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VCE Mathematical Methods ¾
AOS 1 Revision [1.0]
Contour Check (Part 2) Solutions

Contour Checklist

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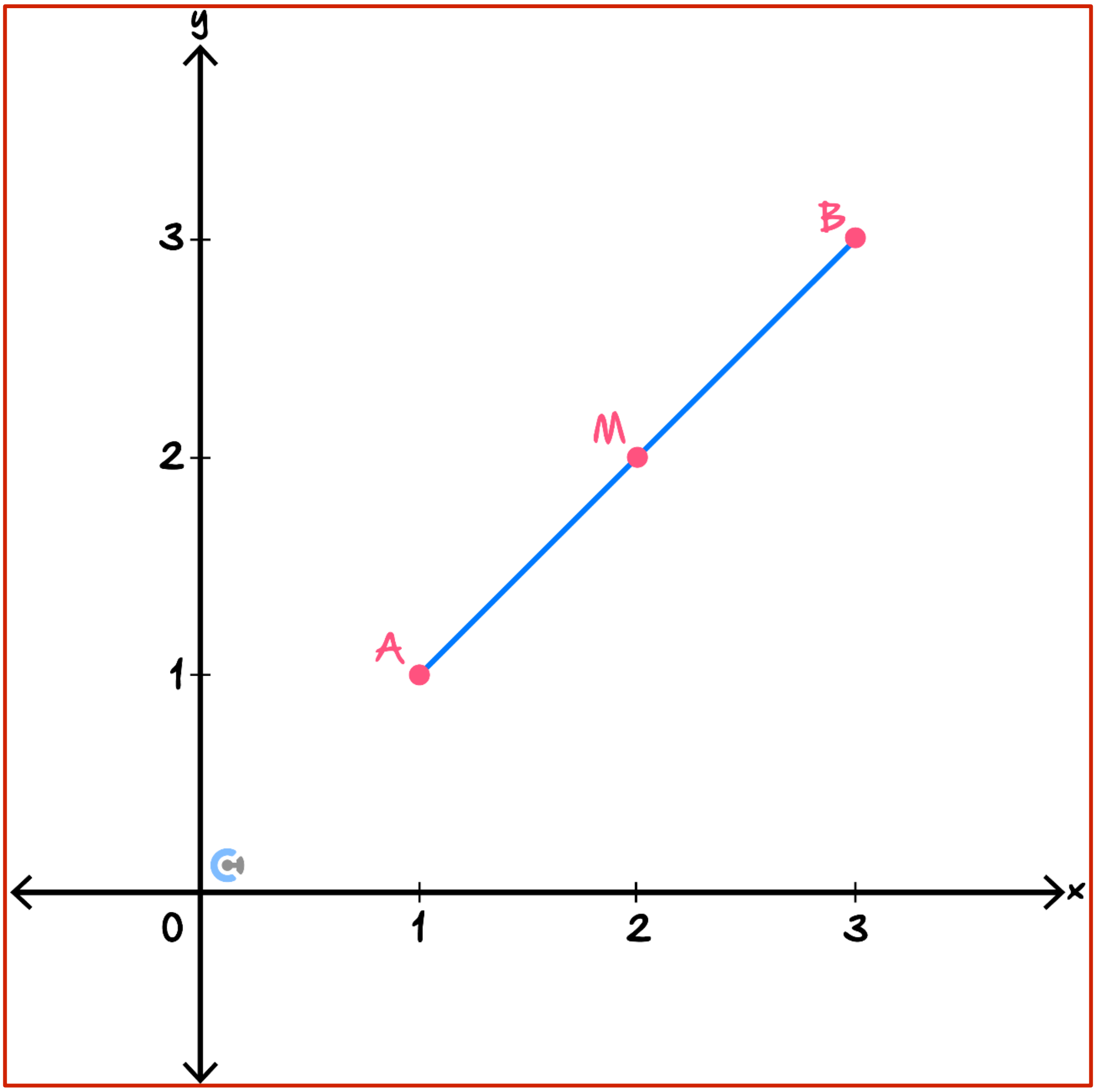
Section A: [1.5] - Linear and Coordinate Geometry (Checkpoints)

Sub-Section [1.5.1]: Finding the Midpoint and Distance Between Points and Functions

Question 88

The line segment AB is shown on the axis below. Draw the midpoint, M of AB.



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Question 89

Find the midpoints of the following points.

a. $A(3,7)$ and $B(5,9)$.

$$\left(\frac{3+5}{2},\frac{7+9}{2}\right)=(4,8)$$

b. $C(-2,-3)$ and $D(6,4)$.

$$\left(\frac{-2+6}{2},\frac{-3+4}{2}\right)=\left(2,\frac{1}{2}\right)$$

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Question 90

The midpoint of points A and B is $M(2,2)$.

a. If the coordinates of A are $(6,-4)$, find the coordinates of B.

Let B have the co-ordinates (x, y).
Then,
$\frac{6+x}{2} = 2 \implies 6 + x = 4 \implies x = -2$ and $\frac{-4+y}{2} = 2 \implies -4 + y = 4 \implies y = 8$
Thus, the coordinates of B are $(-2, 8)$.

Consider the points $C(c, 5)$ and $D(-3, d)$. The midpoint of the line CD is the origin.

b. Find the values of c and d.

We know that $\frac{c-3}{2} = 0$, thus $c = 3$.
Similarly, $\frac{5+d}{2} = 0$, thus $d = -5$.

c. Find the midpoint of $E(x_1, y_1)$ and $F(x_2, y_2)$ in terms of x_1, x_2, y_1, and y_2.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

d. The graph of $y = x^2 + k$ and the line $y = 1$ has a minimum vertical distance of 4. Find the value of k.

The parabola $y = x^2 + k$ has lowest point at $(0, k)$. Therefore $k - 1 = 4 \implies k = 5$.

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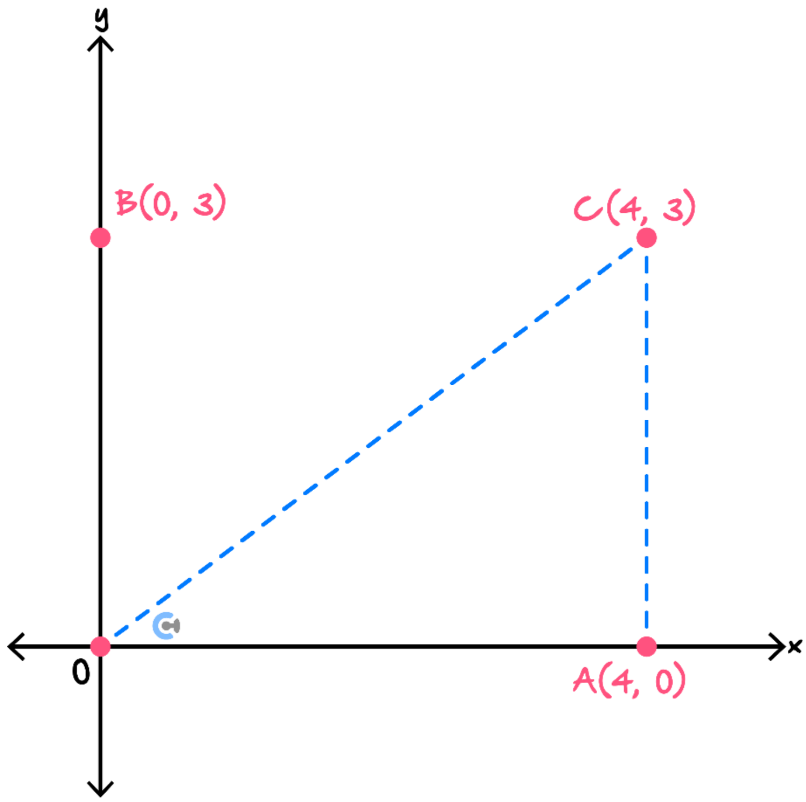
Sub-Section [1.5.2]: Finding Distances Between Points

Question 91

Consider the points, A, B, C as well as the origin drawn below.



a. Find the distance between the origin and point A.

4 units.

b. Find the distance between the origin and point B.

3 units.

c. Use Pythagoras' theorem to find the distance between the origin and point C.

$\sqrt{3^2 + 4^2} = 5$ units.

Question 92

Find the distance between the following pairs of points.

a. $A(2,5)$ and $B(-2,2)$.

$\sqrt{(2-(-2))^2+(5-2)^2}=\sqrt{4^2+3^2}=\sqrt{25}=5$ units.

b. $C(-1,-7)$ and $D(4,5)$.

$\sqrt{(-1-4)^2+(-7-5)^2}=\sqrt{(-5)^2+(-12)^2}=\sqrt{25+144}=\sqrt{169}=13$ units.

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Question 93

A point $P(u, v)$ lies on the line $y = 3 - x$.

a. Express the distance between P and the origin in terms of u only.

We know that $v = 3 - u$, thus the distance of OP is,

$$\sqrt{u^2 + (3-u)^2} = \sqrt{2u^2 - 6u + 9}$$

Consider the points $A(-1, -1), B(5, 7)$ and $C(x, y)$.

The length of AC is equal to the length of BC which is equal to halve the length of AB.

b. Find the coordinates of C.

The two conditions provided in the question ensure that C is the midpoint of A and B.

Thus the coordinates of C are $(2, 3)$.

c. **Tech-Active.** The distance between the point $P(u, v)$ is 3 units away from the origin and 4 units away from the point $Q(1, 4)$. Find the coordinates of P.

The distance between P and the origin is, $\sqrt{u^2 + v^2} = 3$.

The distance between P and Q is, $\sqrt{(u-1)^2 + (v-4)^2} = 4$.

We solve these two equations simultaneously to get the value(s) of u and v. Thus,

$$P = \left(\frac{5 - 32\sqrt{2}}{17}, \frac{20 + 8\sqrt{2}}{17}\right) \quad \text{or} \quad P = \left(\frac{5 + 32\sqrt{2}}{17}, \frac{20 - 8\sqrt{2}}{17}\right)$$

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Sub-Section [1.5.3]: Finding Parallel and Perpendicular Lines

Question 94

State whether the following lines are parallel or perpendicular to each other.

a. $y = 2x + 1$ and $y = 2x + 5$.

$m_1 = m_2 \implies$ parallel

b. $y = 3x + 2$ and $y = -\frac{1}{3}x - 2$.

$m_1 \times m_2 = -1 \implies$ perpendicular

c. $2x + 3y = 5$ and $4x + 6y = 12$.

$$2x + 3y = 5 \implies y = -\frac{2}{3}x + \frac{5}{3}$$

$$4x + 6y = 12 \implies y = -\frac{4}{6}x + \frac{12}{6} \implies y = -\frac{2}{3}x + 2$$

$m_1 = m_2 \implies$ parallel

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Question 95

A line l_1 goes through the points $(2, 3)$ and $(3, 5)$.

a. Find the gradient of l_1.

Let m_1 be the gradient of l_1.
Then, $m_1 = \frac{5-3}{3-2} = 2$.

b. Find the equation of l_1.

$$y = 2(x - 2) + 3$$
$$y = 2x - 1$$

The line l_2 is perpendicular to l_1 and goes through the point $(2, 3)$.

c. Find the gradient of l_2.

$$y - 3 = -\frac{1}{2}(x - 2)$$
$$y = \frac{-1}{2}(x - 2) + 3$$
$$y = \frac{-x}{2} + 4$$

d. Find the equation of l_2.

$$y = \frac{-1}{2}(x - 2) + 3 = \frac{-x}{2} + 4$$

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Question 96

The line l_1 is parallel to the line $l_2 = \{(x, y) \in \mathbb{R}^2 : 2y + 3x = 5\}$ and goes through the origin.

a. Find the equation of l_1.

The equation for l_2 is $y = \frac{-3}{2}x + \frac{5}{2}$. Hence the gradient for l_2 is $\frac{-3}{2}$.

Hence the equation for l_1 is $y = \frac{-3}{2}x$

b. Find the equation of the line that is perpendicular to the line with the equation $y = -5x + 7$ and passes through the point $(2, -5)$.

$$y = -5x + 7$$
$$Slope = -5$$
$$Slope\ of\ the\ perpendicular\ line = \frac{1}{5}$$
$$Required\ line\ y + 5 = \frac{1}{5}(x - 2)$$
$$y = \frac{x}{5} - \frac{27}{5}$$

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Question 97

a. Find the perpendicular bisector of the points $A(2,3)$ and $B(4,9)$.

The line AB has gradient $\frac{9-3}{4-2} = 3$ and midpoint, $\left(\frac{4+2}{2}, \frac{9+3}{2}\right) = (3,6)$.

Hence the perpendicular bisector goes through the point $(3,6)$ and has a gradient of $\frac{-1}{3}$.

Hence the equation of the perpendicular bisector is,

$$y = \frac{-1}{3}(x-3) + 6 = \frac{-x}{3} + 7$$

b. A point $P(u,v)$ lies on the line $y = 2x$.

Find the value of u and v for which the distance between P and the point $Q(0,1)$ is minimum.

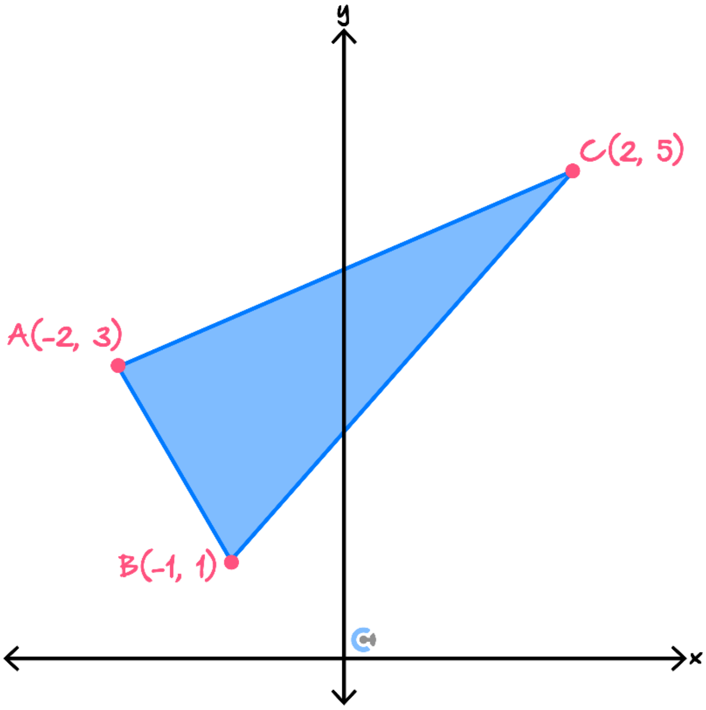
Hint: The line PQ is perpendicular to the line $y = 2x$.

The gradient of PQ must be $\frac{-1}{2}$. As PQ goes through the point Q, it's equation is,

$$y = \frac{-x}{2} + 1.$$

Thus, $v = \frac{-u}{2} + 1 = 2u \implies \frac{5u}{2} = 1 \implies u = \frac{2}{5} \implies v = \frac{4}{5}$.

c. Consider the triangle ABC drawn below.



i. Show that the line AB is perpendicular to the line AC.

The line AC has gradient $\frac{5-3}{2+2} = \frac{1}{2}$.
The line AB has gradient $\frac{3-1}{-2+1} = -2$ As $\frac{1}{2}$ is the negative reciprocal of -2, the lines AB and AC are perpendicular.

ii. Hence, find the area of the triangle ABC.

The length of AB is $\sqrt{2^2+1^2} = \sqrt{5}$.
The length of AC is $\sqrt{2^2+4^2} = \sqrt{20}$
Thus the area of the triangle is $\frac{\sqrt{5}\sqrt{20}}{2} = 5$

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Sub-Section [1.5.4]: Angles Between Lines

Question 98

a. Find the angle of the line $y = x + 1$ makes with the positive direction of the x-axis.

$\tan^{-1}(1) = 45°$

b. Find the equation of the line that passes through the origin and makes an angle of 30 degrees with the positive direction of the x-axis.

The line will have a gradient of $\tan(30°) = \frac{1}{\sqrt{3}}$.

Thus the equation of the line is $y = \frac{x}{\sqrt{3}}$

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Question 99

a. Find the acute angle between the lines $y = \frac{1}{\sqrt{3}}x + 2$ and $y = \frac{-1}{\sqrt{3}}x$.

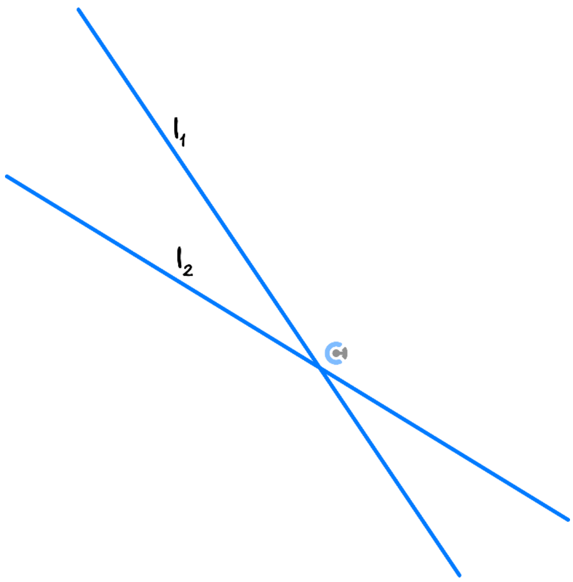
The angle $y = \frac{1}{\sqrt{3}}x + 2$ makes with the x-axis is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$.

The angle $y = \frac{-x}{\sqrt{3}}$ makes with the x-axis is $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -30°$.

Thus the acute angle between our lines is $60°$

b. **Tech-Active.** Consider the line l_1, with the equation $2y + 3x = 5$.

The line l_2 intersects l_1 at an acute angle $25°$. Both l_1 and l_2 are drawn below.



Find the slope of l_2 correct to 2 decimal places.

The angle l_1 makes with the positive direction of the x-axis is $180° + \tan^{-1}\left(\frac{-3}{2}\right) = 180° - 56.31° = 123.69°$.

Thus l_2 makes an angle of $148.69°$ with the positive direction of the x-axis.

Hence the slope of l_2 is $\tan(148.69°) = -0.608.. \approx -0.61$

c. **Tech-Active.** Find the acute angle of intersection between the lines $y = 3x + 5$ and $-2x + 3y = 7$.

Give your answer in degrees correct to the nearest degree.

$$m_1 = 3, m_2 = \frac{2}{3}$$

Angle between lines:

$$\tan(\theta) = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{3 - \frac{2}{3}}{1 + 1}\right|$$

$$\tan\theta = \frac{7}{6}$$

$$\theta = 49.398.. \approx 49°$$

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Question 100

The line l intersects the positive y-axis at $30°$

a. Find the gradient, m of l if $m < 0$.

The acute angle between l and the x-axis is $60°$.

If m is the gradient of l, then $m = \tan(\pm 60°) = \pm\sqrt{3}$.

As m is negative then $m = -\sqrt{3}$.

b. **Tech-Active.** Find the acute angle of intersection between the lines $y = 2x + 3$ and $3x + 5y = -4$.

Give your answer in degrees correct to the nearest degree.

The angle $y = 2x + 3$ makes with the positive direction of the x-axis is, $\tan^{-1}(2) = 63.43°$.

The angle $3x + 5y = -4$ makes with the positive direction of the x-axis is, $\tan^{-1}\left(\frac{-3}{5}\right) = -30.96°$.

Hence an angle between our two lines is $94°$.

As this angle is greater than 90, it's supplementary angle of $86°$ is the acute angle of intersection between our two lines.

c. Find the equation of all lines that intersect the line $y = x + 3$ at the point $(1, 4)$ at an acute angle of $15°$.

The angle $y = x + 3$ makes with the positive direction of the x-axis is $45°$.

For line to intersect $y = x + 3$ at $15°$ it needs to,

a. Make an angle of 60 degrees with the positive direction of the x-axis, thus have a gradient of $\sqrt{3}$.

b. Make an angle of 30 degrees with the positive direction of the x-axis, thus have a gradient of $\frac{1}{\sqrt{3}}$.

Hence our lines are,

$$y = \sqrt{3}(x-1) + 4 = \sqrt{3}x + 4 - \sqrt{3} \quad \text{and} \quad y = \frac{1}{\sqrt{3}}(x-1) + 4 = \frac{x}{\sqrt{3}} + 4 - \frac{1}{\sqrt{3}}$$

Sub-Section [1.5.5]: Simultaneous Equations

Question 101

Solve the following equations simultaneously.

a. $3x + 4y = 7$ and $5x - 2y = 3$.

We add $2 \times$ the right equation to the left equation to get,

$$3x + 10x = 7 + 6 \implies 13x = 13 \implies x = 1$$

We substitute it into the left equation to get,

$$3 + 4y = 7 \implies y = 1$$

Hence $x = y = 1$.

b. $y = 5x + 3$ and $3y + 4x = 8$.

We substitute the left equation into the right equation, yielding,

$$15x + 9 + 4x = 8 \implies 19x = -1 \implies x = \frac{-1}{19}.$$

Substituting this into the left equation yields,

$$y = 5\left(\frac{-1}{19}\right) + 3 = \frac{52}{19}$$

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Question 102

a. Find the point of intersection between the lines $y = 3x + 7$ and $2x + 5y = 1$.

We substitute the first line into the second, yielding,

$$15x + 35 + 2x = 1 \implies 17x = -34 \implies x = -2.$$

We substitute this into the left equation to get $y = 1$.

Hence the point of intersection is $(-2, 1)$

b. Explain why the equations $2x + 4y = 6$ and $3x + 6y = 5$ have no solutions.

Both lines have slope $-\frac{1}{2}$ but the y-intercepts are different. Hence, they are parallel lines. No solution.

c. **Tech-Active.** For each pair of simultaneous equations, state whether they have, no solution, a unique solution or infinitely many solutions.

i. $2x + 5y = 7$ and $3x + 2y = 8$.

Solve on calc, you get 1 solution.
Hence, a unique solution.

ii. $y = -3x + 6$ and $2y + 6x = 6$.

Solve on calc, you get false.
Hence, no solution.

iii. $6x + y = 2$ and $y = -6x + 2$.

Solve on calc, you get $y = 2 - 6x$.
Hence, you have infinitely many solutions.

Question 103

a. Consider the following pair of simultaneous equations,

$$kx - y = 6$$
$$7x + (k - 8)y = 4$$

For what value(s) of k do they have:

i. A unique solution?

ii. No solution?

For the equations to have a unique solution, their gradients must be different.

The gradient of the first equation is k, whilst the gradient for the second equation is $\frac{7}{8-k}$.

We solve,

$$k = \frac{7}{8-k} \implies k^2 - 8k + 7 = (k-7)(k-1) = 0 \implies k = 1, 7$$

Hence our equations have a unique solution if $k \neq 1, 7$.

If $k = 1$ the lines are $x - y = 6$ and $7x - 7y = 4$ and hence there is no solution.

If $k = 7$ the lines are $7x - y = 6$ and $7x - y = 4$ and hence there is no solution.

b. Consider the following pair of simultaneous equations,

$$ax + 3y = 6$$
$$x + (4 - a)y = 2$$

For what value(s) of a do they have:

i. No solution?

ii. Infinitely many solutions?

iii. A unique solution?

The gradient of our first line is $\frac{-a}{3}$, whilst the gradient of our second line is $\frac{1}{a-4}$. Hence we solve,

$$\frac{-a}{3} = \frac{1}{a-4} \implies a^2 - 4a + 3 = (a-3)(a-1) = 0 \implies a = 1, 3$$

If $a = 1$ then our equations are $x + 3y = 6$ and $x + 3y = 2$, which have no solutions.

If $a = 3$, then our equations are $3x + 3y = 6$ and $x + y = 2$, which have infinitely many solutions.

Finally, there is a unique solution for $a \in \mathbb{R} \setminus \{1, 3\}$.

c. **Tech-Active.** Consider the following pair of simultaneous equations,

$$3x + (1-a)y = 2$$
$$ax - 2y = b$$

Find all pairs (a, b) such that the equations have infinitely many solutions.

The gradient of our first line is $\frac{3}{a-1}$, whilst the gradient of our second line is $\frac{a}{2}$.

We equate these two gradients yielding, $a = -2, 3$.

Since the gradients of our two lines will be the same for $a = -2, 3$ if the two lines also share a point they will have infinitely many solutions.

If $a = -2$, the y-axis intercept of the first line is $\left(0, \frac{2}{3}\right)$, thus $b = -2(0) - 2\frac{2}{3} = \frac{-4}{3}$.

If $a = 3$, the y-axis intercept of the first equation is $(0, -1)$, thus $b = 3(0) - 2(-1) = 2$.

Hence our pairs of values are $(a, b) = \left(-2, \frac{-4}{3}\right)$ and $(3, 2)$.

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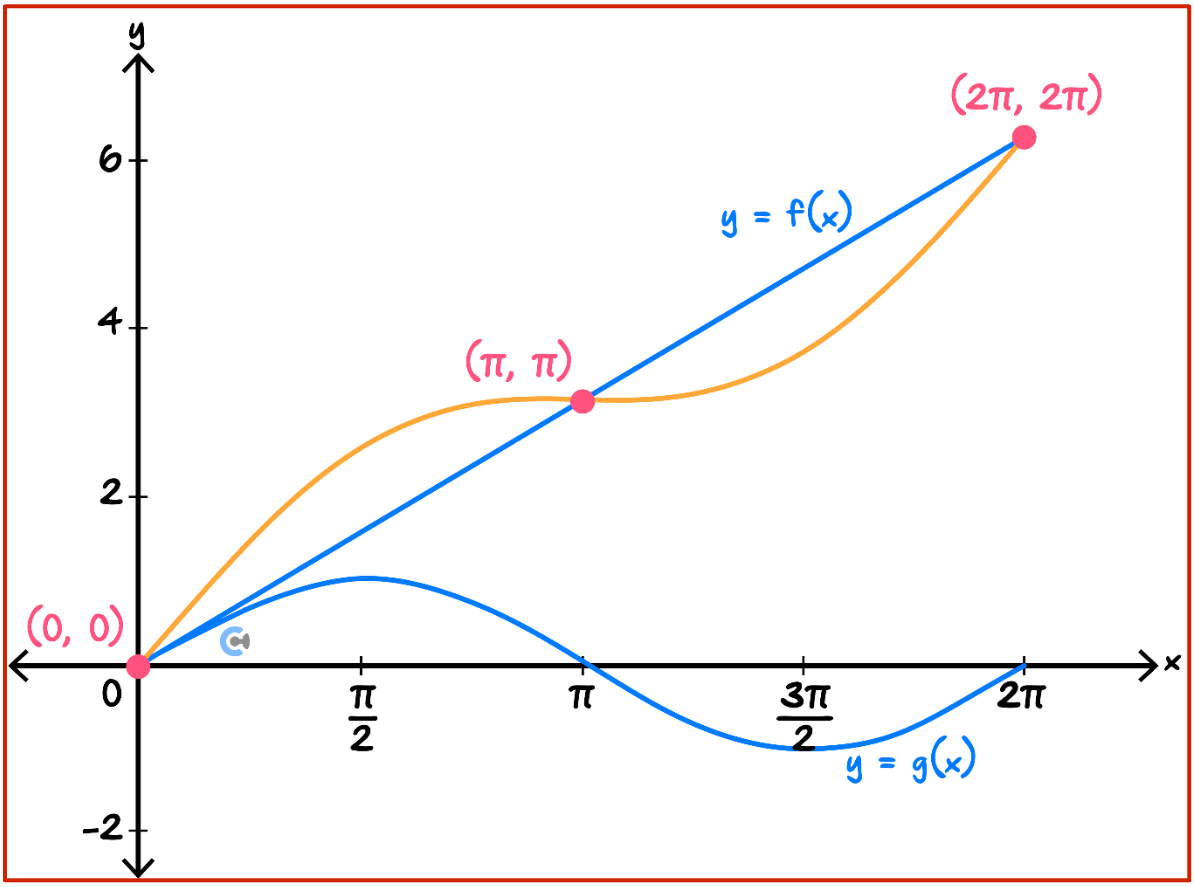
Sub-Section [1.5.6]: Addition of Ordinates

Question 104

The graphs of $f : [0, 2\pi] \longrightarrow \mathbb{R}, f(x) = x$, and $g : [0, 2\pi] \longrightarrow \mathbb{R}, g(x) = \sin(x)$ are drawn below.

Sketch the graph of $h(x) = f(x) + g(x)$ on the axis below, labelling all points of intersection between f and h with their co-ordinates.



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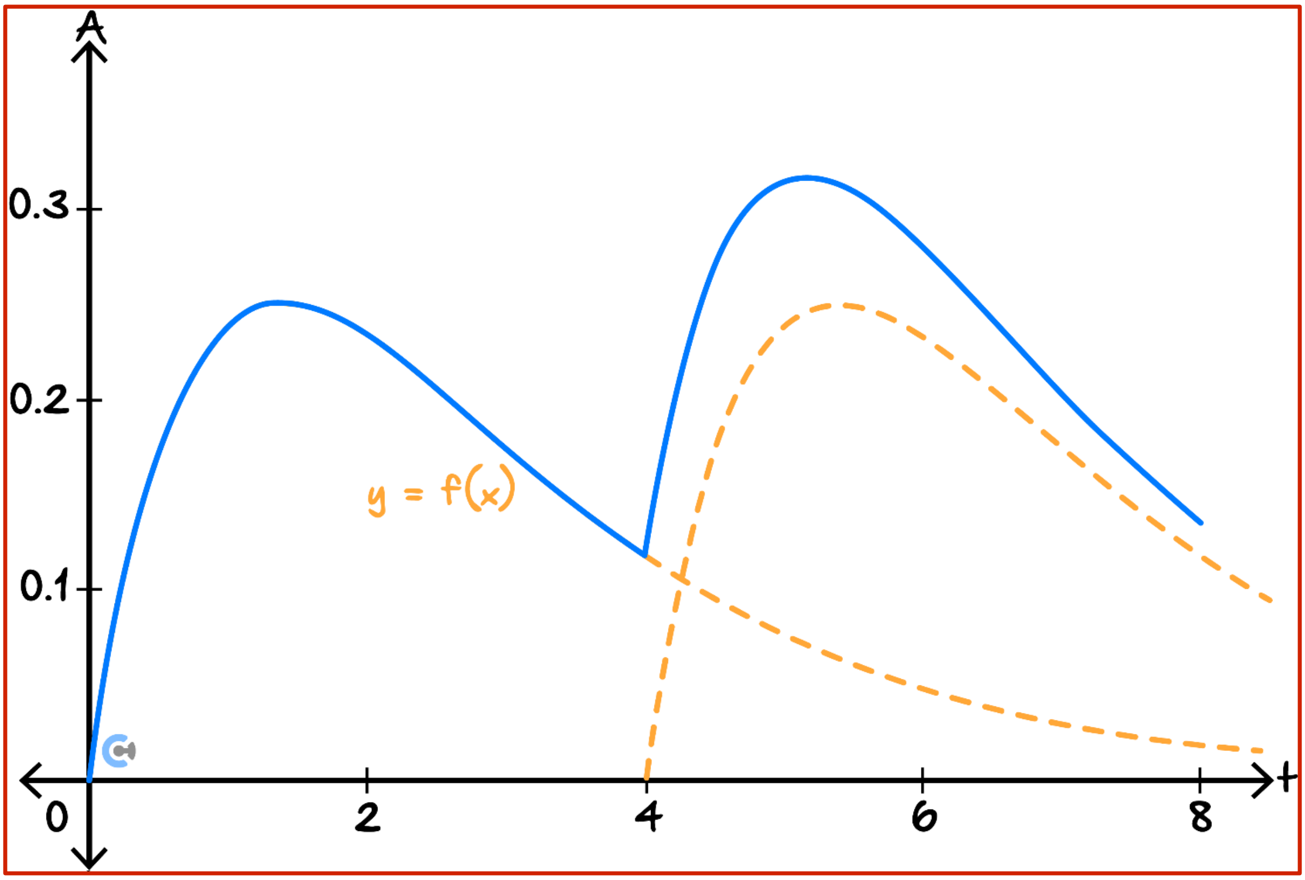
Question 105

t hours after taking a mystery pill, the concentration of dopamine in a patient's bloodstream is $A = f(t)$ milligrams per litre. The graph of f is shown below.

4 hours after taking one mystery pill, the patient takes another mystery pill.

On the axis below, sketch the concentration of dopamine in the patient's bloodstream during the first 8 hours after they take the first mystery pill.



Question 106 Tech-Active.

Let $f(x) = e^x - e^{-2x}$ and $g(x) = e^{x-x^2}$.

How many solutions does the equation $f(x) + g(x) = 0$ have?

Sketch both graphs. For negative x, both graphs are strictly positive. For positive x, $g(x)$ is positive however asymptotes towards 0, whilst $f(x)$ tends towards negative infinity.

By addition of ordinates we see that $f(x) + g(x)$ will have one solution.

Sub-Section [1.5.7]: Boss Question

Question 107

Consider the points $A(1, 0)$ and $B(4, 3)$.

a. Find the equation of the line segment AB.

The gradient of AB is $\frac{3-0}{4-1} = 1$. Hence the equation of AB is,

$$y = 1(x-1) + 0 = x - 1$$

There is another point C, such that A is the midpoint of the line segment CB.

b. Find the coordinates of C.

Let (x, y) be the coordinates of C. Then,

$$\frac{x+4}{2} = 1 \implies x = -2 \quad \text{and} \quad \frac{y+3}{2} = 0 \implies y = -3,$$

thus the co-ordinates of C are $(-2, -3)$

c. Hence or otherwise, find the length of BC.

The length of BC is $\sqrt{(4-(-2))^2 + (3-(-3))^2} = \sqrt{6^2 + 6^2} = 6\sqrt{2}$.

d. Another point $D(u, v)$ has the following properties,

- The length of AD is equal to twice the length of AB.
- The angle between AD and AB is $30°$.
- The gradient of AB is larger than the gradient of AD.
- Both u and v are positive.

Find the values of u and v correct to 3 decimal places.

The line segment AB makes an angle of $45°$ with the x-axis, as the gradient of AB is larger than the gradient of AD, AD must make an angle of $15°$ with the x-axis. Thus a point (x, y) on the line segment AD satisfies,

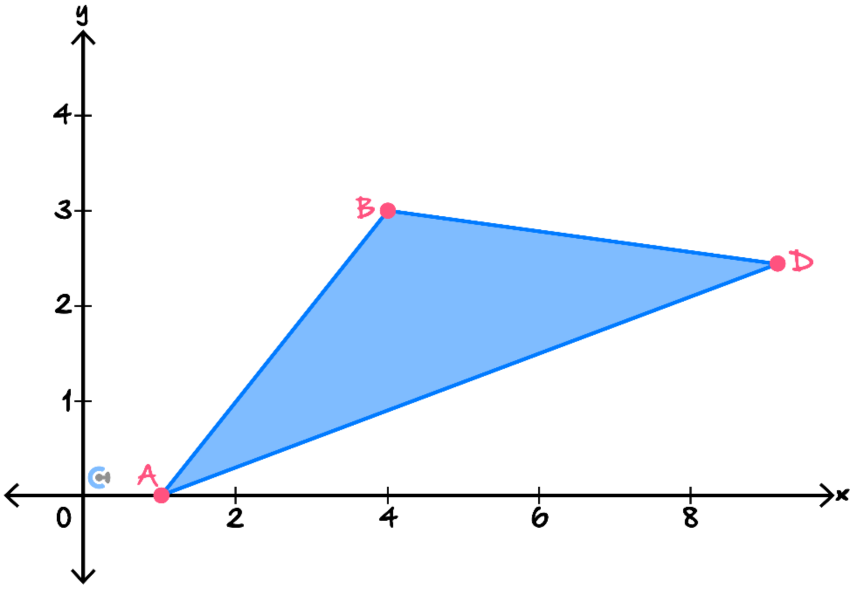
$$y = \tan(15°)(x - 1) = (2 - \sqrt{3})(x - 1)$$

As (u, v) is such a point, along with the fact that the length of AD is $6\sqrt{2}$ we can solve the following two equations simultaneously for u and v,

$$v = (2 - \sqrt{3})(x - 1) \quad \text{and} \quad (u - 1)^2 + v^2 = 72$$

Thus, $u = 9.196$ and $v = 2.196$.

e. The triangle ABD is drawn below.



i. Find the equation of the line, l perpendicular to AD that goes through B.

The gradient of AD is $2-\sqrt{3}$, thus the gradient of l is $\frac{1}{\sqrt{3}-2}$. Thus the equation of l is,

$$y = \frac{1}{\sqrt{3}-2}(x-4)+3$$

ii. Hence or otherwise, find the area of ABD correct to the nearest integer.

We find the point of intersection of l and the line AD by solving,

$$y = \frac{1}{\sqrt{3}-2}(x-4)+3 \quad \text{and} \quad y = (2-\sqrt{3})(x-1)$$

simultaneously. This yields $(x, y) = (4.549, 0.951)$.

Thus the "height" of the triangle, the distance between B and this point is 2.12132 units, whilst the base of the triangle is $6\sqrt{2}$ units.

Hence the area of the triangle is 9 units

Section B: [1.6] - Linear and Coordinate Geometry Exam Skills (Checkpoints)

Sub-Section [1.6.1]: Apply Midpoint to Find a Reflected Point

Question 108

The point $(-1, 5)$ is reflected in the line $y = 2$. Find the coordinates of the reflected point.

$(-1, -1)$

Question 109

The point $(2, -3)$ is reflected in a line to become the point $(-10, -3)$. State the equation of the line.

$x = -4$

Space for Personal Notes

Question 110

Find the perpendicular bisector of the line segment joining the points $(4, -2)$ and $(-1, 0)$.

Midpoint is $\left(\frac{3}{2}, -1\right)$ and the line joining the points is $y = -\frac{2}{5}x - \frac{2}{5}$

So we want a line with gradient $\frac{5}{2}$ through the point $\left(\frac{3}{2}, -1\right)$. Therefore, the perpendicular bisector is

$$y = \frac{5}{2}x - \frac{19}{4}$$

Question 111

The point $(1, -6)$ is reflected in a line to become the point $(5, -4)$. Find the equation of the line.

The line is the perpendicular bisector between $(1, -6)$ and $(5, -4)$

Midpoint is $(3, -5)$ and the line joining the points is $y = \frac{1}{2}x - \frac{13}{2}$

So we want a line with gradient -2 through the point $(3, -5)$. Therefore, the perpendicular bisector is

$$y = -2x + 1$$

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Sub-Section [1.6.2]: Apply Parallel and Perpendicular Lines to Geometric Problems

Question 112

Find the equation of the line that passes through the point $(-2, 3)$ and is perpendicular to $y = x + 7$.

$y = -x + 1$

Question 113

Find the area of the triangle formed by the lines $y = 2x - 8$, $y = 6x - 4$, and $y = 2$.

Lines intersect at $(5,2)$, $(1,2)$ and $(-1,-10)$. Triangle base $= 5 - 1 = 4$ and triangle height $= 2 - (-10) = 12$
Therefore, triangle area is $\frac{1}{2} \times 4 \times 12 = 24$

Space for Personal Notes

Question 114

Find the distance between the point $(2, 7)$ and the line $y = 3x - 1$.

Want the equation of line perpendicular to $y = 3x - 1$ and through the point $(2,7)$. Therefore, the line with gradient $-\frac{1}{3}$ and through $(2,7)$.

$$y = -\frac{1}{3}x + \frac{23}{3}$$

Now $y = 3x - 1$ and $y = -\frac{1}{3}x + \frac{23}{3}$ intersect at the point $\left(\frac{13}{5}, \frac{34}{5}\right)$.

Therefore, the minimum distance is the distance between the points $(2,7)$ and $\left(\frac{13}{5}, \frac{34}{5}\right)$.

$$d = \sqrt{\left(2 - \frac{13}{5}\right)^2 + \left(7 - \frac{34}{5}\right)^2} = \frac{\sqrt{10}}{5}$$

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Question 115

Consider the points $A(2,1)$, $B(1,-2)$, $C(5,0)$ and $D(m,n)$, where $m,n \in \mathbb{R}^+$. It is known that $\angle ABC = 45°$. Find the values of m and n such that $\angle BCD = 135°$.

$\angle ABC$ is supplementary to $\angle BCD$. Therefore, $ABCD$ is a parallelogram.

Equation of AB is $y = 3x - 5$ and equation of BC is $y = \frac{1}{2}x - \frac{5}{2}$.

AD is parallel to BC and goes through A. Therefore, AD is the line with gradient $\frac{1}{2}$ through (2,1).

$$y = \frac{1}{2}x$$

CD is parallel to AB and goes through C. Therefore, CD is the line with gradient 3 through (5,0).

$$y = 3x - 15$$

Now $y = \frac{1}{2}x$ and $y = 3x - 15$ intersect at (6,3).

$m = 6$ and $n = 3$.

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Sub-Section [1.6.3]: Solve Coordinate Geometry Problems with Transformations

Question 116

The area bound by the lines $y = 2x - 4$, $y = -1 - x$, and $y = \frac{1}{2}x + 2$ is $\frac{27}{2}$ square units. Hence, find the area bound by:

a. The lines $y = 8x - 4$, $y = -1 - 4x$ and $y = 2x + 2$.

All lines have been dilated by a factor of $\frac{1}{4}$ from the y axis. Therefore the area is $\frac{27}{8}$ square units

b. The lines $y = -2x + 4$, $y = 1 + x$ and $y = -\frac{1}{2}x - 2$.

All lines have been reflected in the x axis. The area does not change. Therefore the area is $\frac{27}{2}$ square units

c. The lines $y = 6x - 4$, $y = 5 - 3x$, $y = \frac{3}{2}x + 14$.

All lines have undergone a dilation by a factor of 3 from the x axis followed by a translation 8 units up. Therefore the area is $\frac{81}{2}$ square units

Question 117

a. The original function is $f(x) = \frac{2}{(x-5)^2} - 16$, and the tangent line to the graph of $y = f(x)$ at $x = 6$ is $y = -4x + 8$. The graph of $f(x)$ is reflected in the x-axis translated 2 units down, then dilated by a factor of $\frac{1}{2}$ from the x-axis. Find the equation of the tangent to the transformed graph when $x = 6$.

$y = \frac{1}{2}(-(-4x+8)-2)$. Tangent is $y = 2x - 5$

b. The graph of $f(x) = 2x^2 - 3x + 1$ has a tangent line at $x = -1$ with an equation of $y = -7x - 1$. $f(x)$ undergoes a translation 3 units right, followed by a dilation by a factor of 4 from the x-axis. Find the equation of the tangent to the transformed graph when $x = 2$.

$y = 4(-7(x-3)-1)$. Tangent is $y = -28x + 80$

c. Consider the graph $f(x) = x^2 - 6x + 4$. The line $y = 2x - 12$ is a tangent to $f(x)$ at $x = 4$. Find the equation of the tangent to $y = 4x^2 - 28x + 32$ at $x = 4$.

$f(x) = (x-3)^2 - 5$

Let $g(x) = 4x^2 - 28x + 32 = (2x-7)^2 - 17 = \left((2x-4)-3\right)^2 - 5 - 12$

Therefore $g(x) = f(2x-4) - 12$

A dilation by a factor of $\frac{1}{2}$ from the y axis and translation 2 units right maps $x = 4$ to $x = 4$. Therefore the equation of the tangent is

$$y = 2(2x-4) - 12 - 12 = 4x - 32$$

Question 118

a. Find the value of a such that the area bound by the graphs $y = x - 2$, $y = ax + a$ and the y-axis is 2 square units.

The lines intersect when $x - 2 = ax + a \Longrightarrow x = \frac{a+2}{1-a}$. Therefore intersect at $\left(\frac{a+2}{1-a}, \frac{3a}{1-a}\right)$

Take the triangle height as $\frac{a+2}{1-a}$, then the base is $a + 2$. Solve

$$\frac{1}{2} \times \frac{a+2}{1-a} \times (a+2) = 2$$

$$a = -8, 0$$

b. It is known that the triangle formed by the lines $y = 2x + 6$, $y = -x - a$, and the x-axis has an area of 5. Find the values of a.

The lines intersect when $2x + 6 = -x - a$.
Therefore, intersect at $\left(\frac{-a-6}{3}, \frac{-2a-6}{3}\right)$.

Take the triangle height as $\frac{-2a-6}{3}$.
Then the base is $-3 - a$.
Solve;

$$\frac{1}{2} x \frac{-2a - 6}{3} x\,(-3 - a) = 5$$
$$a = -2\sqrt{6}$$

c. Find the values of a where the area between the lines $y = ax$, $y = x - 4$ and the y-axis is 12.

The lines intersect when $ax = x - 4 \Rightarrow x = \frac{4}{1-a}$. Therefore intersect at $\left(\frac{4}{1-a}, \frac{4a}{1-a}\right)$
Take the triangle height as $\frac{4}{1-a}$, then base is 4. Solve

$$\frac{1}{2} \times \frac{4}{1-a} \times 4 = 12$$
$$a = \frac{1}{3}$$

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Question 119

a. The shape bound by the lines $y = -\frac{1}{2}x - 1$, $y = x + 5$ and $y = ax - 1$ has an area of 8 square units. Find the value of a if $a \in (-\infty, 1)$.

The lines intersect at $(0, -1)$, $(-4, 1)$ and $\left(\frac{6}{a-1}, \frac{5a+1}{a-1}\right)$

Take the base of the triangle as $\sqrt{\left(-4 - \frac{6}{a-1}\right)^2 + \left(1 - \frac{5a+1}{a-1}\right)^2} = \frac{2\sqrt{2}(a+1)}{a-1}$

Height of triangle is perpendicular to base and goes through $(0, -1)$. Therefore, want a line with a

Gradient -1 and through $(0, -1)$.

$$y = -x - 1$$

Intersection between $y = x + 5$ and $y = -x - 1$ is $(-3, 2)$.

Height of triangle is $\sqrt{(0 - (-3))^2 + ((-1) - 2)^2} = 3\sqrt{2}$

Solve $\frac{1}{2} \times \frac{2\sqrt{2}(2a+1)}{a-1} \times 3\sqrt{2} = 8$

$$a = -\frac{7}{2}$$

b. Hence or otherwise, find the values of m and c such that the area bound by the graphs $y = -2x + 2$, $y = 4x + 8$, and $y = mx + c$ is 2 square units. Assume $m, c \in (1, \infty)$.

All lines have undergone a dilation by a factor of $1/4$ from the y-axis and a translation 3 units up, making the new area $\frac{1}{4} \times 8 = 2$.

The equation of the original lines was $y = -\frac{7}{2}x - 1$.

Therefore, the equation of the new line is $y = 7 - \frac{7}{2}(4x) - 1 + 3 = -14x + 2$.

$$m = -14 \text{ and } c = 2.$$

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Sub-Section: Exam 1 Questions

Question 120

Consider the simultaneous linear equations:

$$2ax - (a+1)y = -1$$

$$\frac{x}{2a+1} + 3y = 4a + 5$$

Where a is a real constant.

a. Find the values of a for which there is a unique solution to the set of equations.

The lines must have different gradients. Solve

$$\frac{2a}{a+1} \neq -\frac{1}{6a+3}$$

$$a \neq -\frac{1}{3}, -\frac{1}{4}$$

Therefore, unique solution for $a \in \mathbb{R} \backslash \left\{-\frac{1}{3}, -\frac{1}{4}\right\}$

b. Find the value of a for which there are no unique solutions.

The y intercepts must not be equal. Solve

$$\frac{1}{a+1} \neq \frac{4a+5}{3}$$

$$a \neq -2, -\frac{1}{4},$$

Therefore, unique solution for $a = -\frac{1}{3}$

c. Find the value of a for which there are infinitely many solutions.

$$a = -\frac{1}{4}$$

Question 121

Consider the points $A(8, -2)$ and $B(2, 6)$.

a. Find the equation of the line that is parallel to the line segment AB, and also contains the point $C(6,9)$.

Gradient of AB is $\frac{-2-6}{8-2} = -\frac{4}{3}$

Line with gradient $-\frac{4}{3}$ and through $(6,9)$

$y = -\frac{4}{3}x + 17$

b. Find the equation of the perpendicular bisector of AB.

Midpoint of AB is $(5,2)$

Perpendicular bisector has gradient $\frac{3}{4}$ and goes through $(5,2)$

$$y = \frac{3}{4}x - \frac{7}{4}$$

c. Find the coordinates of D, the point of intersection between the lines found in **part a.** and **b.**

$(9, 5)$

d. Find the area of the quadrilateral $ABCD$.

$ABCD$ is a trapezium

$|AB| = \sqrt{(8-2)^2 + (-2-6)^2} = 10$

$|CD| = \sqrt{(9-6)^2 + (5-9)^2} = 5$

Height $= \sqrt{(9-5)^2 + (5-2)^2} = 5$

Therefore area $ABCD = \frac{1}{2}(10+5) \times 5 = \frac{75}{2}$

e. Let $E\left(\frac{8}{3}, -4\right)$, $F\left(\frac{2}{3}, 12\right)$, $G(2, 18)$, and $H(3, 10)$. Find the area of $EFGH$.

The points A, B, C, D are dilated by a factor of 2 from the x axis and by a factor of $\frac{1}{3}$ from the y axis

Therefore, area $EFGH = \frac{75}{2} \times 2 \times \frac{1}{3} = 25$

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Question 122

The point $P(4, 1)$ is reflected in the line $y = 2x - 2$ to become the point P'.

a. Find the coordinates of P'.

Want the line with gradient $-\frac{1}{2}$ passing through the point (4,1).

$$y = -\frac{1}{2}x + 3$$

Now we find the intersection of the lines $y = 2x - 2$ and $y = -\frac{1}{2}x + 3$.

∴ Intersection at (2,2).
Therefore, we have $P'(0, 3)$

b. Find the point of intersection between the lines $y = 2x - 2$ and $y = 7x - 27$.

$(5, 8)$

c. The line $y = 7x - 27$ is reflected in the line $2x - 2$. Find the equation of the new line.

The line passes through the points (5,8) and (0,3)

$$y = x + 3$$

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Question 123

At $x = -2$, the graph $y = f(x)$ has a tangent line with the equation $y = 3 - 2x$, and a normal line is given by $y = \frac{1}{2}x + 8$.

a. Find the area bounded by the tangent line, normal line, and the x-axis.

When $x = -2, y = 7$;
Normal line has x-intercept $(-16, 0)$ and tangent has x-intercept $\left(\frac{3}{2}, 0\right)$.
Therefore, area of triangle is $\frac{1}{2} \times 7 \times \left(\frac{3}{2} + 16\right) = \frac{245}{4}$

The graph of $f(x)$ is translated down 3 units, dilated by a factor of 2 from the x-axis, and dilated by a factor of 5 from the y-axis to become the graph $g(x)$.

b. Find the equation of the normal line to $y = g(x)$ at $x = -4$.

Let $t(x) = \frac{1}{2}x + 8$, then the equation of the tangent at $x = -4$ is given by

$$y = 2\left(t\left(\frac{1}{5}x\right) - 3\right) = \frac{1}{5}x + 10$$

c. Find the area bounded by the x-axis, the tangent line and the normal line of the graph $y = g(x)$ at $x = -4$.

$$\text{Area} = 2 \times 5 \times \frac{245}{4} = \frac{1225}{4}$$

Sub-Section: Exam 2 Questions

Question 124

The set of simultaneous equations:

$$\frac{5}{3k-4}y - \frac{x}{2} = \frac{3}{8}k + \frac{3}{2}$$

$$(k-6)x + 2ky = \frac{4}{3} - k$$

Has no solutions for:

A. $k = 3$ or $k = -\frac{10}{3}$

B. $k = -\frac{10}{3}$

C. $k = 3$

D. $k \neq -\frac{2}{3}$ or $k \neq -\frac{10}{3}$

Question 125

The area of the triangle formed by the points $(2, 3)$, $(-4, 7)$ and $(4, 6)$ is:

A. 13 square units.

B. 25 square units.

C. 26 square units.

D. 19 square units.

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Question 126

The graph $f(x) = x^2 - 4x + 3$ has a tangent line and a normal line constructed at $x = 1$. The area bound by the tangent line, the normal line, and the y-axis is $\frac{5}{4}$ square units. The area bound by the y-axis, tangent line, and normal line to the graph $y = -\frac{1}{2}x^2 + 4x - 3$ at $x = -2$ is:

A. $\frac{5}{8}$ square units.

B. $\frac{5}{4}$ square units.

C. 5 square units.

D. 8 square units.

Question 127

The acute angle formed between the lines $y = 3x - 1$ and $y = mx + 5$ is at least 45° when:

A. $m \in \left[\frac{1}{2}, \infty\right)$

B. $m \in \left[-2, \frac{1}{2}\right]$

C. $m \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

D. $m \in \left[-2, 0\right) \cup \left(0, \frac{1}{2}\right]$

Question 128

The equation of the tangent line to $f(x)$ at $x = 2$ is $y = 1 - 4x$. The equation of the normal line to $f(x)$ at $x = 2$ is:

A. $y = \frac{1}{4}x - \frac{15}{2}$

B. $y = -\frac{1}{4}x + 1$

C. $y = 4x - 2$

D. Cannot be determined.

Question 129

Consider the points $A(6,-2)$ and $B(3,4)$.

a. Find the perpendicular bisector of AB.

Midpoint is $\left(\frac{9}{2},1\right)$ and the line joining the points is $y=-2x+10$.
So we want a line with gradient $\frac{1}{2}$ through the point $\left(\frac{9}{2},1\right)$. Therefore, the perpendicular bisector is

$$y=\frac{1}{2}x-\frac{5}{4}$$

b. Find the values of m such that the line $y=mx$ forms a $45°$ angle with the line segment AB.

$|\arctan(-2)-\arctan(m)|=45$
$m=-\frac{1}{3},3$

c. Point $C(m,n)$ and point $D(p,q)$ are different points that lie on the perpendicular bisector of AB, where m, $n\in\mathbb{R}^+$. Find the coordinates of C and D such that the triangles ABC and ABD are both right-angle triangles.

We want a line with gradient 3 and goes through $(6,-2)$. Therefore, the line is $y=3x-20$
Intersection between $y=\frac{1}{2}x-\frac{5}{4}$ and $y=3x-20$ is $\left(\frac{15}{2},\frac{5}{2}\right)$
We want a line with gradient $-\frac{1}{3}$ and goes through $(6,2)$. Therefore, the line is $y=-\frac{1}{3}x$
Intersection between $y=\frac{1}{2}x-\frac{5}{4}$ and $y=-\frac{1}{3}x$ is $\left(\frac{3}{2},-\frac{1}{2}\right)$
Therefore, $C\left(\frac{15}{2},\frac{5}{2}\right)$ and $D\left(\frac{3}{2},-\frac{1}{2}\right)$

d. The point C can be mapped onto the point D by a reflection in the line $y = a$ followed by a reflection in the line
$x = b$. State the values of a and b.

$a = 1$ and $b = \frac{9}{2}$

e. Find the area of $ACBD$.

$$|AC| = \sqrt{\left(6 - \frac{15}{2}\right)^2 + \left(-2 - \frac{9}{2}\right)^2}$$

$$|AC| = \frac{\sqrt{178}}{2}$$

$$\text{Area} = \left(\frac{\sqrt{178}}{2}\right)^2 = \frac{89}{2}$$

f. Find the area of the square that has opposite corners at $(7, -4)$ and $(1, 8)$.

A, B have been dilated by a factor of 2 from the x axis and dilated by a factor of 2 from the y axis and translated 5 units left

Area $= \frac{89}{2} \times 2 \times 2 = 178$

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Question 130

The function $f(x) = 2(x+3)^2 - 5$ has a tangent line with the equation $y = 4x + 5$.

a. Show that $y = 4x + 5$ is a tangent to $f(x)$ at the point $(-2, -3)$.

$f(x)$ and the tangent line will intersect at the point of tangency
$2(x+3)^2 - 5 = 4x + 5$
$2x^2 + 8x + 8 = 0$
$2(x+2)^2 = 0$
$x = -2$
$f(-2) = -3$
Therefore, the point of tangency is $(-2, -3)$

b. Find the equation of the normal line to $f(x)$ at $x = -2$.

Normal line has a gradient $-\frac{1}{4}$ and passes through $(-2, -3)$

$$y = -\frac{1}{4}x - \frac{7}{2}$$

c. State the obtuse angle formed between the line $y = 4x + 5$ and the x-axis, correct to 2 decimal places.

$\theta = 180 - \arctan(4) = 104.04^o$

d. Find the area enclosed by the tangent line, the normal line, and the x-axis.

Normal line has x-intercept $(-14, 0)$ and tangent has x-intercept $\left(-\frac{5}{4}, 0\right)$.
Therefore, the triangle has area $\frac{1}{2} \times 3 \times \frac{51}{4} = \frac{153}{8}$

The graph of $y = f(x)$ is translated 4 units right, dilated by a factor of 4 from the x-axis, and dilated by a factor of $\frac{2}{3}$ from the y-axis to become the graph $y = g(x)$.

e. Find the equation of the tangent line to $y = g(x)$ at $x = \frac{4}{3}$.

$g(x) = 4f\left(\frac{3}{2}x - 4\right)$. The tangent when $x = \frac{4}{3}$ is given by
$$y = 4\left(4\left(\frac{3}{2}x - 4\right) + 5\right) = 24x - 44$$

f. State the obtuse angle formed between the new tangent of $y = g(x)$ at $x = \frac{4}{3}$, correct to 2 decimal places.

$$\theta = 180 - \arctan(24) = 92.39^o$$

g. Find the area of the triangle formed between the x-axis, the tangent, and the normal line to $y = g(x)$ at $x = \frac{4}{3}$.

$$\text{Area} = 4 \times \frac{2}{3} \times \frac{153}{8} = 51$$

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Section C: [1.7] – Polynomials (Checkpoints)

Sub-Section [1.7.1]: Applying Factor and Remainder Theorems

Question 131

a. State the remainder when $x^2 + 5x - 3$ is divided by $x + 2$.

We can write $x^2 + 5x - 3 = (x+2)Q(x) + r$ for some quadratic Q. Hence,

$$r = (-2)^2 + 5(-2) - 3 = -9$$

b. Is $x - 2$ a factor of $f(x) = x^4 - 16$?

$x - 2$ is a factor of $f(x)$ if and only if $f(2) = 0$. As

$$f(2) = 2^4 - 16 = 0,$$

$x - 2$ is a factor of f.

c. Is $x + 4$ a factor of $g(x) = x^3 + 4x^2 + 2$?

$x + 4$ is a factor of $g(x)$ if and only if $g(-4) = 0$. As

$$g(-4) = (-4)^3 + 4(-4)^2 + 2 = -64 + 64 + 2 = 2 \neq 0$$

$x + 4$ is not a factor of g.

Question 132

Let $f(x) = 2x^3 + ax^2 + ax + 3$. Find the value of a such that $f(x)$ has a factor of $2x + 3$.

We require $f\left(-\frac{3}{2}\right) = -\frac{27}{4} + \frac{9a}{4} - \frac{3a}{2} + 3 = 0$. Multiplying our expression by 4 yields,

$$-27 + 9a - 6a + 12 = 0 \implies 3a = 15 \implies a = 5$$

Question 133

Let $f(x) = x^2 + ax + b$. Find the values of a and b such that f has a factor of -1, and when f is divided by $2x - 3$, it has a remainder of -5.

Since -1 is a root of f, we see that, $f(-1) = 0$.

Since $f/(2x-3)$ yields a remainder of -5, we see that, $f\left(\frac{3}{2}\right) = -5$. From here we construct a pair of simultaneous equations,

$$f(-1) = 1 - a + b = 0 \quad (1)$$

$$f\left(\frac{3}{2}\right) = \frac{9}{4} + \frac{3}{2}a + b = -5 \quad (2)$$

Subtracting (2) from (1) yields,

$$-\frac{5}{4} - \frac{5a}{2} = 5 \implies -10a = 25 \implies a = \frac{-5}{2}.$$

Substituting this value of a into (1) yields, $b = a - 1 = \frac{-7}{2}$

Question 134

A cubic polynomial, $g(x)$ has the following properties:

1. $g(x) - 3$ has a factor of $(x-2)^2$.

2. $g(x)$ divided by $x^2 - 1$ leaves a remainder of 2.

Find the rule for $g(x)$.

The first statement implies that $g(x) - 3 = a(x-2)^2(x-b)$ for some real numbers a and b.

Since $x^2 - 1 = (x-1)(x+1)$, the second statement implies that $g(1) = g(-1) = 2$.

Hence, $a(1-2)^2(1-b) = a(1-b) = 2 - 3 = -1$ and $a(-1-2)^2(-1-b) = -9a(1+b) = 2 - 3 = -1$. Equating these two expressions yields,

$$a - ab = -9a - 9ab \implies 10a = -8ab \implies b = -\frac{5}{4}$$

Substituting this value into $a(1-b) = -1$ yields, $\frac{9a}{4} = -1 \implies a = -\frac{4}{9}$. Hence,

$$g(x) = -\frac{4}{9}(x-2)^2\left(x + \frac{5}{4}\right) + 3$$

Sub-Section [1.7.2]: Finding Factored Forms of Polynomials

Question 135

Factorise the following polynomials:

a. $x^3 - 8$

We apply difference of cubes with $a = x$ and $b = 2$, thus,

$$x^3 - 8 = (x-2)(x^2 + 2x + 4)$$

As $2^2 - 4 \times 4 < 0$ we cannot factorise our expression any further.

b. $x^3 - 7x^2 + 10x$

$$x^3 - 7x^2 + 10x = x(x^2 - 7x + 10) = x(x-2)(x-5)$$

c. $x^3 + 3x^2 - 4x - 12$

$$x^3 + 3x^2 - 2x - 6 = x^2(x+3) - 4(x+3) = (x+3)(x^2-4) = (x+3)(x-2)(x+2)$$

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Question 136

a. Factorise $f(x) = x^3 + x^2 - 17x + 15$.

By the rational root theorem, our possible roots are $\pm1, \pm3, \pm5$ and ±15.
After some testing we see that $f(1) = 0$ hence $x - 1$ is a factor.
Thus $f(x) = (x-1)(x^2 + 2x + 15) = (x-1)(x+5)(x-3)$

b. Factorise $g(x) = x^3 - 4x^2 + x + 6$.

$g(x) = (x+1)(x-2)(x-3)$

c. Find all of the real roots of $h(x) = x^3 - 3x^2 + 4$.

By the rational root theorem, possible rational roots of h are $\pm1, \pm2, \pm4$.
After some testing we see that -1 is a root hence $x + 1$ is a factor of h.
Thus $h(x) = (x+1)(x^2 - 4x + 4) = (x+1)(x-2)^2$.
Hence the real roots of $h(x)$ are $-1, 2$.

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Question 137

a. Factorise $f(x) = x^3 - 5x^2 - 29x + 105$.

$(x - 7)(x - 3)(x + 5)$

b. Factorise $g(x) = 18x^3 - 3x^2 - 28x - 12$.

$(2 + 3x)^2(2x - 3)$

c. Factorise $h(x) = 2x^3 + 14x^2 - 10x - 150$.

$$2(x-3)(x+5)^2$$

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Question 138

Let $f(x) = ax^2 + bx + c$ with a, b, c being co-prime non-zero integers, and assume that $\frac{p}{q}$ is a root of f with p and q co-prime and both non-zero.

a. Show that p divides c.

We know that $f\left(\frac{p}{q}\right) = a\frac{p^2}{q^2} + b\frac{p}{q} + c = 0$.
After subtracting c from both sides and multiplying both sides by q^2 we have that,

$$-cq^2 = p(ap + bq)$$

Hence p is a factor of cq^2. Since q is coprime to p it follows that p divides c.

b. Show that q divides a.

Like in part **a.** we can rearrange $f\left(\frac{p}{q}\right) = 0$ to get the equation,

$$-ap^2 = q(bp + cq)$$

Thus we see that q divides ap^2. Since p is coprime to q it follows that q divides a.

c. If a, b, c are not co-prime integers, where would your arguments for **parts a.** and **b.** breakdown?

In the equation $-cq^2 = p(ap + bq)$ we assume that $ap + bq$ is an integer, hence p is a factor of $-cq^2$. If $ap + bq$ is not an integer, this may no longer be the case.

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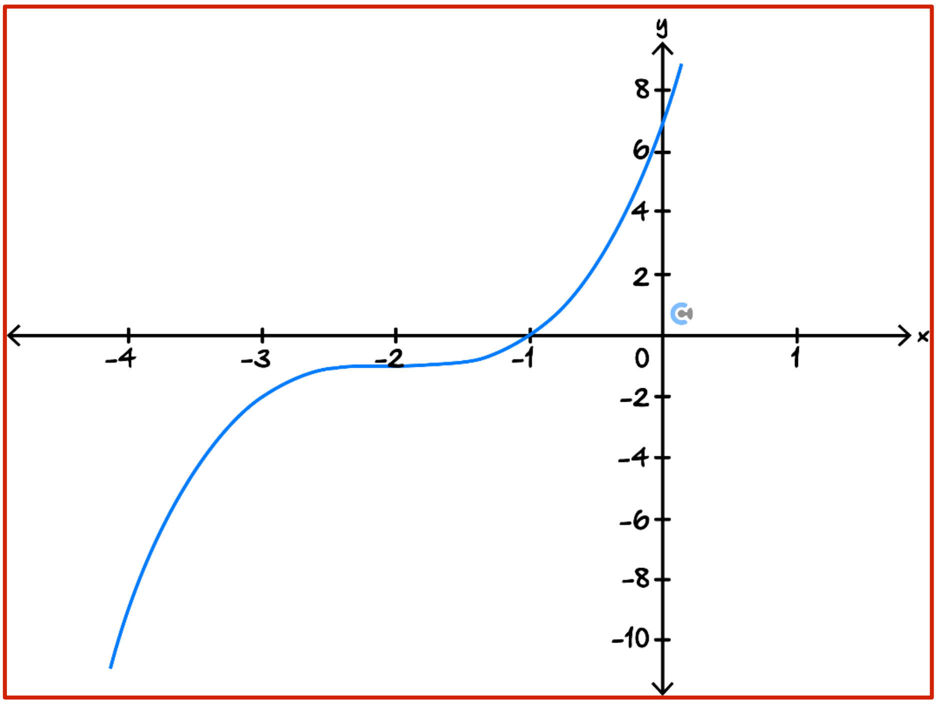
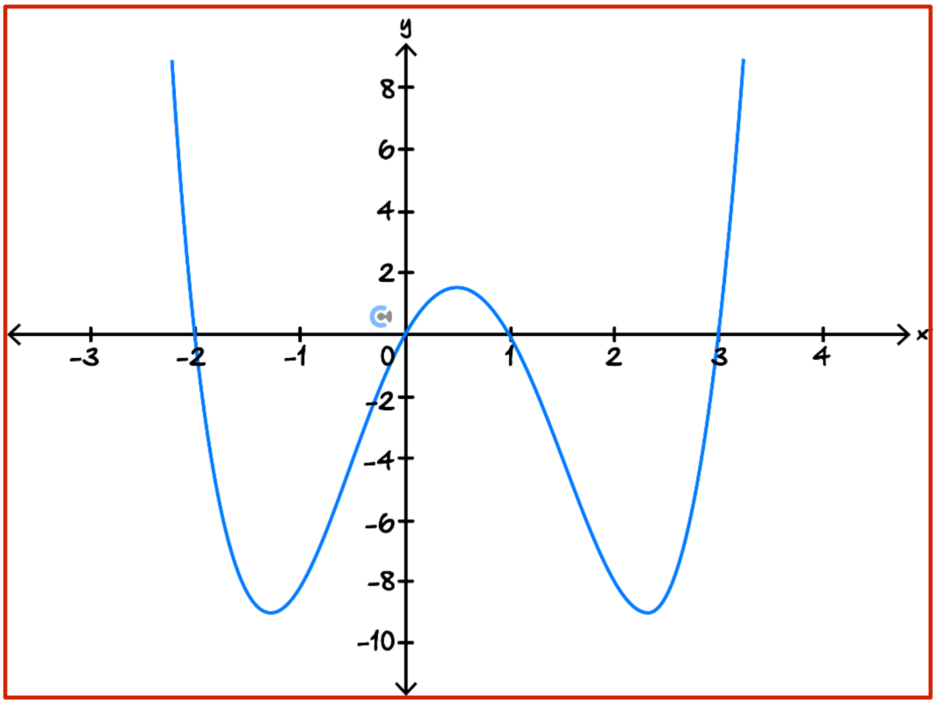
Question 139

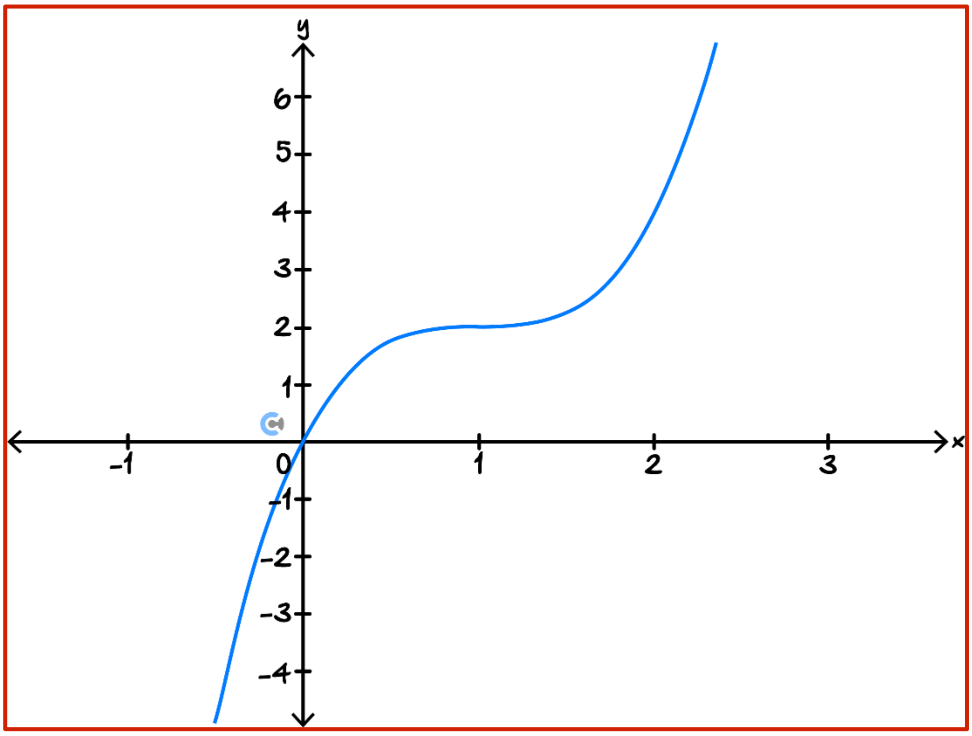
a. Sketch the graph of $y = (x + 2)^3 - 1$ on the axis below.



b. Sketch the graph of $y = x(x - 1)(x + 2)(x - 3)$ on the axis below.

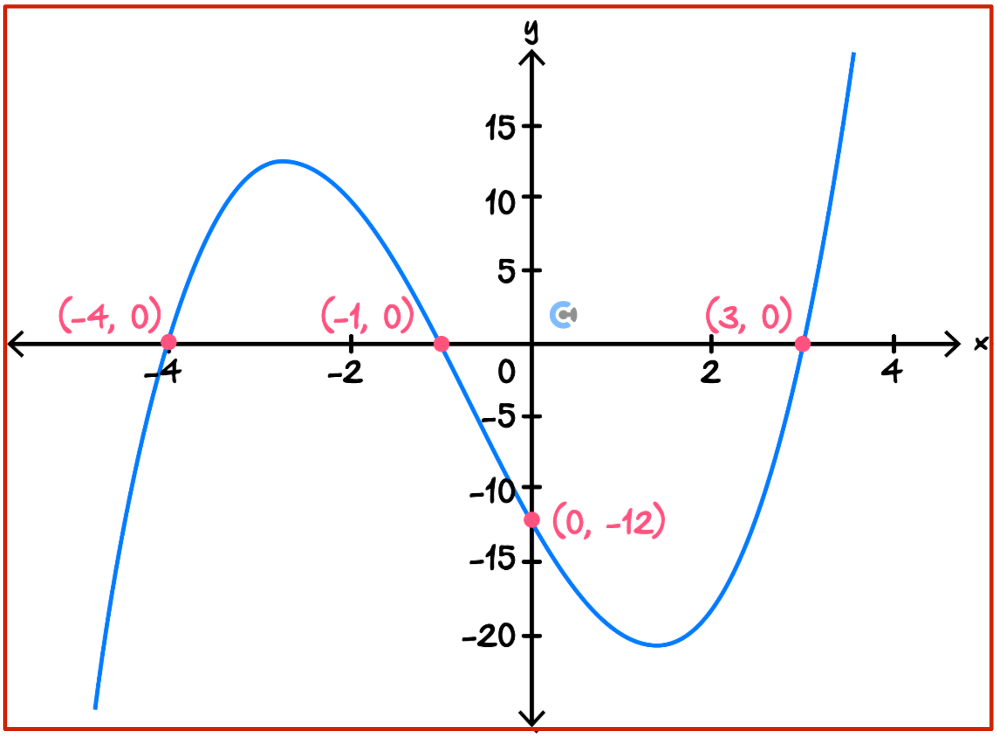


c. Sketch the graph of $y = 2(x-1)^3 + 2$ on the axis below.

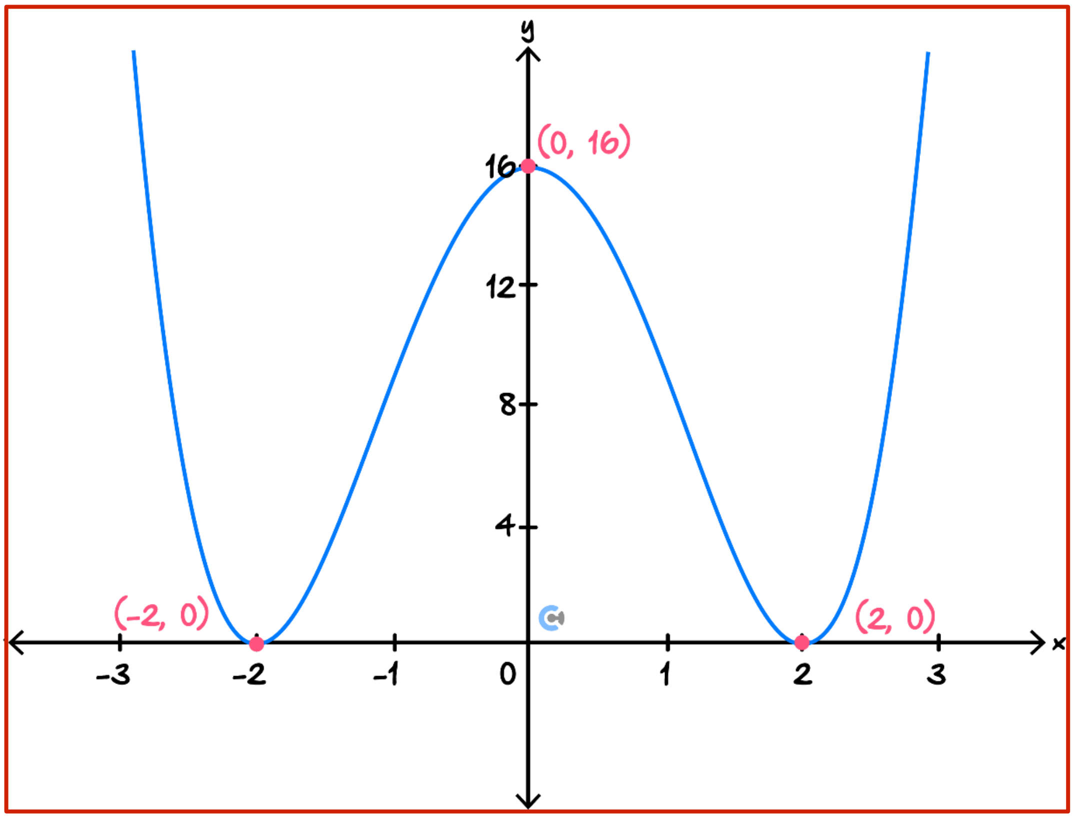


Question 140

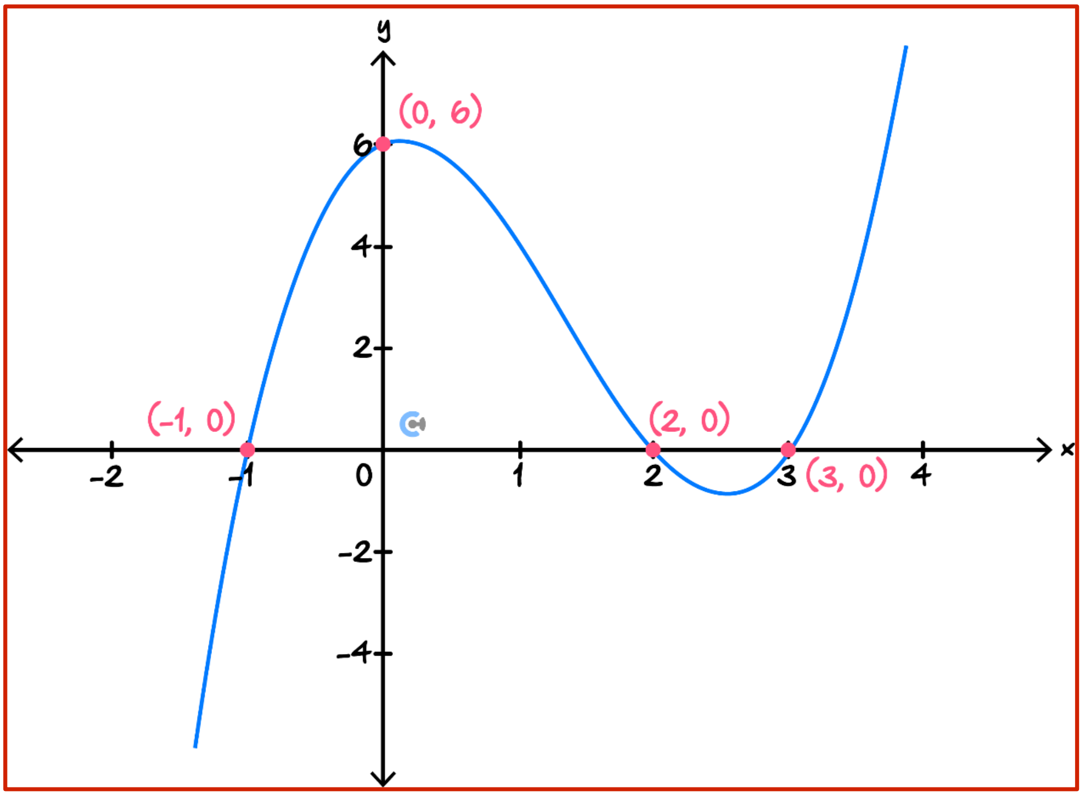
a. Sketch the graph of $y = x^3 + 2x^2 - 11x - 12$ on the axis below, the labelling axis intercepts with their coordinates.



b. Sketch the graph of $y = x^4 - 8x^2 + 16$ on the axis below, the labelling axis intercepts with their coordinates.

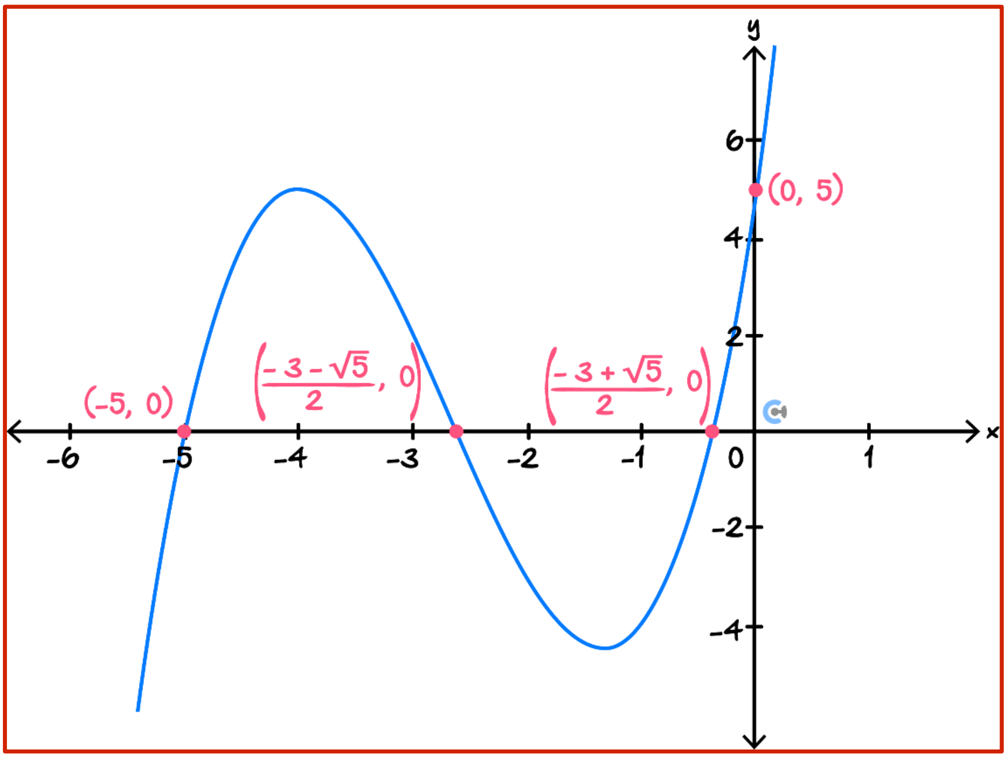


c. Sketch the graph of $y = x^3 - 4x^2 + x + 6$ on the axis below, the labelling axis intercepts with their coordinates.

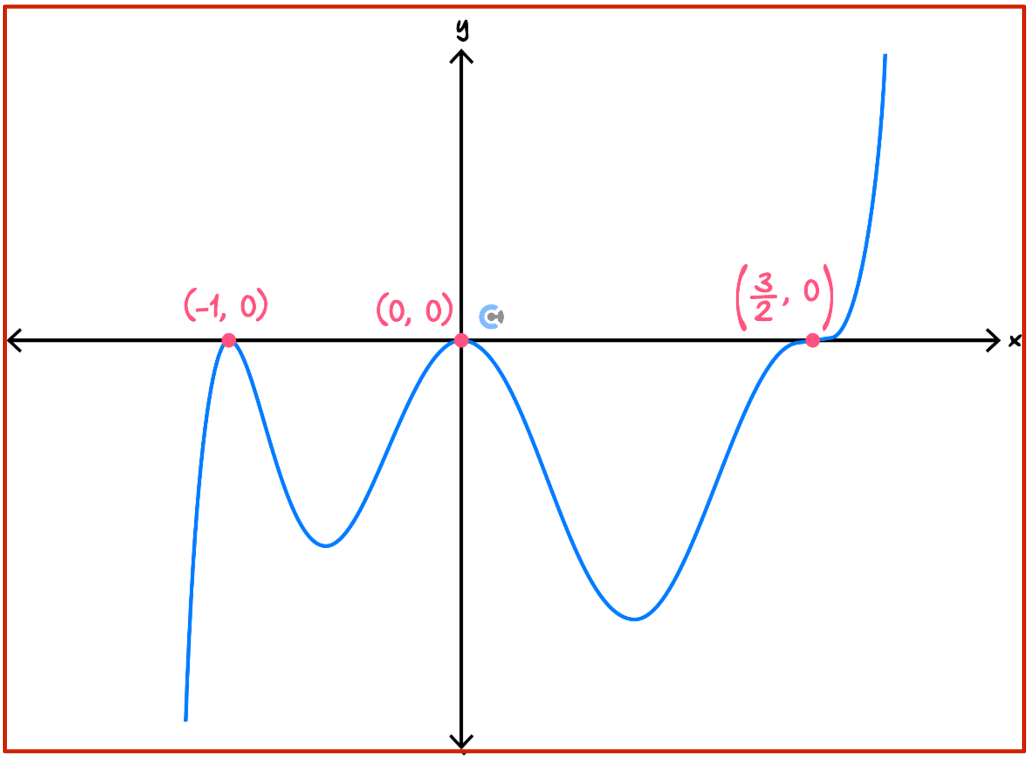


Question 141

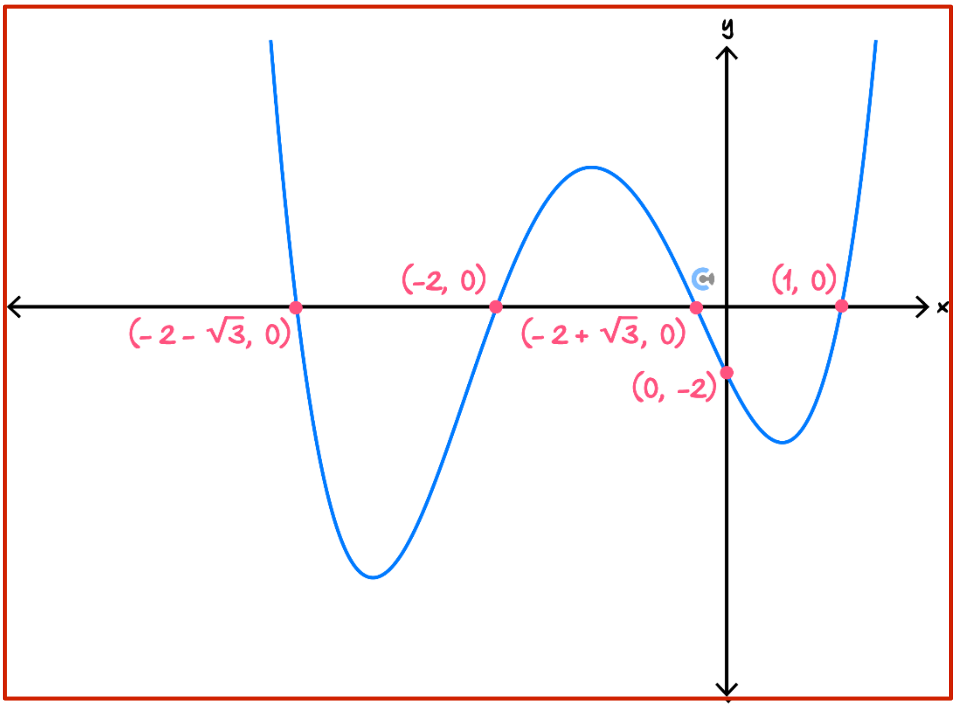
a. Sketch the graph of $y = x^3 + 8x^2 + 16x + 5$ on the axis below, the labelling axis intercepts with their coordinates.



b. Sketch the graph of $y = x^2(2x - 3)^3(x + 1)^2$ on the axis below, the labelling axis intercepts with their coordinates.

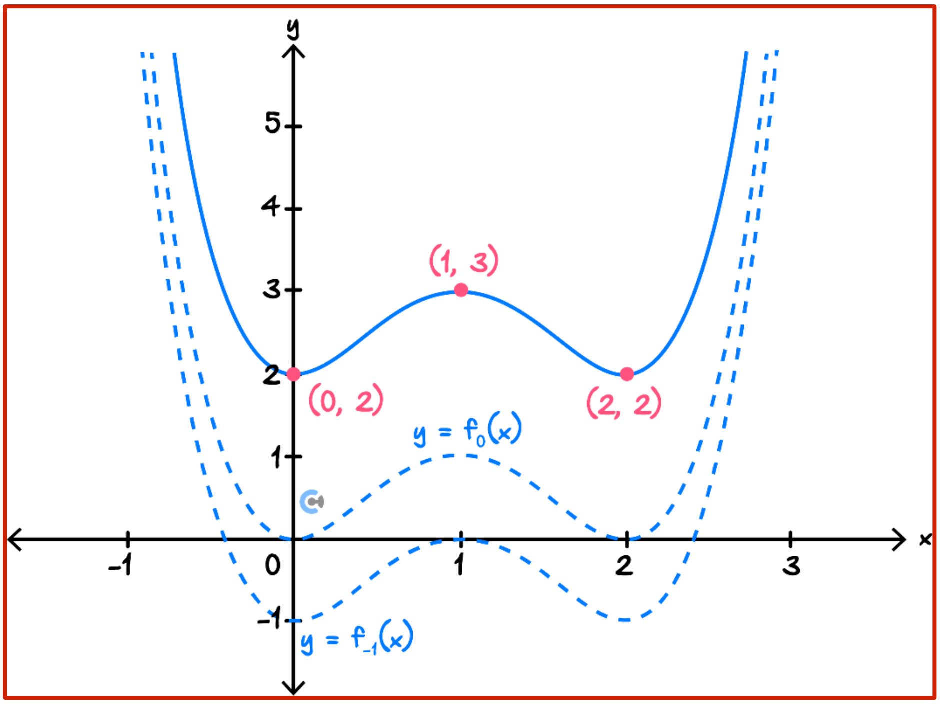


c. Sketch the graph of $y = x^4 + 5x^3 + 3x^2 - 7x - 2$ on the axis below, the labelling axis intercepts with their coordinates.



Question 142

Let $f_k(x) = x^4 - 4x^3 + 4x^2 + k$. By considering f_0 and f_{-1}, sketch the graph of f_2 on the axis below, the labelling axis intercepts and turning points with their coordinates.



Sub-Section [1.7.4]: Identify Odd and Even Functions

Question 143

a. Let $f(x)$ and $g(x)$ both be an odd function.

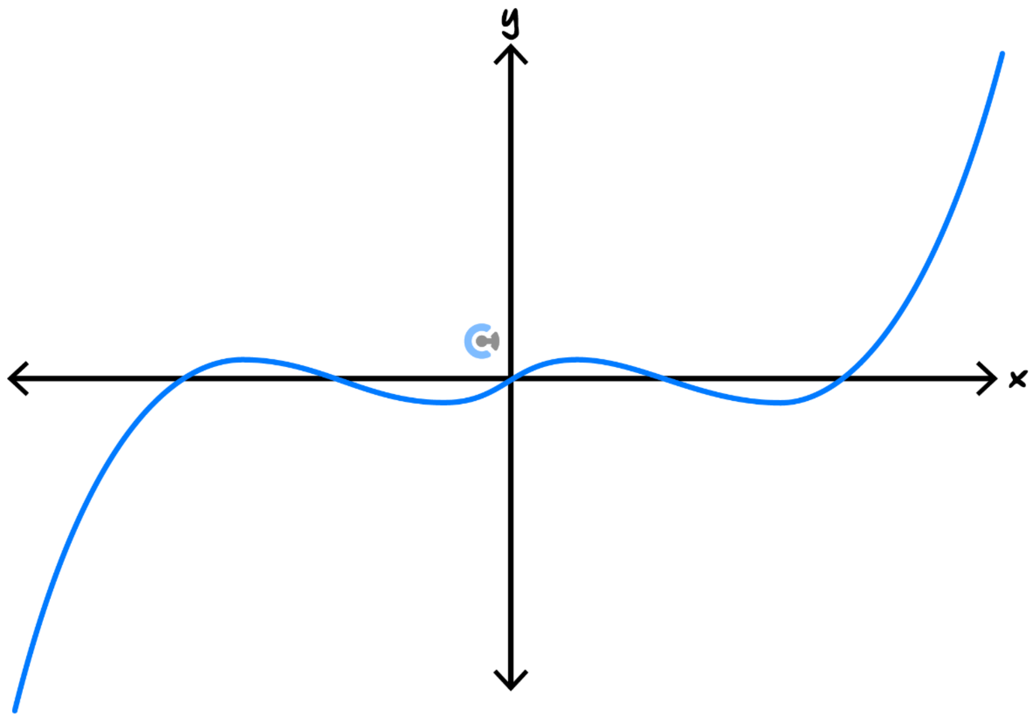
i. State whether $f(x) + g(x)$ is an even or an odd function.

As $f(-x) + g(-x) = -f(x) - g(x) = -[f(x) + g(x)]$, $f(x) + g(x)$ is an odd function.

ii. State whether $(f(x))^2 + 2f(x)g(x) + (g(x))^2$ is an even or an odd function.

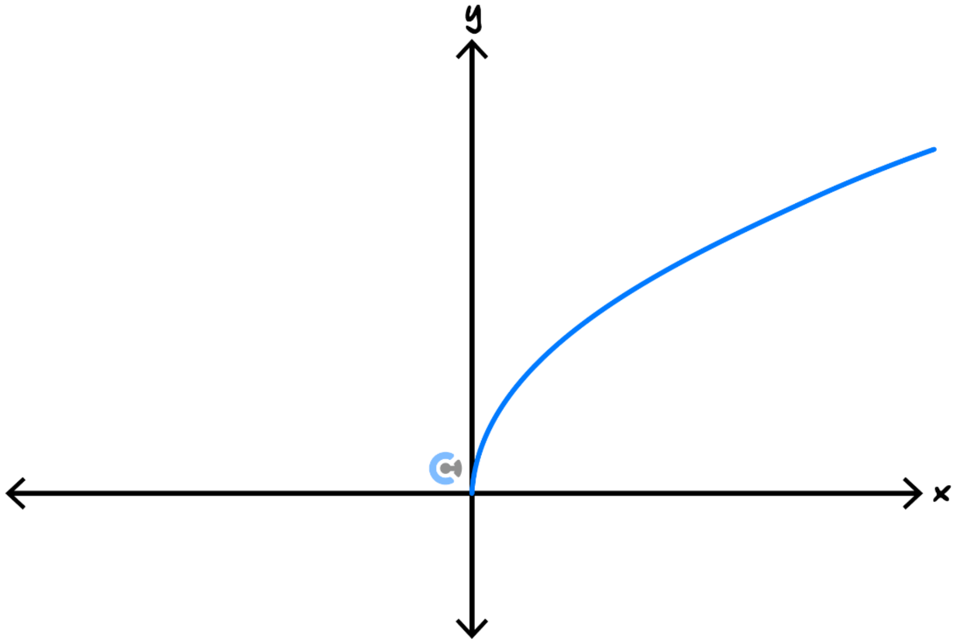
$(f(x))^2 + 2f(x)g(x) + (g(x))^2 = (f(x) + g(x))^2$ and is hence an even function.

b. Part of the graph of $f(x)$ is drawn below. State whether f is an odd or an even function.



$f(x)$ is an odd function.

c. Part of the graph of $y = x^{\frac{m}{n}}$ is drawn below where m and n are co-prime.



State whether m and n are even or odd.

As the domain of our graph only positive numbers, n is even.

As m and n are co prime, m is thus odd.

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Question 144

a. Show that $f(x) = x^4 - 2x^3$ is neither an even nor an odd function.

Observe that $f(-x) = x^4 + 2x^3 \neq x^4 - 2x^3$, hence f is not an even function.
Observe that $f(-x) = x^4 + 2x^3 \neq -x^4 + 2x^3$, hence f is not an odd function.

b. Describe a translation that maps the graph of $y = x^2 + 6x + 7$ onto the graph of an even function.

$x^2 + 6x + 7 = (x+3)^2 - 2$. Thus we want to map our graph onto the graph of $y = x^2 - 2$, hence we simply need to translate our graph 3 units to the right.

c. Consider the function $f(x)$. It is known that $f(2x + 3)$ is an odd function.

If $f(5) = 4$ and $f(-1) = -3$, find the value of $f(1)$.

Let $g(x) = f(2x + 3)$. Then,

$$f(1) = g(-1) = -g(1) = -f(5) = -4.$$

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Question 145

a. Let $f(x)$ be a strictly increasing function with $f(0) = 0$.

If $\left(f(x)\right)^2$ is an even function, show that $f(x)$ is an odd function.

Since $(f(x))^2 = (f(-x))^2$ we know that for all $x > 0$ that $f(x) = \pm f(-x)$.
However since f is an increasing function it must be one to one, and thus $f(x) \neq f(-x)$ for $x > 0$.
Hence $f(x) = -f(-x)$ for $x > 0$, which implies that $f(x) = f(-x)$ for $x \neq 0$. Lastly we see that $f(0) = -f(-0) = 0$.

b. Let $f(x) = x^4 + 2x^3 + x^2$.

Describe a transformation that maps the graph of f onto the graph of an even function.

Observe that $f(x) = x^2(x+1)^2$.
We can map the graph of f onto the graph of $y = \left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{2}\right)^2$ which is an even function by translating it $\frac{1}{2}$ units right.

c. Let $f(x)$ be an even function.
The function,

$$g(x) = \begin{cases} f(x) + c & x \geq 0 \\ -f(x) + d & x < 0 \end{cases}$$

is an odd function.

Find the values of c and d.

We require $g(0) = -g(-0) = g(0) = f(0) + c = 0$ for g to be an odd function.
Hence $c = -f(0)$.
Now for $x > 0$ we require that $f(x) + c = g(x) = -g(-x) = f(-x) - d = f(x) - d$.
Hence $d = -c = f(0)$.

Question 146

Let $f(x) = x^4 - 4x^3 + x^2 + 6x + k$, where k is a real number.
The function $g(x) = f(x - h)$ is an even function.
Find the value of h.

We observe that as k represents a vertical translation, it does not affect whether or not g is an even function.
Thus for simplicity we set $k = 0$ and factorise f.
Thus $f(x) = x(x^3 - 4x^2 + x + 6) = x(x + 1)(x - 2)(x - 3)$.
Hence $f(x + 1) = (x + 1)(x + 2)(x - 1)(x - 2)$ is an even function and thus $h = -1$.

Section D: [1.8] - Polynomials (Checkpoints)

Sub-Section [1.8.1]: Apply Transformations to Restrict the Number of Positive/Negative x-Intercept(s)

Question 147

Let $f(x) = (x-1)(x+4)(x-2)^2$. Find the values of k such that $f(x+k)$ has no positive x-intercepts.

We want $2-k \leq 0$, so $k \geq 2$.

Question 148

Let $f(x) = x^3 - 2x^2 - 5x + 6$. Find the values of k such that $f(x+k)$ has exactly one negative x-intercept.

Note that $f(x) = (x-1)(x+3)(x+2)$.
For exactly one negative x-intercept, we need $-2-k < 0$ and $1-k \geq 0$.
Hence, $-2 < k \leq 1$.

Question 149

Let $f(x) = 2x^2 - 15x + 14$ and $g(x) = x^2 - 10x + 8$. Find the values of k such that $f(x+k)$ and $g(x+k)$ have exactly two intersections with negative x-coordinates.

Note that $f(x) = g(x)$ is equivalent to $f(x) - g(x) = x^2 - 5x + 6 = (x-2)(x-3) = 0$.

Therefore, $f(x+k) = g(x+k)$ will have exactly two intersections with negative x-coordinates for $3-k < 0$, therefore $k > 3$.

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Question 150

Let $f(x) = \frac{1}{2}x + 3$ and $g(x) = 2x^2 - 4x - 22$. Find the values of k such that $f\big(g(x+k)\big)$ has exactly one negative x-intercept.

Note that $f(g(x)) = x^2 - 2x - 8 = (x-4)(x+2)$. Therefore, $f(g(x+k))$ will have exactly one negative x-intercepts for $-2 < k \leq 4$.

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Sub-Section [1.8.2]: Apply Discriminant to Solve Number of Solutions Questions

Question 151

Find the values of k such that the equation $x^2 - 2^k x + 4$ has no solutions.

$$(-2^k)^2 - 4 \cdot 1 \cdot 4 < 0 \implies 2^{2k} < 2^4 \implies 2k < 4 \implies k < 2.$$

Question 152

Find the values of k such that the equation $x^2 - 2kx + 5k$ has exactly two solutions.

$$(-2k)^2 - 4 \cdot 5k > 0 \implies 4k^2 - 20k > 0 \implies k^2 - 5k > 0 \implies k < 0 \text{ or } k > 5.$$

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Question 153

Find the values of k such that the equation $(x^2 - kx + 4)(x^2 - 2\sqrt{3}x + k) = 0$ has exactly three solutions.

Either $x^2 - kx + 4$ gives two solutions (which requires $k < -4$ or $k > 4$) and $x^2 - 2\sqrt{3}x + k$ gives one solution (so that $k = 3$), or $x^2 - kx + 4$ gives one solution (so that $k = \pm 4$) and $x^2 - 2\sqrt{3}x + k$ gives two solutions (so that $k < 3$). Therefore, $k = -4$ is the only acceptable value.

Question 154

Let $f(x) = x^2 - 4x + 3$ and $g(x) = x^2 - 6x + k$. Find the values of k such that $f\big(g(x)\big)$ has exactly four solutions.

The equation $f\big(g(x)\big) = 0$ gives $g(x) = x^2 - 6x + k = 1$ or $g(x) = x^2 - 6x + k = 3$.
These two equations in total will result in four solutions if both equations individually have two solutions.
The discriminant for each equation is positive whenever $(-6)^2 - 4(k-1) > 0$ and also $(-6)^2 - 4(k-3) > 0$, i.e.,, $k < 10$.

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Sub-Section [1.8.3]: Apply Shape/Graph to Solve Number of Solutions Questions

Question 155

Suppose $f(x) = x^2 - kx + 3$. Find the value of $k > 0$ so that $f(x) = k$ has exactly one solution.

We may solve $(-k)^2 - 4 \cdot 1 \cdot 3 = 0$ to find $k = \pm 2\sqrt{3}$. We reject $k = -2\sqrt{3}$ because $k > 0$ is specified in the question. Hence, $k = 2\sqrt{3}$.

Question 156

It is known that the quartic $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 8.5$ has turning points at $(1, -0.5)$, $(2, 0.5)$ and $(3, -0.5)$. Find the values of k such that $f(x) = k$ has exactly two solutions.

By inspection of the graph, $k > \frac{1}{2}$ or $k = -\frac{1}{2}$.

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Question 157

It is known that the quartic $f(x) = x^4 - 4x^3 - 8x^2 + 48x + 3$ has turning points at $(-2, -77), (2, 51)$ and $(3, 48)$. Find the values of k such that $f(x) = k$ has exactly two solutions.

By inspection of the graph, $-77 < k < 48$ or $k > 51$.

Question 158

Let $f(x) = x^4 - 16x^3 + 46x^2 - 48x + 20$ and $g(x) = -x^4 + 2x^2 + 3$. It is known that the quartic $h(x) = 2x^4 - 16x^3 + 44x^2 - 48x + 17$ has turning points at $(1, -1), (2,1)$ and $(3, -1)$. Hence or otherwise, find the value of k such that $f(x) = g(x) + k$ has exactly three solutions.

First note that $f(x) = g(x) + k$ is equivalent to $f(x) - g(x) = k$, i.e. $2x^4 - 16x^3 + 44x^2 - 48x + 17 = k$. By inspection, we conclude $k = 1$

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Sub-Section [1.8.4]: Apply Odd and Even Functions

Question 159

Show that the function given by $f(x) = x^5 - 2x^2 + 1$ is neither even nor odd.

Observe that $f(1) = 0$ and $f(-1) = -2$. Notice that $f(1) \neq f(-1)$ therefore f is not even. Similarly, $f(1) \neq -f(-1)$, so f is not odd either.

Question 160

Let $f(x) = x^4 - (k^2 - 5k + 6)x^3 + k^3x^2 + 10$. Find the value(s) of k so that $f(x)$ is an even function.

Set k so that the coefficients of the odd power terms are zero. Hence $k^2 - 5k + 6 = 0$, i.e. $k = 2$ or $k = 3$.

Question 161

The tangent to the graph of $f(x) = x^2 - 4$ at the point $x = 2$ is given by $h(x) = 4x - 8$. Denote the tangent to $f(x)$ at $x = -2$ by $k(x)$. Find the rule for $k(x)$ by applying a reflection to $h(x)$.

Note that $f(x)$ is an even function, so it is symmetric about the y-axis. The tangent at $x = -2$ may obtained as a reflection of $h(x)$ across the y-axis. Hence, $k(x) = 4(-x) - 8 = -4x - 8$.

Question 162

The tangent to the graph of $f(x) = x^3 - 3x$ at the point $x = 2$ is given by $h(x) = 9x - 16$. Denote the tangent to $f(x)$ at $x = -2$ by $k(x)$. The rule for $k(x)$ can be obtained from the rule of $h(x)$ via the following sequence of transformations:

- A translation of a units in the positive direction of the x-axis.
- A translation of b units in the positive direction of the y-axis.

State the values of a and b and hence or otherwise, find the rule of $k(x)$.

We want to map the point $(2, 2)$ to $(-2, -2)$, so we need $a = -4, b = -4$ and $k(x) = 9(x + 4) - 16 - 4 = 9x + 16$.

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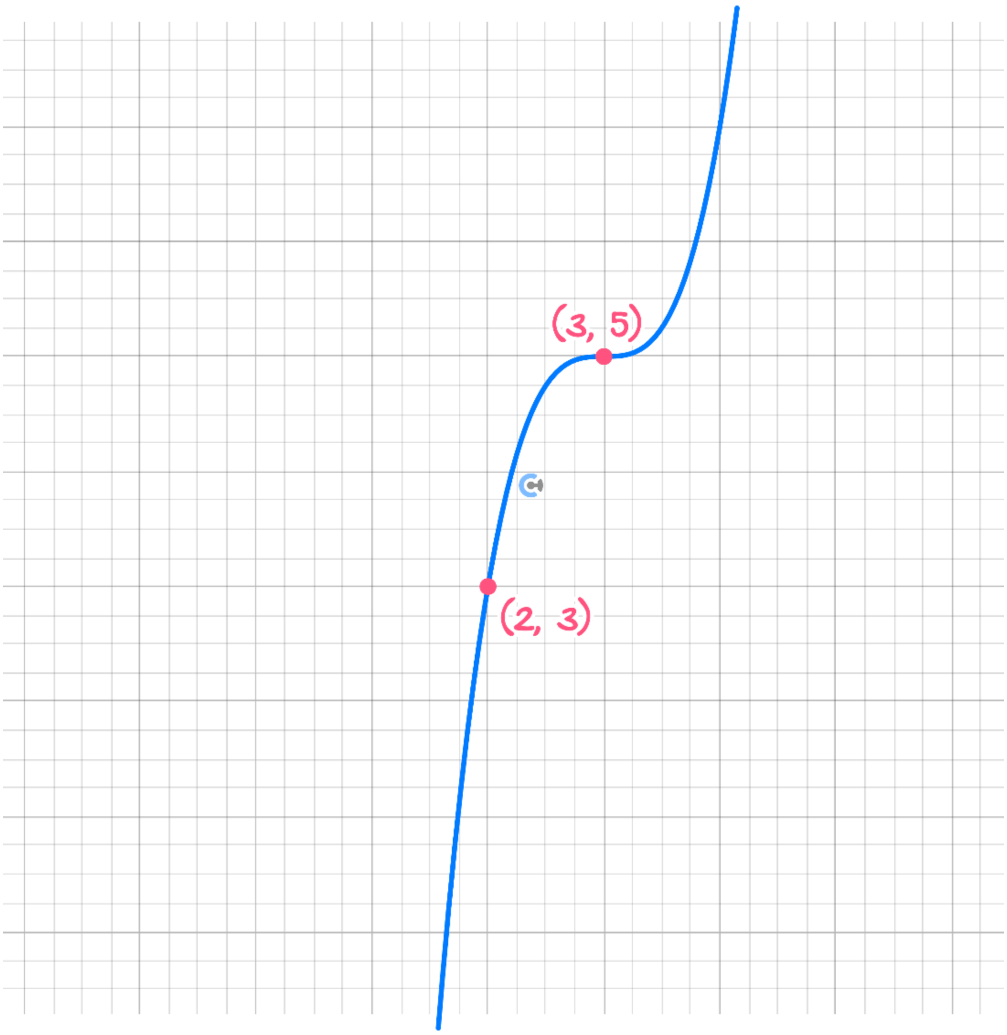
Sub-Section [1.8.5]: Identify Possible Rule(s) from a Graph

Question 163

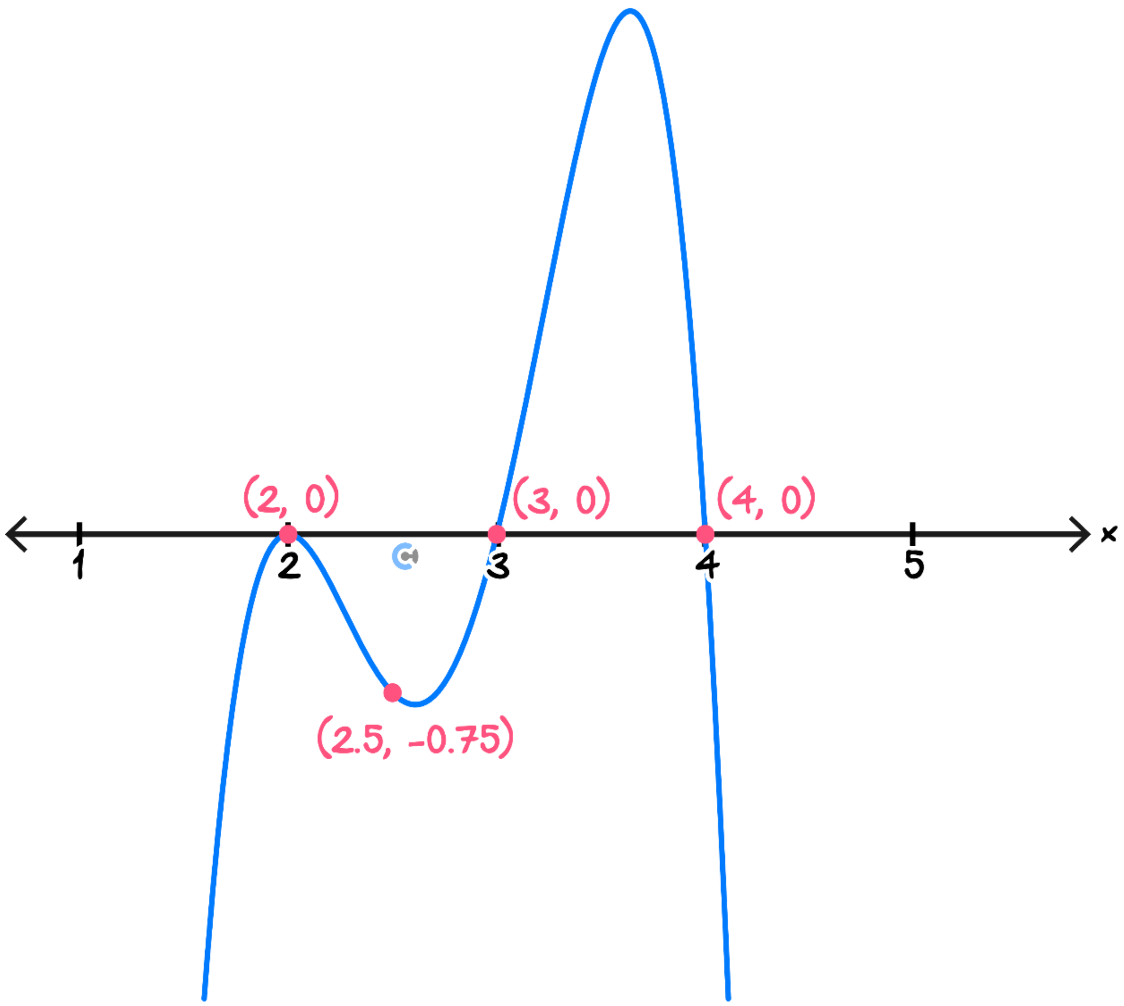
Part of the graph of $f(x)$ is plotted below. The point $(3,5)$ is a stationary point of inflection. Find a possible rule for the function.



The function could be $f(x) = 2(x-3)^3 + 5$.

Question 164

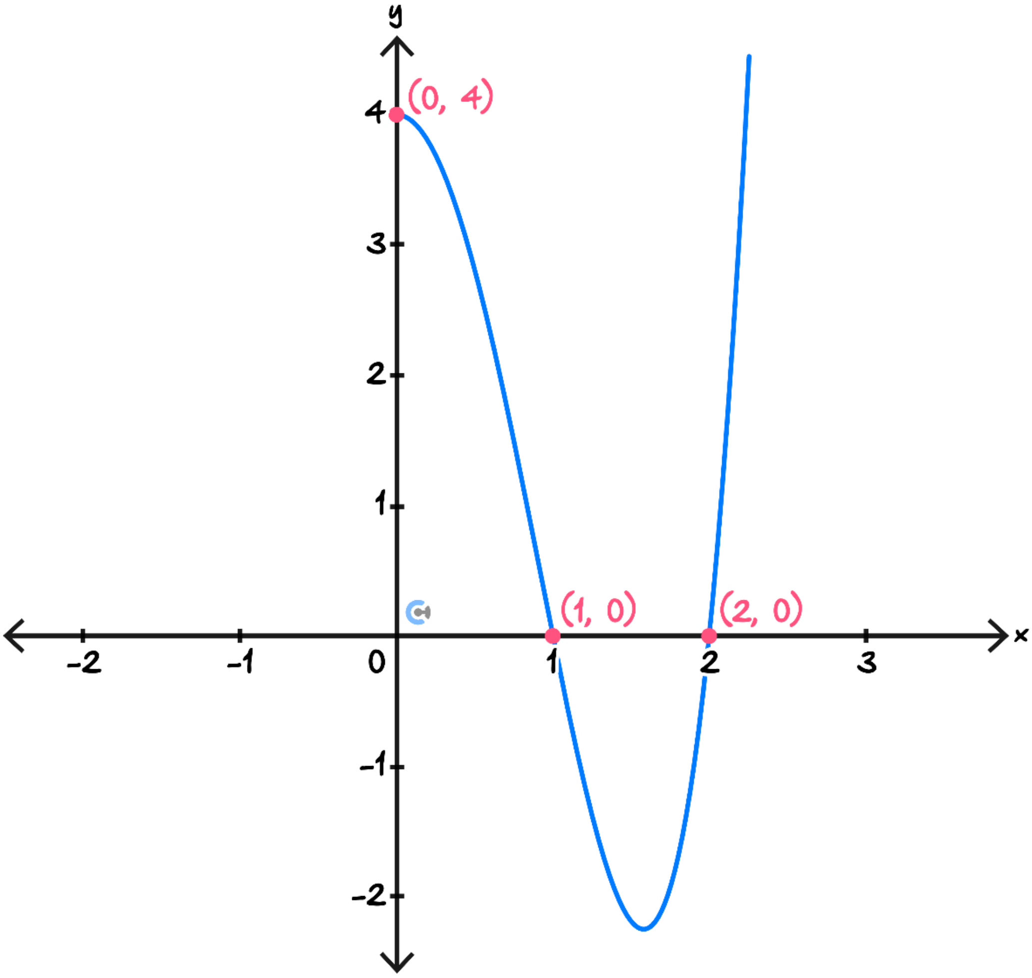
Part of the graph of $f(x)$ is plotted below. Find a possible rule for the function.



The rule for this function could be $f(x) = -4(x-2)^2(x-3)(x-4)$.

Question 165

Part of the graph $f(x)$ is plotted below. Find a possible rule for the function if the function is known to be even.



The rule could be given by $f(x) = (x-1)(x-2)(x+1)(x+2)$.

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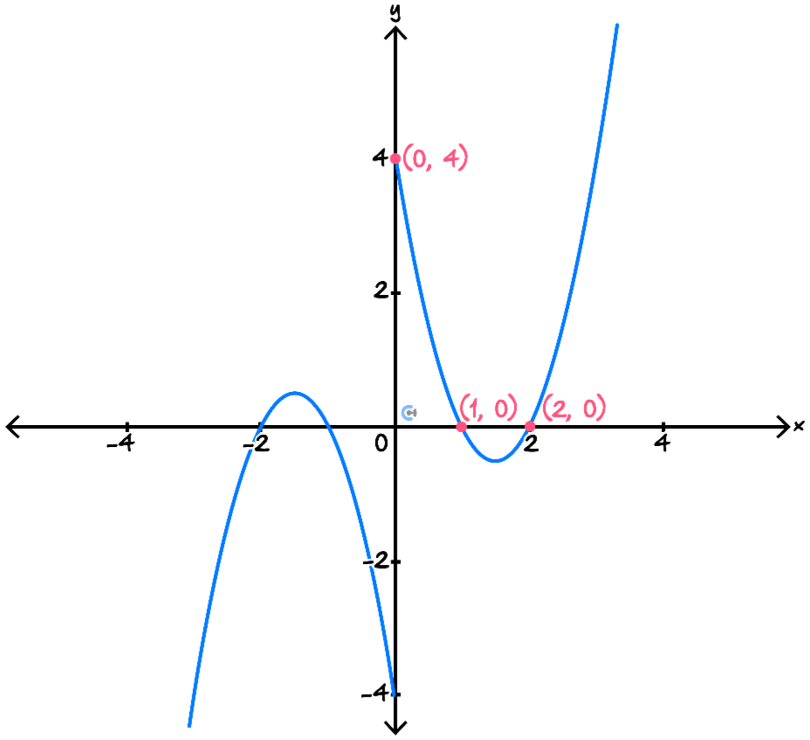
Question 166

Part of the graph $f(x)$ is plotted below.



Find a possible rule for the function if the function is known to be odd. Write your answer in the form.

$$f(x) = \begin{cases} f_1(x), & x < 0 \\ f_2(x), & x > 0 \end{cases}$$

The function could have the rule

$$f(x) = \begin{cases} 2(x-1)(x-2), & x > 0 \\ -2(-x-1)(-x-2), & x < 0 \end{cases}.$$

Note that the rule for $x < 0$ can be obtained by applying a reflection in the y-axis, followed by a reflection in the x-axis to the rule for $x > 0$.

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Sub-Section: Exam 1 Questions

Question 167

Find the value(s) of k so that the equation $(x^2 - kx + 16)(x^2 - 2\sqrt{7}x + k) = 0$ has:

a. Exactly one solution.

Either $x^2 - kx + 16$ has no solutions and $x^2 - 2\sqrt{7}x + k$ has one solution, or $x^2 - kx + 16$ has one solution and $x^2 - 2\sqrt{7}x + k$ has no solutions. Hence $k = 7$ or $k = 8$.

b. Exactly four solutions.

We need both $x^2 - kx + 16$ and $x^2 - 2\sqrt{7}x + k$ to have two solutions. Therefore we need $-8 < k$ or $k > 8$ and also $k < 7$. Hence, $k < -8$.

Question 168

Suppose that $f(x) = x^2 - 7x + 6$ and $g(x) = x^2 - kx + 1$. Find the values of k so that the equation $f(g(x))$ has:

a. Exactly two solutions.

The equation $f(g(x)) = 0$ gives $g(x) = 1$ or $g(x) = 6$. Therefore, there will be two solutions if

- $x^2 - kx + 1 = 1$ gives two solutions and $x^2 - kx + 1 = 6$ gives no solutions: This never happens because $x^2 - kx + 1 = 6$ will always give two solutions.
- $x^2 - kx + 1 = 1$ gives no solutions and $x^2 - kx + 1 = 6$ gives two solutions: Note that this never happens because $x^2 - kx$ always leads to a solution (the discriminant is never negative).
- $x^2 - kx + 1 = 1$ and $x^2 - kx + 1 = 6$ gives a single solution each. Note that this never happens because $x^2 - kx - 5 = 0$ always gives two solutions.

Hence, no such value of k exists.

b. Exactly four solutions.

Similar to above, now both $g(x) = 1$ and $g(x) = 6$ need to give two solutions each. Therefore, $k \neq 0$.

Question 169

Suppose that $f(x)$ is an odd function such that $f(x) = (x-2)^2$ for $x > 0$.

a. Write down a possible rule for $f(x)$ in the form:

$$f(x) = \begin{cases} f_1(x), & x < 0 \\ f_2(x), & x > 0 \end{cases}$$

$$f(x) = \begin{cases} (x-2)^2, & x > 0 \\ -(-x-2)^2, & x < 0 \end{cases}.$$

We can apply reflections across the x-axis and y-axis to obtain the rule for $x < 0$. Note that for $x < 0$, one can simplify the rule further to obtain $-(x+2)^2$.

b. It is known that the tangent to $f(x)$ at the point $x = 3$ is given by the rule $h(x) = 2x - 5$. By applying an appropriate sequence of transformations to $h(x)$, find the rule for the tangent at the point $x = -3$.

By sketching out the function, we can notice that the tangent at $x = -3$ can be obtained through translating the tangent at $x = 3$ so that $(3, 1)$ is mapped to $(-3, -1)$. Therefore, we should translate 6 units to the left and 2 units down. Thus, $k(x) = 2(x+6) - 5 - 2 = 2x + 5$. Alternatively, one could also apply reflections across both the x- and y-axes.

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Question 170

Consider a quartic of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. It is known that the quartic satisfies the following conditions:

- $f(1) = 0$
- $f(2) = 0$
- $f(0) = 4$
- Also, $f(x)$ is even.

a. Find the values of a, b, c, d and e.

We require $b = 0$ and $d = 0$ since $f(x)$ is even. Furthermore, $f(1) = 0$ tells us that $a + c + e = 0$, $f(2) = 0$ tells us that $16a + 4c + e = 0$ and $f(0) = 4$ tells us that $e = 4$. Therefore, $a + c = -4$ and $16a + 4c = -4$. Solving this system of equations, we conclude $a = 1$ and $c = -5$.

b. Verify that $f(x)$ can be factorised to $(x - 1)(x + 1)(x - 2)(x + 2)$.

We have found previously $f(x) = x^4 - 5x^2 + 4$. Now, we can factorise $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2)$.

c. Find the values of k so that $f(x + k)$ has exactly two positive x-intercepts.

Setting $1 - k > 0$ and $-1 - k \leq 0$ gives $-1 \leq k < 1$.

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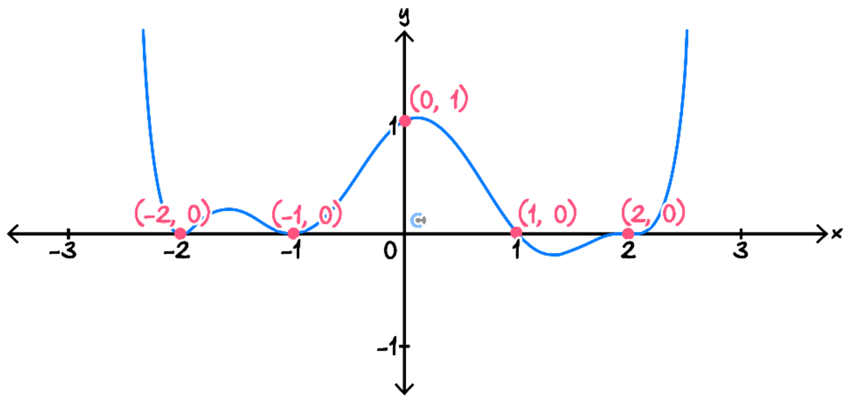
Sub-Section: Exam 2 Questions

Question 171

The minimum degree of the following polynomial is:



A. 2

B. 4

C. 6

D. 8

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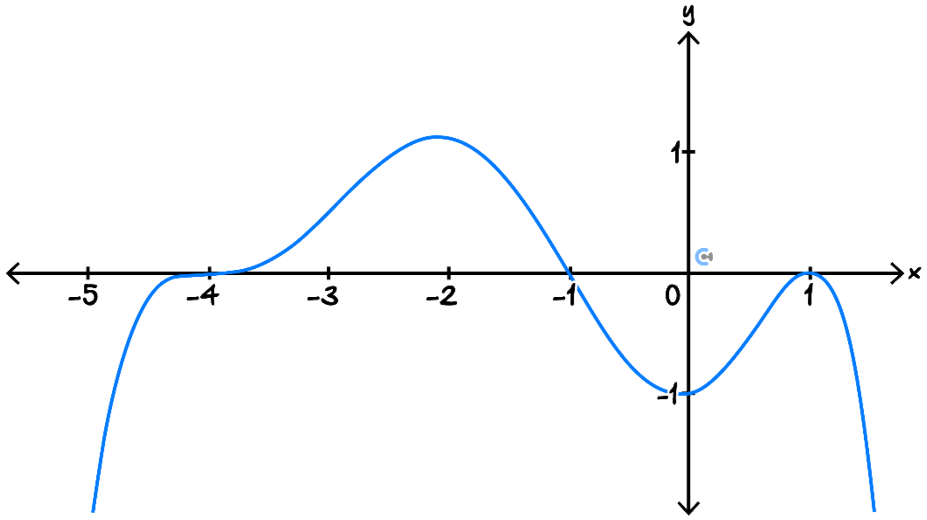
Question 172

A possible rule for the following function given below is:



A. $a(x-1)^3(x+4)^2(x+1)$ where $a<0$.

B. $a(x-1)^3(x+4)^2(x+1)^3$ where $a>0$.

C. $a(x-1)^2(x+4)^3(x+1)$ where $a<0$.

D. $a(x-1)\ (x+4)^3(x+1)$ where $a>0$.

Question 173

Let $f(x)=x^3-(k^2-5k+6)x^2-(k^3+5k)x$. If $f(x)$ is odd, then k must equal:

A. 1 or 3

B. 1 or 2

C. 2 or 3

D. 2 or 6

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Question 174

Let $g(x) = (x - 1)^2(x - 5)^2 - 4$. There will be exactly four solutions to the equation given by $g(x) = k$ whenever:

A. $-16 < k < 8$

B. $-4 < k < 12$

C. $-4 < k < 0$

D. $-4 < k < 16$

Question 175

Let $h(x) = x^4 - 10x^2 + 9$. The function $h(x + k)$ will have exactly three negative x-intercepts whenever:

A. $1 < k \leq 3$

B. $1 \leq k \leq 3$

C. $-3 < k \leq 1$

D. $-3 \leq k \leq 1$

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Question 176

Consider a cubic of the form $f(x) = ax^3 + bx^2 + cx + d$. Suppose that $f(x)$ satisfies the following conditions:

- $f(0) = 4$
- $f(1) = 0$
- $f(-2) = 0$
- $f(4) = 0$

a. Calculate the values of a, b, c and d.

Since $f(1) = 0, f(-2) = 0$, and $f(4) = 0$.

$\Rightarrow f(x) = n(x-1)(x+2)(x-4) \quad (1)$

Now, at $x = 0, f(x) = 4$.

So, sub $x = 0$ and $f(x) = 4$ into (1)

$$4 = n(0-1)(0+2)(0-4)$$

$$\Rightarrow n = \frac{1}{2}$$

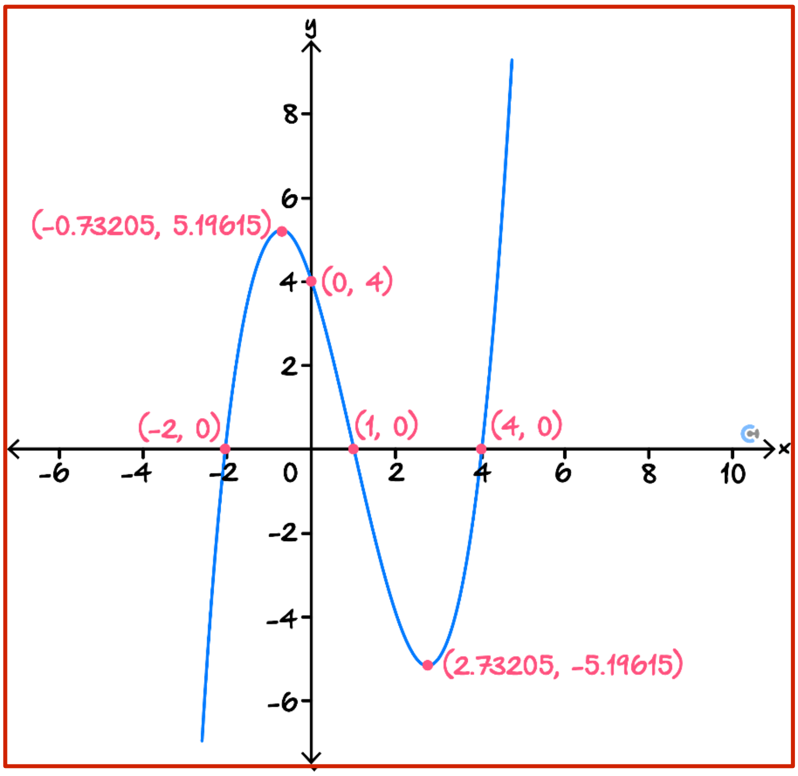
Sub $n = \frac{1}{2}$ into $(1)$and expand.

$$f(x) = \frac{1}{2}(x-1)(x+2)(x-4)$$

$$f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 - 3x + 4$$

Thus, $a = \frac{1}{2}, b = -\frac{3}{2}, c = -3$, and $d = 4$.

b. Sketch the graph of the function $y = f(x)$, labelling all turning points and intercepts.



c. Find the value(s) of k such that $f(x) - k = 0$ has exactly:

i. 2 solutions.

$k = 5.19615$ or $k = -5.19615$

ii. 3 solutions.

$-5.19615 < k < 5.19615$

d. Let $g(x) = x^2 - kx + 5$. State the values of k such that $f(g(x)) = 0$ gives the maximum number of solutions possible.

The equation $f(g(x)) = 0$ is solved whenever $g(x) = 1$, or $g(x) = -2$, or $g(x) = 4$. These three equations give exactly two solutions each if $k < -2\sqrt{7}$ or $k > 2\sqrt{7}$.
Therefore, the maximum number of solutions is six, and we have six solutions whenever $k < -2\sqrt{7}$ or $k > 2\sqrt{7}$.

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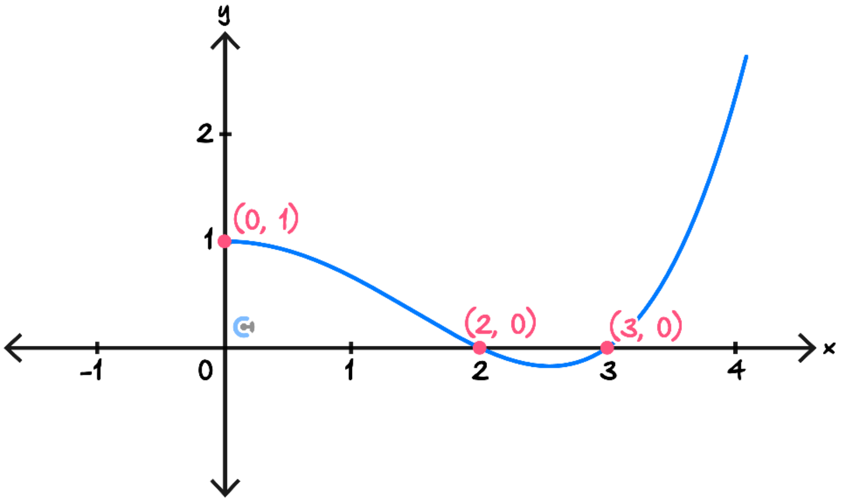
Question 177

The part of the graph of $f(x)$ is shown below. Furthermore, it is known that the function $f(x)$ is a quartic and also even.



a. State the rule for $f(x)$.

$$f(x) = \frac{1}{36}(x-2)(x+2)(x+3)(x-3)$$

b. The tangent to the graph of $f(x)$ at $x = 3$ is given by $y = \frac{5}{6}x - \frac{5}{2}$.

i. Describe a sequence of transformation(s) that can be applied to $h(x)$ to obtain the tangent to the graph of $f(x)$ at $x = -3$.

Reflection across the y-axis.

ii. Hence, write down the rule for the tangent to the graph of $f(x)$ at $x = -3$.

$$y = -\frac{5}{6}x - \frac{5}{2}$$

c. State the values of k so that $f(x-k)$ has exactly:

i. 3 positive x-intercepts.

$2 < k \leq 3$

ii. 3 negative x-intercepts.

$-3 \leq k < -2$

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