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VCE Mathematical Methods ¾
Functions & Relations [1.1]
Workbook

Outline:

Section A: Domain of Functions

Sub-Section: Maximal Domains

Functions and Relations

- Our topics today:
  1. Domain
  2. Composite Functions
  3. Inverse Functions
- None of these can be understood without being able to find a domain of a function.
- Today's class will get progressively harder so be sure to ask questions when you have.

Starting with domain!

Maximal Domain

- **Definition**: The largest possible set of input values (elements of the domain) for which the function is well-defined.
- **Three Important Rules:**

Functions	Maximal Domain
$\sqrt{z}$	
$\log(z)$	
$\frac{1}{z}$	

Steps

1. Find the restriction of the inside.
2. Sketch the graph if needed.
3. Solve for domain.

Let's have a look at a question together!

Question 1 Walkthrough.

Find the maximal domain of each of the following functions.

a. $f(x) = 3\sqrt{4x+3} - 2$

b. $h(x) = \log_2(-x^2 + 16)$

Active Recall: Steps to Find Maximal Domain

1. Find the restriction of the _____.
2. Sketch the _____ if needed.
3. Solve for _____.

Your turn!

Question 2

Find the maximal domain of the following functions.

a. $f(x) = -\sqrt{-2x-4}+1$

b. $\frac{1}{x^2-9}$

c. $h(x) = -\log_2(x^2 + 4x - 5)$

Question 3 Extension.

State the maximal domain of the following function.

$$y = \frac{1}{\sqrt{x^2 + 3x + 2}}$$

Space for Personal Notes

Sub-Section: Domain of Sum, Difference, and Product of Functions

What about a domain of the sum of two functions?

Analogy: Students

- Consider two students with different availabilities:

Student	Function	Availability
	f	$10\text{ A. M.} - 2\text{ P. M.}$
	g	$11\text{ A. M.} - 5\text{ P. M.}$

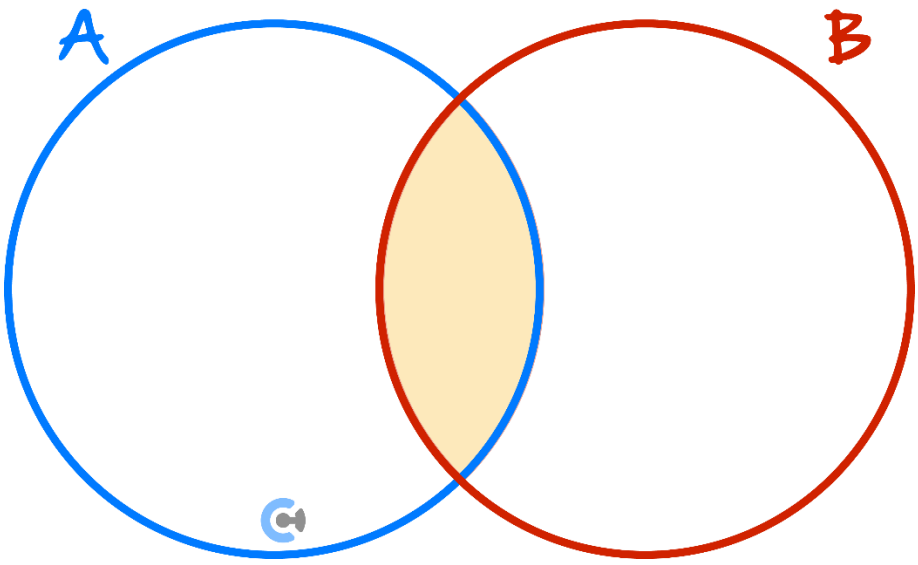
- When can these two meet?

Meeting	$f+g$	

This is the same as finding the domain of the sum of two functions!

Exploration: Domain of Sum, Differen

- If the domain of f is A and the domain of g is B, what would be the **domain of $f + g$**?



- For $f + g$ to be defined, do both f and g be defined? [Yes] / [No]
- How can both f and g defined? (Hint: Look at the diagram above.)

$$Dom\ f + g = \_\_\_\_\_$$

- Will this work for $f - g$? [Yes] / [No]
- Will this work for $f \times g$? [Yes] / [No]
- Will this work for $\frac{f}{g}$? [Yes] / [No]

Discussion: Why not for $\frac{f}{g}$?

Sums, Differences, and Products of Functions

- Rules:

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(f \times g)(x) = f(x) \times g(x)$$

- Idea:

Domain of sum or product of two functions = Intersection of the two domains

- Steps:

1. Find the domain of each function
2. Find the intersection (draw number line if needed)

Let's look at some questions together!

Question 4 Walkthrough.

Find the maximal domain of the following function:

$$g(x) = \sqrt{x-4} + \log_3(10-x)$$

TIP: Read the inequalities out loud to avoid making mistakes!

Recall!

Active Recall: To find the maximal domain we

- Find the _____ of each function
- Find the _____ of the function domains

Your turn!

Question 5

Find the maximal domain of each of the following functions.

a. $\sqrt{10-x}+\frac{-1}{x-4}$

b. $\log_3(x^2-4)+\frac{3}{x^2-1}$

Question 6 Extension.

State the maximal domain of the following function.

$$y = \sqrt{4 - x} - \log_3\left(\frac{1}{x + 4}\right)$$

Key Takeaways

- Inside of a log > 0.
- Inside of a root ≥ 0.
- Denominator $\neq 0$.
- Domain of sum or product of two functions = Intersection of the two domains.

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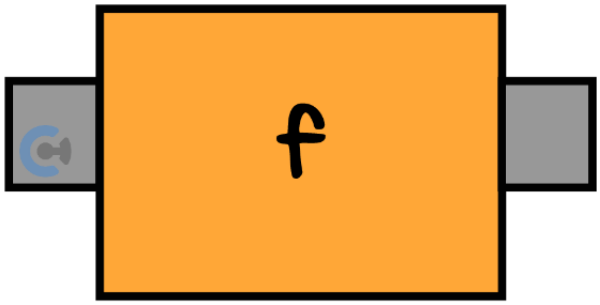
Section B: Composite Functions

Sub-Section: Basics of Composition

Analogy: Function and Machines

- Functions can be thought of as a simple machine.
  - Takes an _____.
  - Performs some _____ on that input.
  - Returns an _____.



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What would happen if we stacked two functions one after another?

Composite Functions

- **Definition**: A_____ of functions.
- **Representation of the Above:**

$$y = \_\_\_\_\_$$

NOTE: Inside Function = 1^{st} function in the series.

Try this question!

Question 7

Consider two functions $f(x) = 2x + 1$ and $g(x) = x^2$ performed in order. That is, the **output from f becomes the input of g**.

What would be the output of the combined function if the initial input is $x = 4$?

Sub-Section: Validity of Composite Functions

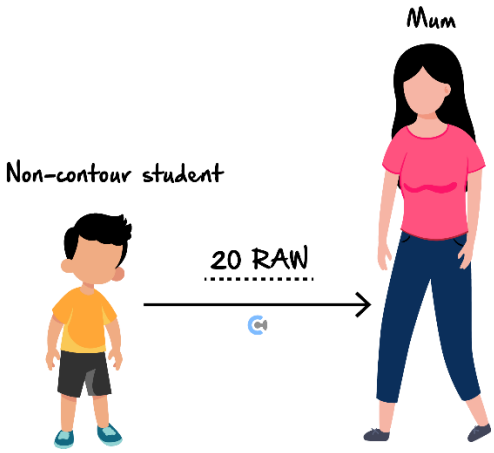
Do composite functions work all the time?

Analogy: Non-Contour Student Getting a 20 Raw.

- **Let's consider a Non-Contour student giving their study score to their mum.**
- Their mum is only willing to accept $[40\ Raw, 50\ Raw]$

Mum: "Anything below is outside my domain!"

- What would happen if the Non-Contour student gave their 20 Raw to their mum?



- Would this composition work? [Yes] / [No]

RIP Non-Contour Student

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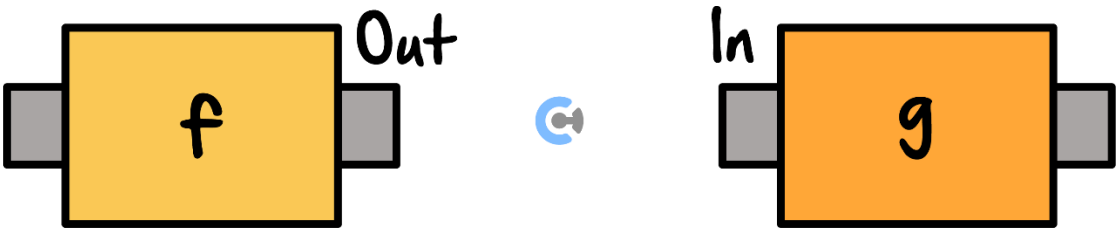
Let's turn the above analogy into a mathematical one!

Exploration: Validity of Composition Function.

- Consider $g(f(x))$ for the following functions:

$$f: R \to R, f(x) = x^2 - 1$$

$$g: [0, \infty) \to R, g(x) = \sqrt{x}$$



- What range of values does $f(x)$ **produce**? _____
- What range of values can $g(x)$ **accept**? _____
- So, can $g(x)$ take in **everything** that is outputted by $f(x)$? [Yes] / [No]
- Hence, can this composite function exist? [Yes] / [No]

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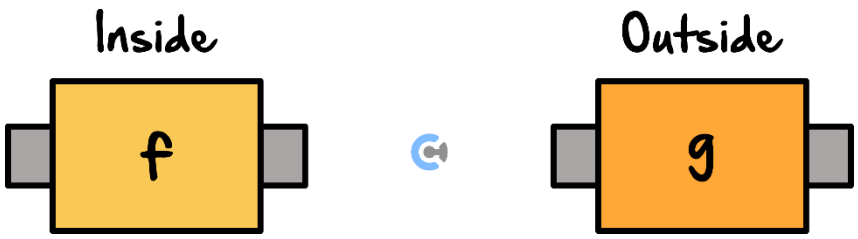
Validity of Composite Functions



- Output of $f(x)$: _____ *(Label Above)*
- Input of $g(x)$: _____ *(Label Above)*
- Composite Function is only valid if:

_____ $\subseteq$ _____

- Acronym:

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Let's look at some questions together!

Question 8 Walkthrough.

Consider the functions $f(x) = x^2 - 4$ and $g(x) = \log_e(x)$ defined over their maximal domain.

a. Does $f(g(x))$ work?

b. Does $g(f(x))$ work?

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Your turn!

Question 9

Consider the functions $f(x) = x^3$ and $g(x) = 3^x - 1$ defined over their maximal domain.

a. Does $f(g(x))$ work?

b. Does $g(f(x))$ work?

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Question 10 Extension.

Consider the functions $f(x) = \sqrt{x+4}$ and $g(x) = x^2 - a$ defined over their maximal domain.

Given that $f(g(x))$ is defined, state the largest value of a.

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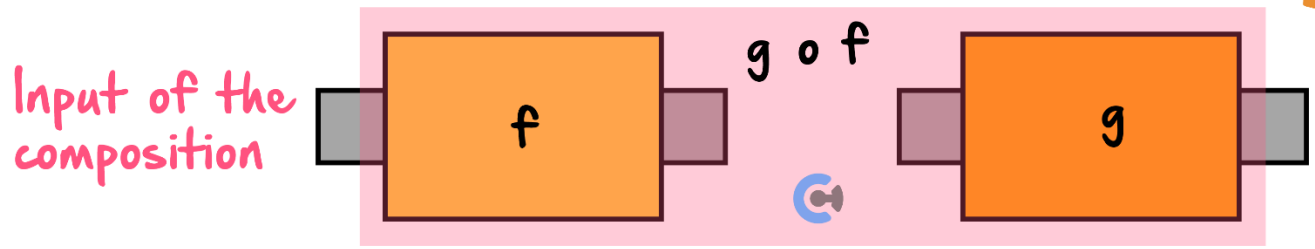
Sub-Section: Domain of Composite Functions

How do we find the domain of a composite function?

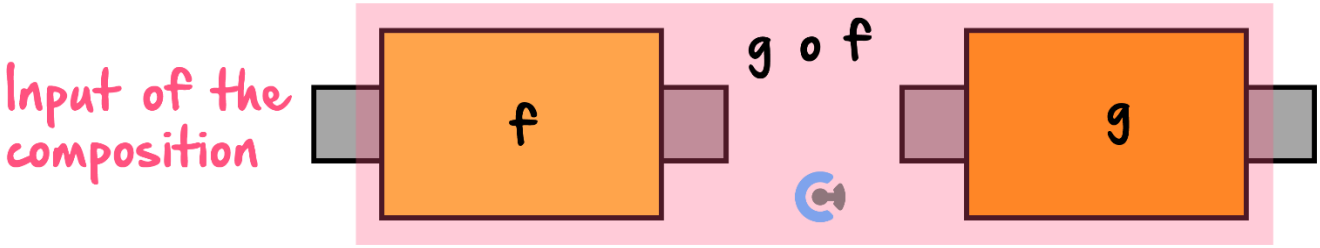
Exploration: Domain of the Composite Function



- Composite function input based on: [First] / [Second] Function
- Composite function domain based on: [First] / [Second] Function

$$Domain\ of\ Composite = Domain\ of\ [Inside]\ /\ [Outside]$$

Domain of Composite Functions



$$Domain\ of\ Composite = Domain\ of\ Inside$$

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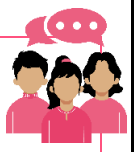
Try the following question!

Question 11

Consider the functions $f(x) = \sqrt{x+4}$ and $g(x) = x^2 + 4$ defined over their maximal domain.

State the domain of the composite function $g(f(x))$.



Discussion: What would the range of the composite function be then?

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Sub-Section: Range of Composite Functions

Misconception

"The range of a composite function must be the range of the outside function."

TRUTH: The range of the composite function is a subset of the range of the outside function.

Exploration: Range of Composite Functions

- Consider the following diagram:

- Consider that: Range of f: $[0, 2]$, Domain of g: $[0, 5]$
  - Does the composite function work? [Yes] / [No]
  - Does $f(x)$ give $g(x)$ every possible value $g(x)$ can take? [Yes] / [No]
  - Does the function g in the composite function produce the entire range of g? [Yes] / [No]
  - Why? _____
  - Would the range of the composite function equal to the range of $g(x)$? [Yes] / [No]

Range Comp _____ Range Outside

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Range of the Composite Functions

Range of Composite ⊆ Range of the Outside

- **Finding the range of composition function**: Use the domain and the rule, just like another function.

Your turn!

Question 12

Consider the functions:

$$f: R \to R, f(x) = x^2 + 4$$
$$g: R \to R, g(x) = x + 6$$

a. For the composite function $g(f(x))$, state the rule and domain.

b. State the range of $g\big(f(x)\big)$.

c. State the range of $g(x)$.

d. Explain why the range of g is not the same as the range of $g \circ f$.

Discussion: To make $\boldsymbol{Range\ Comp = Range\ Outside}$, what must be the range of inside equal to?

Key Takeaways

- $f(g(x)) = f \circ g(x)$.
- Range (output) of Inside $\subseteq$ Domain (Input) of Outside.
- Domain of Composite = Domain of Inside (1^{st}) Function.
- Range of Composite $\subseteq$ Range of the Outside.

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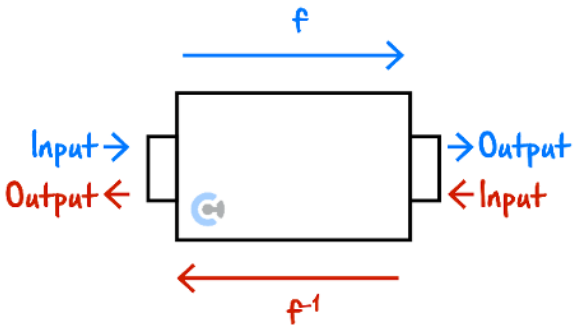
Section C: Inverse Functions

Sub-Section: Basics of Inverses

What does "Inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the _____



Discussion: What would be the inverse of $f(x) = x + 2$?

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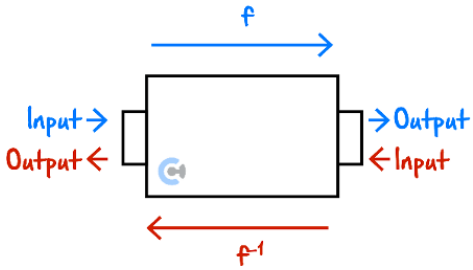
Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

Solving for an Inverse Relation

- Swap x and y.



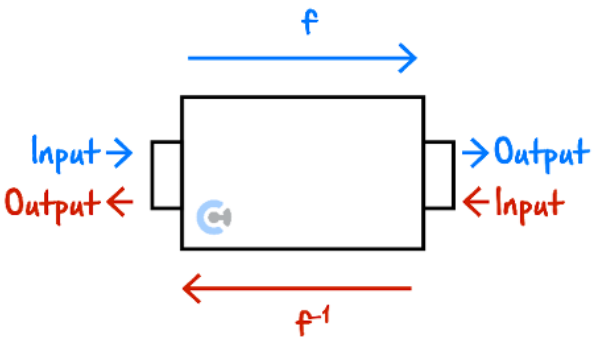
Question 13

Find the inverse of $f(x) = 3x - 1$ by swapping x and y.

NOTE: $f(x) = y$.

Discussion: Hence, what would happen to the domain and range of the function when we find its inverse?

Domain and Range of Inverse Functions



$Dom\ f^{-1} =$ _____

$Ran\ f^{-1} =$ _____

Question 14 Walkthrough.

Consider the function $f(x) = \sqrt{x+2} - 1$ defined for its maximal domain.

a. Find the rule for the inverse function.

b. State the domain and range of the inverse function.

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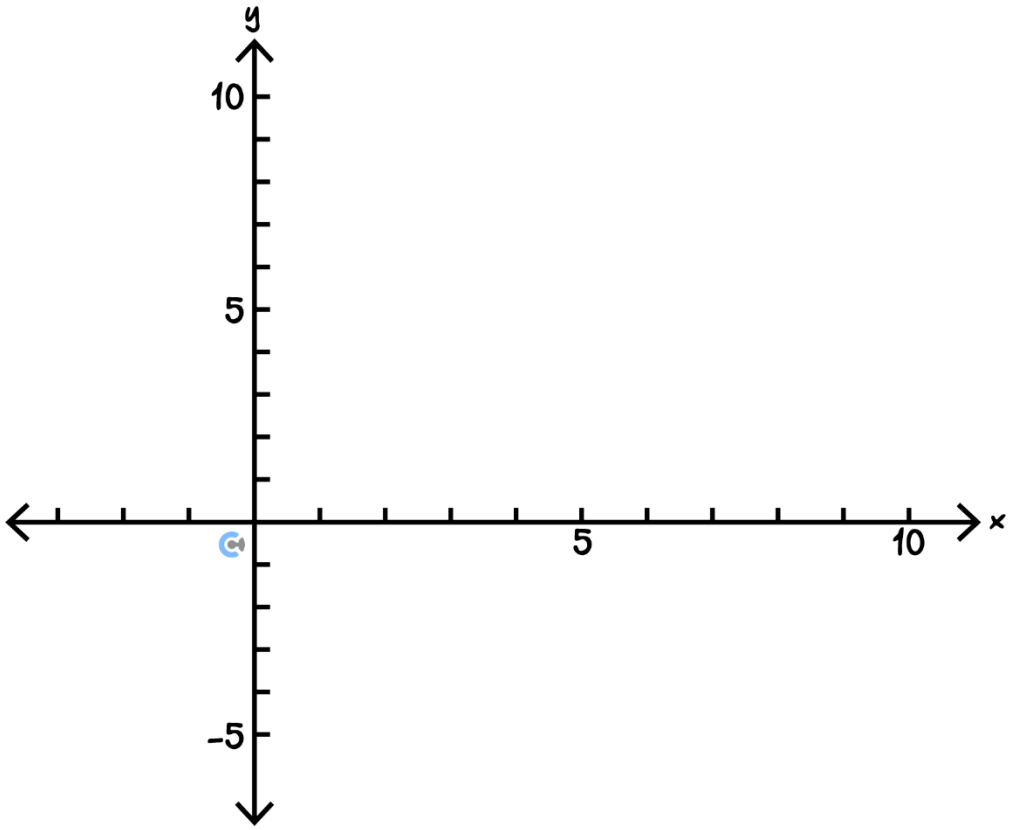
Question 15

Consider the function $f: [0, 4] \rightarrow R, f(x) = 2x + 1$.

a. Find the rule for the inverse function.

b. State the domain and range of the inverse function.

c. Sketch the $f(x)$ and $f^{-1}(x)$ on the axis below.



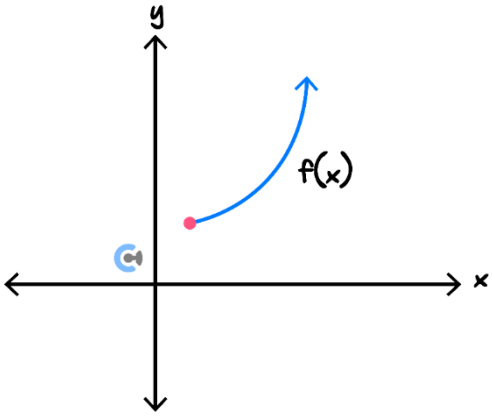
Discussion: In the previous question, which line were the two inverses symmetrical along?

Sub-Section: Symmetry Around $y = x$

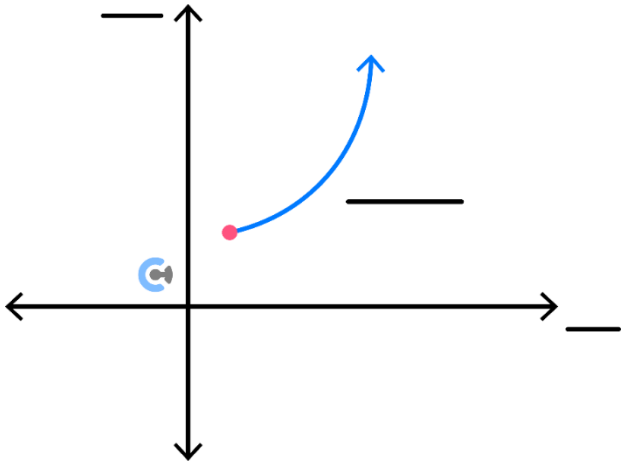
Why does this happen?

Exploration: Symmetry Around $y = x$

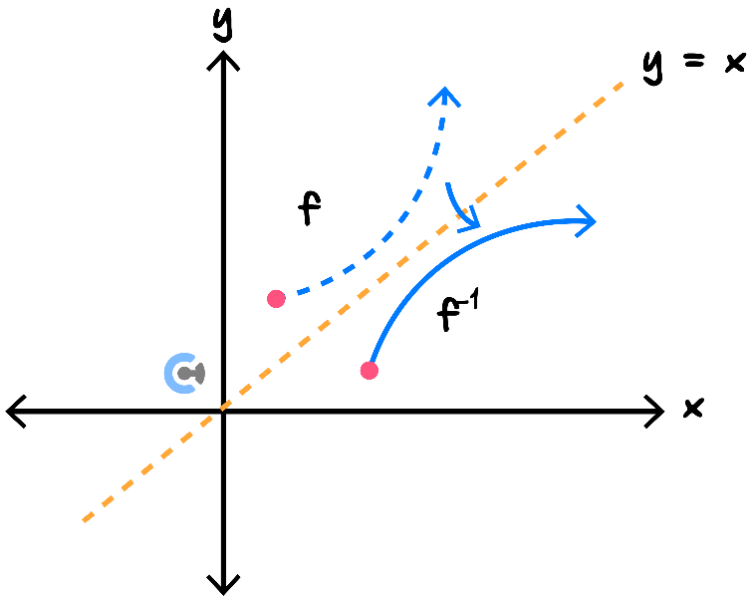
- Consider the following function:



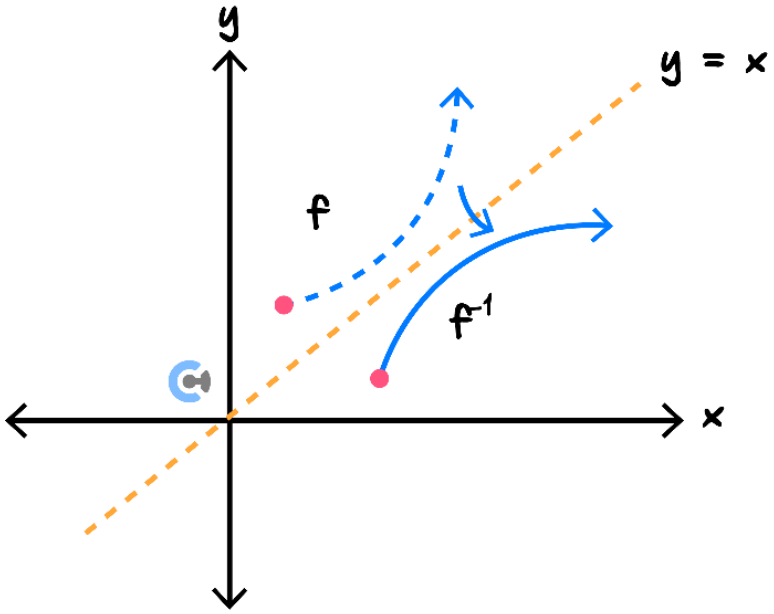
- What happens if you swap the x- and y-axis on the label on our graph?



- Wait... do we want the x-axis to be the vertical one? [Yes/No]
- How should we reflect the graph so that the x- and y-axis become horizontal and vertical again?



Symmetry of Inverse Functions



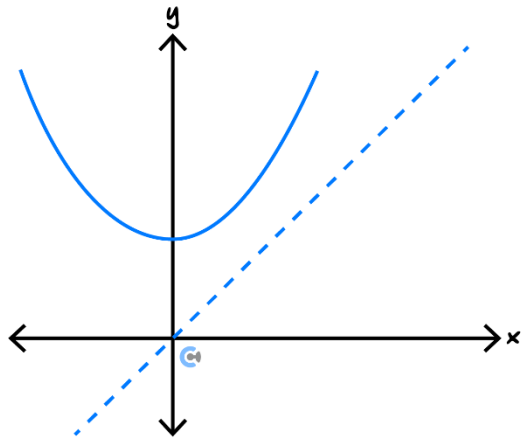
- Inverse functions are always symmetrical around $y = x$.

Sub-Section: Validity of Inverse Function

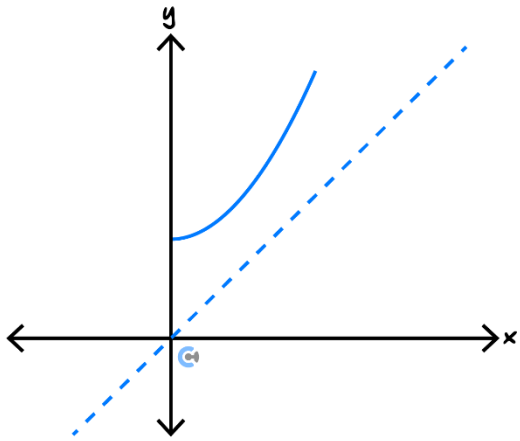
Does an inverse function always exists?

Exploration: Validity of Inverse Functions

- Consider the many to one and one to one functions.



Many to one



One to one

- Reflect them around $y = x$ and sketch the inverse! *(Label Above)*
- Which inverse is a function? (Passes through a vertical line test?)

[neither] / [left] / [right] / [both]

- For an inverse **function** to exist, what must the original function be? [many to one] / [one to one]

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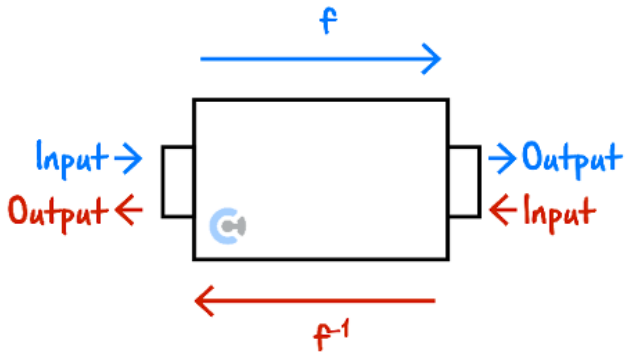
Validity of Inverse Functions



- **Requirement for Inverse Function**:

f needs to be _____

Question 16 Walkthrough.

Consider the function $f: (-\infty, a] \to \mathbb{R}, f(x) = 3(x-2)^2 - 4$.

a. Find the largest possible value of a such that the inverse function f^{-1} exists.

b. Find the inverse function.

NOTE: Finding function means to find the rule AND the domain.

TIP: Always try sketching the function to find the domain such that an inverse function can exist!

NOTE: You will need to complete the square when finding the inverse of quadratic functions!

Your turn!

Question 17

Consider the function $g: (-\infty, b] \to \mathbb{R}, g(x) = -2x^2 - 8x + 1$.

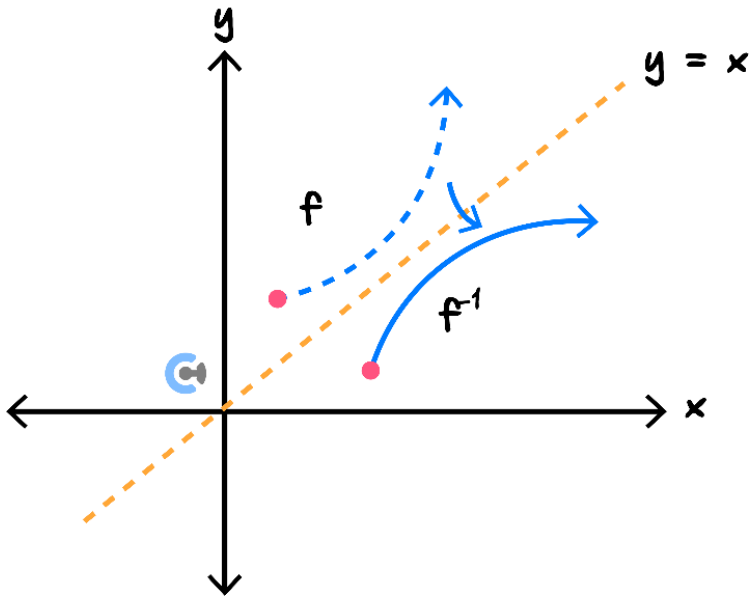
a. Find the largest possible value of b such that the inverse function g^{-1} exists.

b. Find the inverse function.

Sub-Section: Intersection Between Inverses

Where do inverses meet?

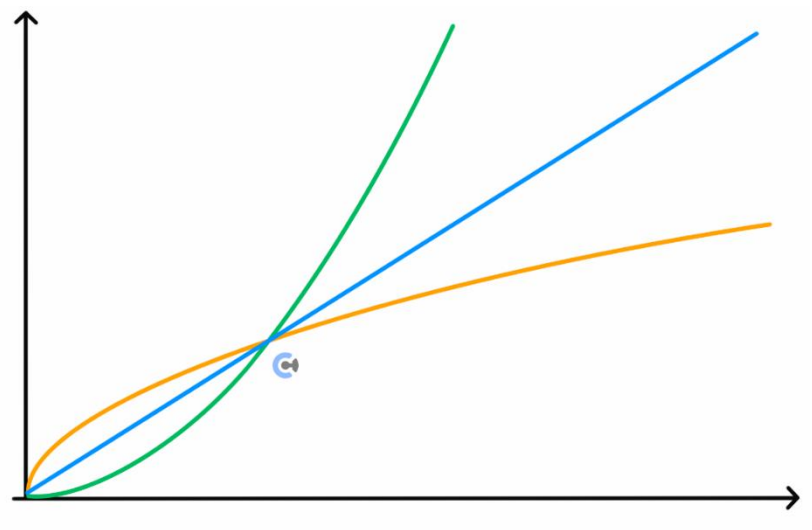
Active Recall: Symmetry Around $y = x$



- Inverse functions are always symmetrical around_____.

Discussion: Where could a function and its inverse meet?

Intersection Between a Function and its Inverse



- Equate with _____ instead.

$$f(x) = x \text{ OR } f^{-1}(x) = x$$

Question 18

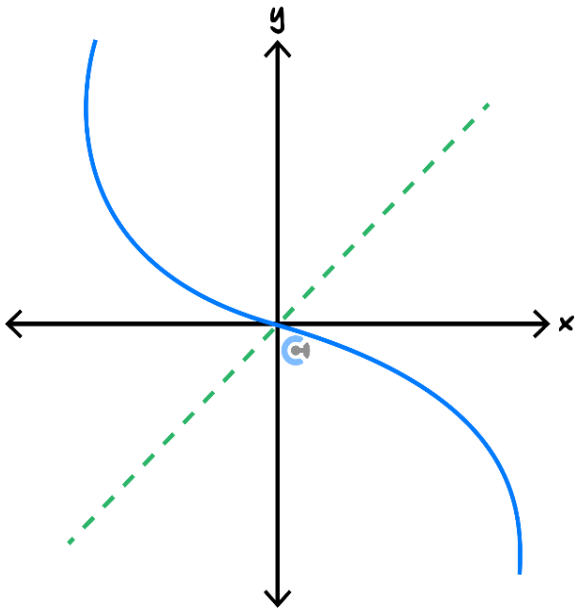
Find the intersection(s) between $f: [0, \infty) \to R, f(x) = x^3$ and its inverse, without finding the inverse.

NOTE: This only works for an increasing function, however in VCAA, this is always the case.

Does this always work?

Extension: Intersections Not on $y = x$ (Not Tested on Exams, but Maybe on SACs!)

- Consider the following:



  - What does the inverse function look like? *(Sketch Above)*
  - Are these intersections on $y = x$? [Yes] / [No]

ALSO NOTE: For SACs is that there **could** be intersections that are **not** on $y = x$.

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Sub-Section: Composition of Inverses

Analogy: Inverse function is your annoying sibling.

- Who has siblings? Is it just me or do your siblings want to do the opposite of what you want to do?
- **Example:**
  - James: Wants to turn the AC up by 5 degrees.
  - Danis (James' brother): Turns the AC _____

This is basically an inverse function relationship!

Discussion: So, now what would happen if we have a function and its inverse happening one after another? (Composite function of inverse)

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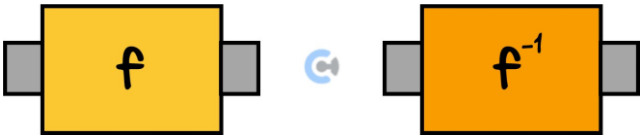
Composition of Inverse Functions



$f \circ f^{-1}(x) = \_\_, \quad for\ all\ x \in \_\_\_\_\_$

$f^{-1} \circ f(x) = \_\_, \quad for\ all\ x \in \_\_\_\_\_$

NOTE: Domain = Domain of Inside

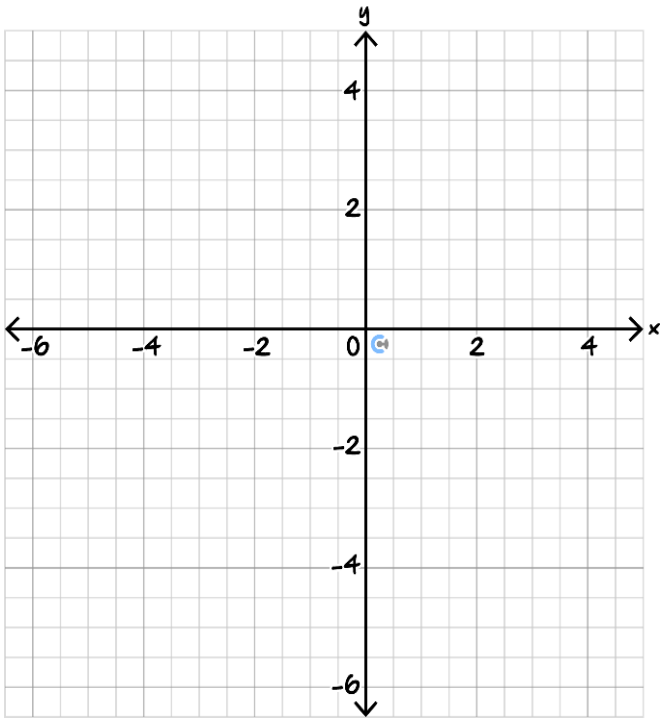
Try this question!

Question 19 (4 marks)

Consider the function $f(x) = \frac{1}{x-1} - 3$.

a. Find the rule and domain for $f^{-1}(f(x))$. (2 marks)

b. Sketch the graph of $y = f^{-1}(f(x))$ on the axes below. (2 marks)



Key Takeaways

- ☑ f needs to be 1: 1 f or f^{-1} to exist.
- ☑ Domain and Range Swaps.
- ☑ Symmetrical around $y = x$.
- ☑ For intersections: $f(x) = x$ or $f^{-1}(x) = x$.
- ☑ Composite function of inverses is always equal to x.

Space for Personal Notes

Contour Check

Learning Objective: [1.1.1] – Find Maximal Domain and Range

Key Takeaways

- Inside of a $\log$ must be _____.
- Inside of a root must be _____.
- Denominator _____.
- Domain of sum or product of two functions is equal to_____ of the two domains.

Learning Objective: [1.1.2] – Find the Rule, Domain and Range of a Composite Function (Range Does Not Require Splitting to Find as the Function is Easy to Draw)

Key Takeaways

- $f(g(x)) =$____ $\circ$____ (x).
- For composite function to exist, _____ $\subseteq$ _____.
- Domain of Composite is equal to the Domain of _____ Function.
- Range of Composite is a _____ of the Range of the Outside.

Learning Objective: [1.1.3] – Find the Rule, Domain, and Range of Inverse Functions

Key Takeaways

- f needs to be _____ f or f^{-1} to exist.
- Domain of the inverse function equals to _____ **and** vice versa.
- Symmetrical around _____.
- For intersections of inverses, we can equate the function to _____.

Learning Objective: [1.1.4] – Find the Composite Function of Inverse Function

Key Takeaways

- Composite function of inverses is always equal to _____.

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