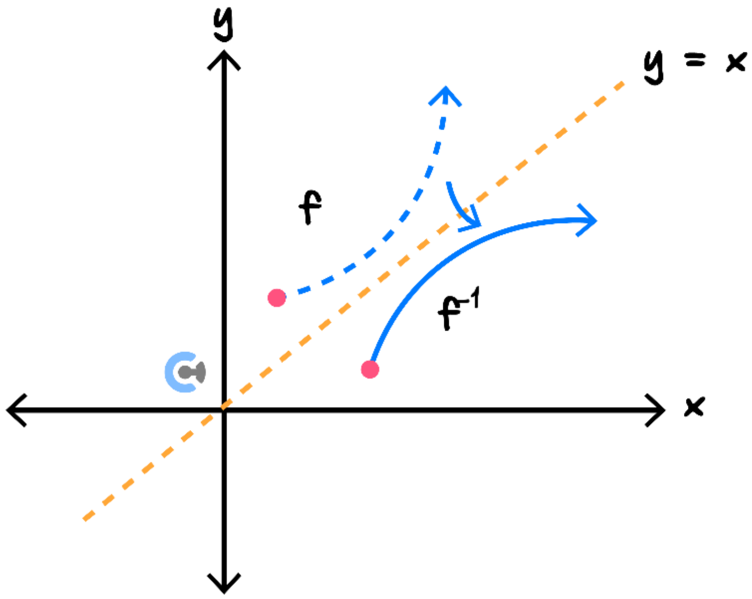


- What happens if you swap the x- and y-axis on the label on our graph?

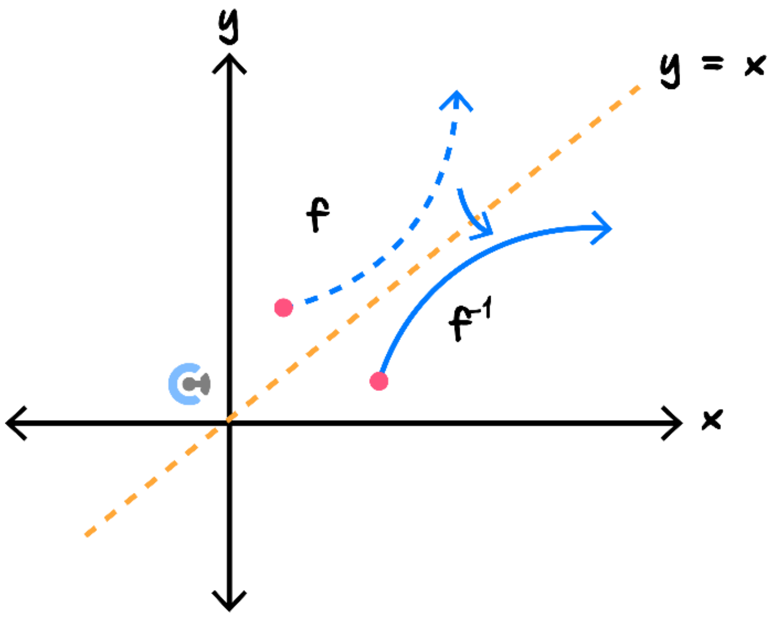


- Wait... do we want the x-axis to be the vertical one? [Yes/No]

- How should we reflect the graph so that the x- and y-axis become horizontal and vertical again?



<u>Symmetry of Inverse Functions</u>



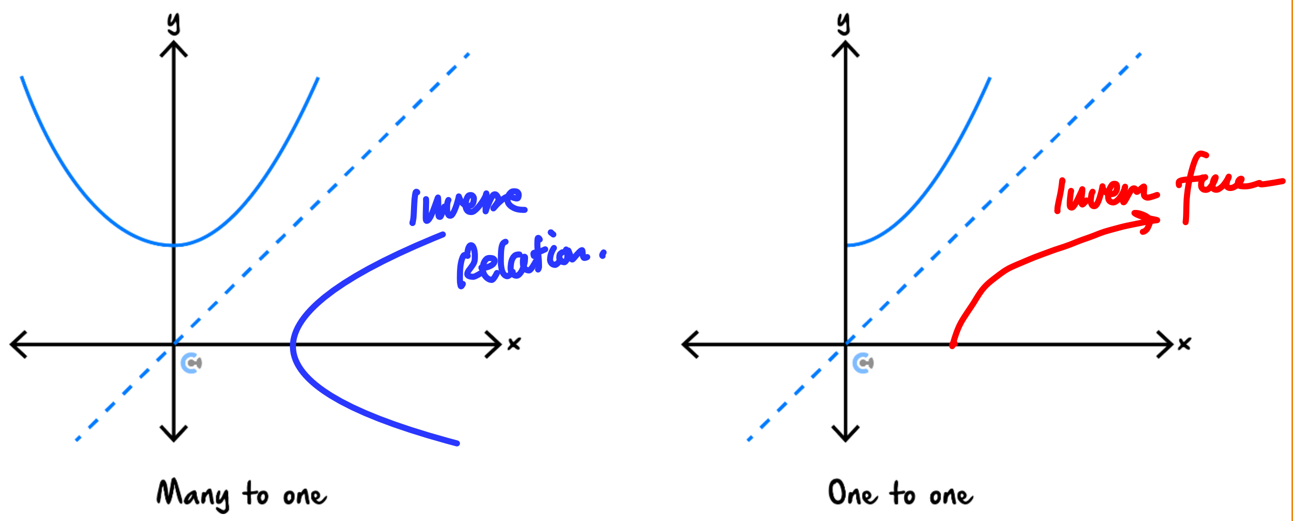
- Inverse functions are always symmetrical around $y = x$.

Sub-Section: Validity of Inverse Function

Does an inverse function always exists?

Exploration: Validity of Inverse Functions

- Consider the many to one and one to one functions.



- Reflect them around $y = x$ and sketch the inverse! *(Label Above)*
- Which inverse is a function? (Passes through a vertical line test?)

[neither] / [left] / [right] / [both]

- For an inverse **function** to exist, what must the original function be? [many to one] / [one to one]

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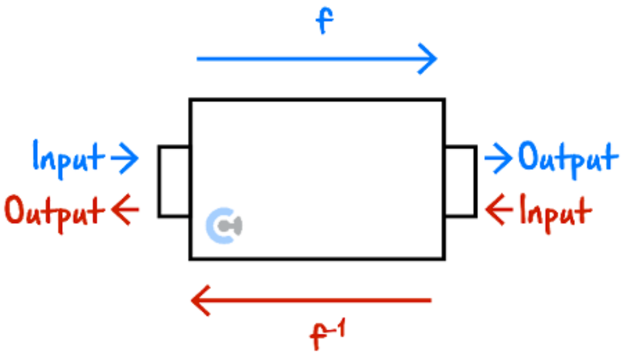
Validity of Inverse Functions



- **Requirement for Inverse Function:**

f ***needs to be*** 1:1 (t.p)

Question 16 Walkthrough.

Consider the function $f: (-\infty, a] \rightarrow \mathbb{R}, f(x) = 3(x-2)^2 - 4$.

a. Find the largest possible value of a such that the ~~inverse function f^{-1} exists.~~

f is 1:1

$a = 2$

b. Find the inverse function.

rule + domain.

Swap x & y for inv.

$x = 3(y-2)^2 - 4$

$\frac{x+4}{3} = (y-2)^2$

$\pm\sqrt{\frac{x+4}{3}} = y - 2$

$y = 2 \pm \sqrt{\frac{x+4}{3}}$

Range f^{-1} = Dom f

$= (-\infty, 2]$.

$f^{-1}(x) = 2 - \sqrt{\frac{x+4}{3}}$,

Dom f^{-1} = Range f

$= [-4, \infty)$

TIP: Always try sketching the function to find the domain such that an inverse function can exist!

NOTE: You will need to complete the square when finding the inverse of quadratic functions!

Your turn!

Question 17

Consider the function $g: (-\infty, b] \to \mathbb{R}, g(x) = -2x^2 - 8x + 1$.

$-2[x^2+4x]$

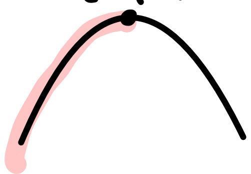
$= -2[(x+2)^2-4]$

$= -2(x+2)^2+8$

$+1$

a. Find the largest possible value of b such that the inverse function g^{-1} exists.

$(-2, 9)$



$g(x) = -2(x+2)^2+9$

$b = -2$

b. Find the inverse function

Swap x & y for inverse

$x = -2(y+2)^2 + 9$

$\frac{x-9}{-2} = (y+2)^2$

$\pm\sqrt{\frac{x-9}{-2}} = y+2$

$-2 \pm \sqrt{\frac{x-9}{-2}} = y$

as Range f^{-1} = Dom f

$= (-\infty, -2]$

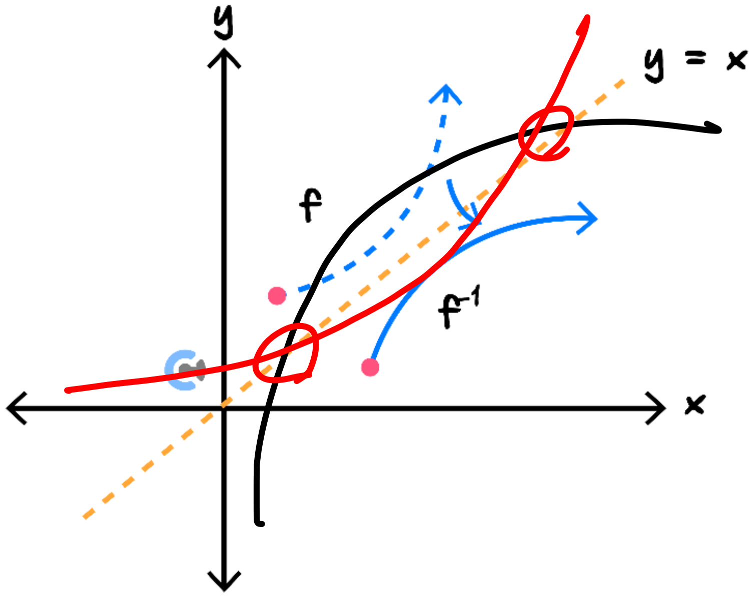
$\therefore f^{-1}(x) = -2 - \sqrt{\frac{x-9}{-2}}$

Dom f^{-1} = Range f $= (-\infty, 9]$

Sub-Section: Intersection Between Inverses

Where do inverses meet?

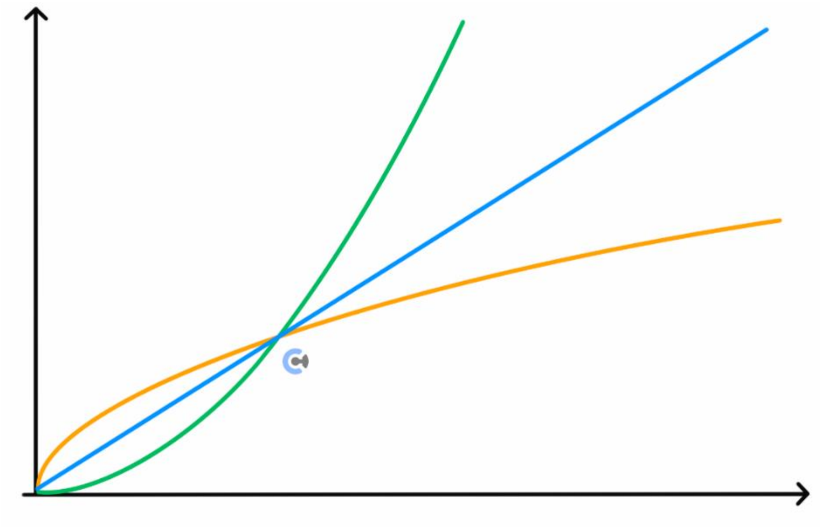
Active Recall: Symmetry Around $y = x$



- Inverse functions are always symmetrical around $y=x$.

Discussion: Where could a function and its inverse meet?

Intersection Between a Function and its Inverse



Delete

- Always equate with $y = x$ instead.

$$f(x) = x \text{ OR } f^{-1}(x) = x$$

Question 18

Find the intersection between $f: [0, \infty) \to R, f(x) = x^3$ and its inverse, without finding the inverse.

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x = 0, \pm 1$$

$$x \neq -1$$

as $x \in [0, \infty)$

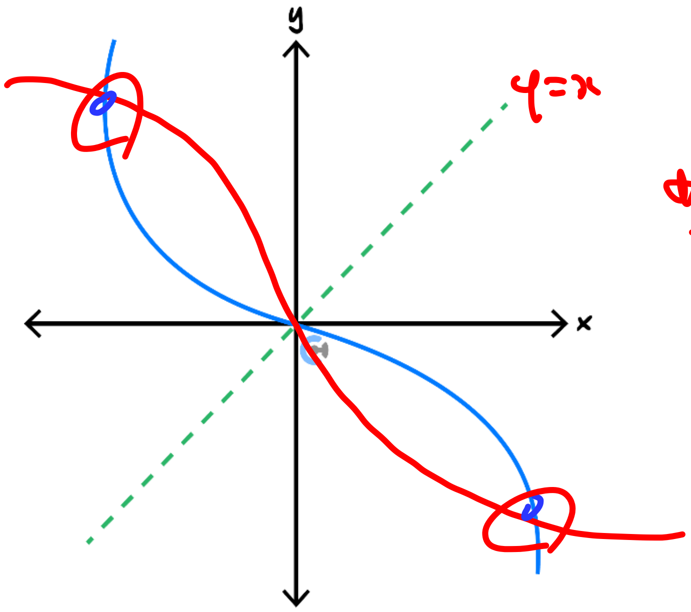
$$\boxed{(0,0), \ (1,1)}$$

NOTE: This only works for an increasing function, however in VCAA, this is always the case.

Does this always work?

Extension: Intersections Not on $y = x$ (Not Tested on Exams, but Maybe on SACs!)

- Consider the following:



*Decreasing fun

- What does the inverse function look like? *(Sketch Above)*
- Are these intersections on $y = x$? [Yes] / [No]

ALSO NOTE: For SACs is that there **could** be intersections that are **not** on $y = x$.

Space for Personal Notes

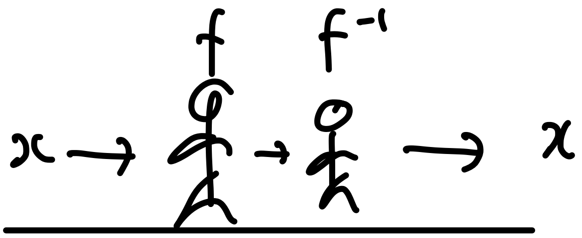
Sub-Section: Composition of Inverses

Analogy: Inverse function is your annoying sibling.

- Who has siblings? Is it just me or do your siblings want to do the opposite of what you want to do?
- **Example:**
  - ~~James~~ Jayesh: Wants to turn the AC up by 5 degrees.
  - ~~Danis~~ Aadi (James' brother): Turns the AC down by 5.

This is basically an inverse function relationship!

Discussion: So, now what would happen if we have a function and its inverse happening one after another? (Composite function of inverse)



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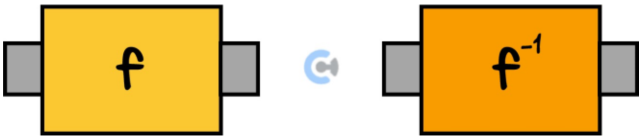
Definition

Composition of Inverse Functions



$f \circ f^{-1}(x) = x$, for all $x \in$ Dom f^{-1}

$f^{-1} \circ f(x) = x$, for all $x \in$ Dom f

NOTE: Domain = Domain of Inside

Try this question!

Question 19 (4 marks)

Consider the function $f(x) = \frac{1}{x-1} - 3$.

a. Find the rule and domain for $f^{-1}(f(x))$. (2 marks)

$$f^{-1}(x) = \frac{1}{x+3} + 1$$

$$f^{-1}(f(x)) = \frac{1}{f(x)+3} + 1$$

$$= \frac{1}{\frac{1}{x-1} - 3 + 3} + 1$$

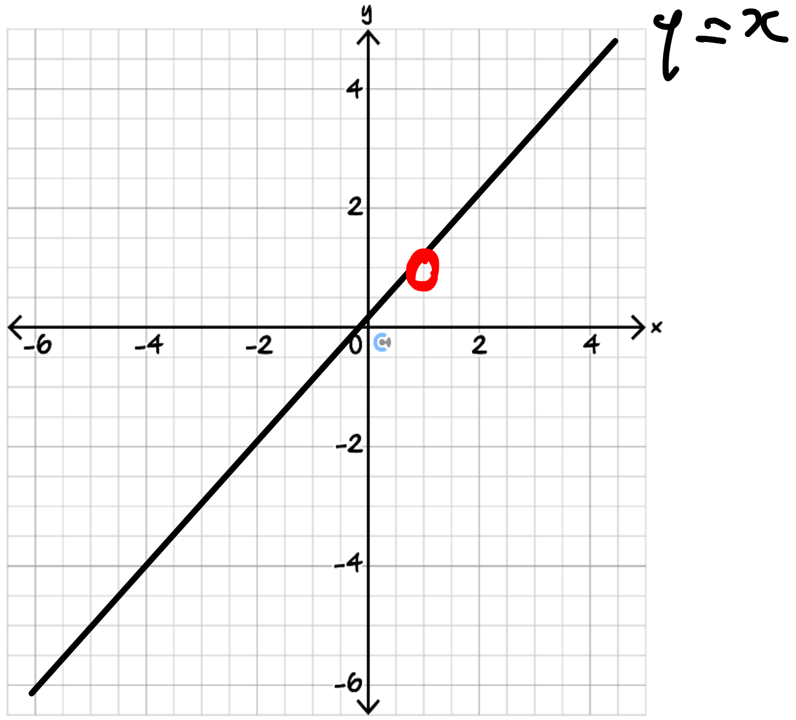
$$= \frac{1}{\frac{1}{x-1}} + 1$$

$$= x - 1 + 1 = x$$

Dom = Dom f

$= \mathbb{R} \setminus \{1\}$

b. Sketch the graph of $y = f^{-1}(f(x))$ on the axes below. (2 marks)



Key Takeaways

- f needs to be $1:1$ for f^{-1} to exist.
- Domain and Range Swaps.
- Symmetrical around $y = x$.
- For intersections: $f(x) = x$ or $f^{-1}(x) = x$.
- Composite function of inverses is always equal to x.

Space for Personal Notes

Contour Check

Learning Objective: [1.1.1] – Find Maximal Domain and Range

Key Takeaways

- Inside of a $\log$ must be positive.
- Inside of a root must be non-negative.
- Denominator cannot be zero.
- Domain of sum or product of two functions is equal to intersection. of the two domains.

Learning Objective: [1.1.2] – Find the Rule, Domain and Range of a Composite Function (Range Does Not Require Splitting to Find as the Function is Easy to Draw)

Key Takeaways

- $f(g(x)) = f \circ g\,(x)$.
- For composite function to exist, Range inside $\subseteq$ Dom outside.
- Domain of Composite is equal to the Domain of Inside Function.
- Range of Composite is a subset of the Range of the Outside.

Learning Objective: [1.1.3] – Find the Rule, Domain, and Range of Inverse Functions

Key Takeaways

- f needs to be 1:1 $\text{for } f^{-1}$ to exist.
- Domain of the inverse function equals to Range of origin and vice versa.
- Symmetrical around $y=x$.
- For intersections of inverses, we can equate the function to $y=x$.

Learning Objective: [1.1.4] – Find the Composite Function of Inverse Function

Key Takeaways

- Composite function of inverses is always equal to $y=x$.