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VCE Mathematical Methods ¾
AOS 1 Revision [1.0]
Contour Check (Part 1) Solutions

Contour Checklist

Section A: [1.1] - Functions and Relationships (Checkpoints)

Sub-Section [1.1.1]: Find the Maximal Domain and Range of Functions

Question 1

Find the maximal domain of the following functions:

a. $f(x) = \sqrt{x^2 + 1}$

Need $x^2 + 1 \geq 0$. This holds for all $\mathbb{R}$ since $x^2 \geq 0$ for all $x \in \mathbb{R}$. Therefore domain $= \mathbb{R}$

b. $f(x) = \log_e(x + 4)$

Need $x + 4 > 0$ therefore domain $= (-4, \infty)$

c. $f(x) = \frac{1}{x+2} - 3$

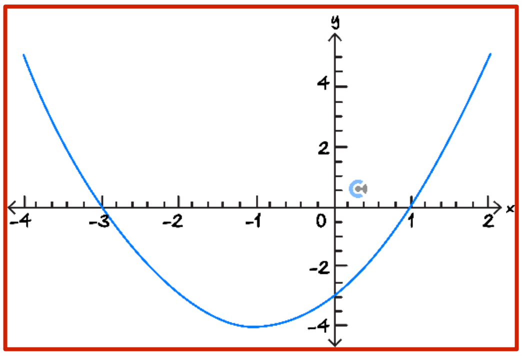
Need $x + 2 \neq 0$, therefore domain $= \mathbb{R} \setminus \{-2\}$

Question 2

Find the maximal domain of the following functions:

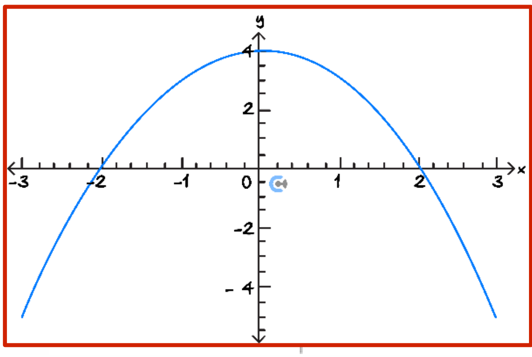
a. $f(x) = \sqrt{(x + 1)^2 - 4}$

We need $x^2 + 2x - 3 \geq 0 \implies (x + 3)(x - 1) \geq 0$, therefore domain $= (-\infty, -3] \cup [1, \infty)$



b. $f(x) = \log_e(4 - x^2)$

We require $4 - x^2 > 0 \implies (2-x)(2+x) > 0$ therefore domain $= (-2, 2)$.



c. $f(x) = \frac{3+x^2}{x^2+5x+6}$

We require that $x^2 + 5x + 6 \neq 0 \implies (x+2)(x+3) \neq 0$, therefore domain $= \mathbb{R} \setminus \{-3, -2\}$.

Question 3

Find the maximal domain of the following functions:

a. $f(x) = \cos(x)\log_e(2x) + \frac{1}{x^2-5}$

cos is defined for all $\mathbb{R}$ but for the log we require $2x > 0 \implies x > 0$ and for the fraction we require $x^2 - 5 \neq 0 \implies x \neq \pm\sqrt{5}$. Therefore the domain is $(0, \sqrt{5}) \cup (\sqrt{5}, \infty)$.

b. $f(x) = \sqrt{\frac{x-3}{x+1}}$

We require that $\dfrac{x-3}{x+1} \geq 0$ and that $x \neq -1$.
If $x \geq 3$ then numerator ≥ 0 and denominator > 0 therefore $f(x)$ defined.
If $x \in (-1, 3)$ then numerator < 0 and denominator > 0 therefore $f(x)$ not defined.
If $x = -1$ then division by zero so $f(x)$ not defined.
If $x < -1$ then both numerator and denominator < 0 so $f(x)$ is defined.
Therefore domain $= (-\infty, -1) \cup [3, \infty)$.

c. $f(x) = \frac{1}{2-x} \times \sqrt{x^2 - 4}\log_e(x^2 - 1)$

From the fraction we require that $x \neq 2$
from the square root we require that $x^2 - 4 \geq 0 \implies x \in \mathbb{R} \setminus (-2, 2)$
from the log we require that $x^2 - 1 > 0 \implies x \in \mathbb{R} \setminus [-1, 1]$.
Therefore domain $= \mathbb{R} \setminus (-2, 2] = (-\infty, -2] \cup (2, \infty)$.

Question 4

Find the maximal domain and range of $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$.

The denominator is never zero so dom $f = \mathbb{R}$.
To find the range consider what happens as $x \to \pm\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{2x}}{e^{2x}} = 1$$

$$\lim_{x\to-\infty} f(x) = \frac{0-1}{0+1} = -1$$

The range is $(-1, 1)$.

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Plot[(Exp[2 x] - 1)/(Exp[2 x] + 1), {x, -10, 10}]
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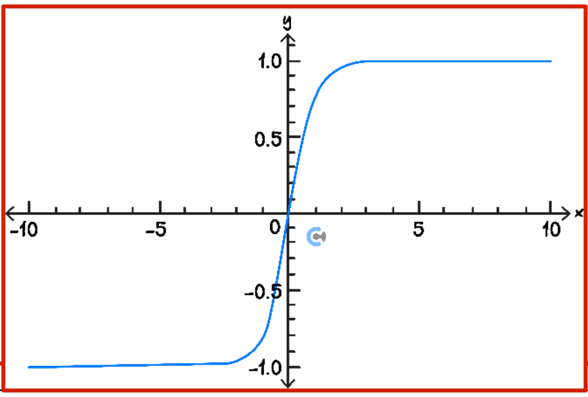

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Sub-Section [1.1.2]: Existence, Rule, Domain, and Range of Composite Functions

Question 5

The following functions are defined over their maximal domain:

$$f(x) = x^2 \text{ and } g(x) = 3 - x$$

a. Determine whether $f(g(x))$ and $g(f(x))$ exist.

Both compositions exist since both functions have domain $= \mathbb{R}$.

b. Find the rule of any composition that exists.

$$f(g(x)) = (g(x))^2 = (3 - x)^2$$
$$g(f(x)) = 3 - f(x) = 3 - x^2$$

c. State the domain of any composition that exists.

$$\text{dom } f \circ g = \text{dom } g = \mathbb{R}$$
$$\text{dom } g \circ f = \text{dom } f = \mathbb{R}$$

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Question 6

The following functions are defined over their maximal domain:

$$f(x) = e^{2x} \text{ and } g(x) = \log_e(2x)$$

a. Determine whether $f(g(x))$ and $g(f(x))$ exist.

dom $f = \mathbb{R}$ and ran $f = (0, \infty)$
dom $g = (0, \infty)$ and ran $g = \mathbb{R}$
Therefore, both compositions exist.

b. Find the rule of any composition that exists.

$$f(g(x)) = e^{2g(x)} = e^{2\log_e(2x)} = (2x)^2 = 4x^2$$
$$g(f(x)) = \log_e(2f(x)) = \log_e(2e^{2x}) = 2x + \log_e(2)$$

c. State the domain of any composition that exists.

dom $f \circ g$ = dom $g = \mathbb{R}^+$
dom $g \circ f$ = dom $f = \mathbb{R}$.

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Question 7

For the following functions:

$$f(x) = x^2 + 1 \text{ and } g(x) = \frac{1}{x^2-4}$$

a. Determine whether $f(g(x))$ and $g(f(x))$ exist.

$f(g(x))$ exists since ran $g = \left(-\infty, -\frac{1}{4}\right] \cup (0, \infty) \subseteq \text{dom } f = \mathbb{R}$.
$g(f(x))$ does not exist since ran $f = [1, \infty) \nsubseteq \text{dom } g = \mathbb{R} \setminus \{-2, 2\}$.

b. Find the rule of any composition that exists.

$$f(g(x)) = (g(x))^2 + 1 = \frac{1}{(x^2-4)^2} + 1.$$

c. State the domain of any composition that exists.

$$\text{dom } f \circ g = \text{dom } g = \mathbb{R} \setminus \{-2, 2\}.$$

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Question 8

Functions are defined over their maximal domain unless specified otherwise.

For the functions f and g, determine whether $f(g(x))$ and $g(f(x))$ exist. State the rule and the domain of the composite function that do exist.

$$f(x) = e^{x} - e^{-x}$$

$$g(x) = \frac{1}{x(x-2)}$$

dom $f = \mathbb{R}$ and ran $f = \mathbb{R}$
dom $g = \mathbb{R} \setminus \{0, 2\}$ and ran $g = (-\infty, -1] \cup (0, \infty)$
Therefore, $f(g(x))$ does exist since ran $g \subseteq$ dom f.
$g(f(x))$ does not exist since ran $f \not\subseteq$ dom g.

$$f(g(x)) = e^{\frac{1}{x^2-2x}} - e^{\frac{1}{2x-x^2}}$$

$$f(g(1)) = \frac{1}{e} - e$$

dom $f(g(x)) =$ dom $g = \mathbb{R} \setminus \{0, 2\}$ and ran $f(g(x)) = \left(-\infty, \frac{1}{e} - e\right] \cup (0, \infty)$.

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Plot[g[x], {x, -2, 3}]
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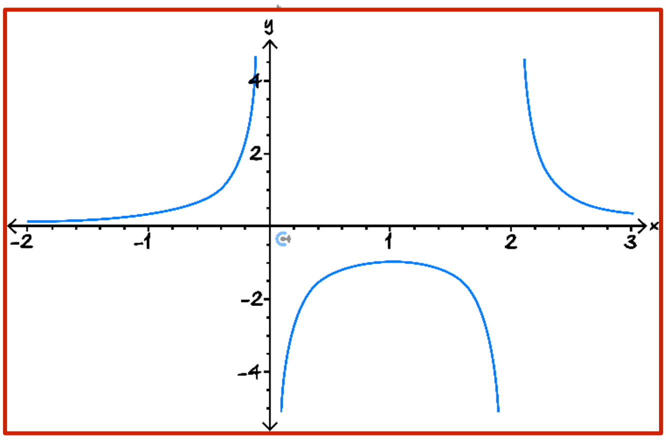


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Plot[f[g[x]], {x, -5, 5}]
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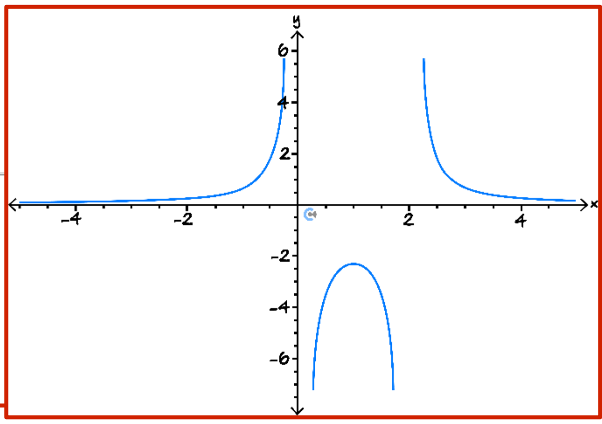

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Sub-Section [1.1.3]: Finding the Rule, Domain, and Range of Inverse Functions

Question 9

For the function:

$$f : (0, \infty) \to \mathbb{R}, f(x) = \log_e(3x)$$

a. Fully define the inverse function.

Swap x and y. $x = \log_e(3y) \implies 3y = e^x \implies y = \frac{1}{3}e^x$.

Now dom $f^{-1} = \text{ran } f = \mathbb{R}$

$$f^{-1} : \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = \frac{e^x}{3}.$$

b. Find the range of the inverse function.

$$\text{ran } f^{-1} = \text{dom } f = (0, \infty).$$

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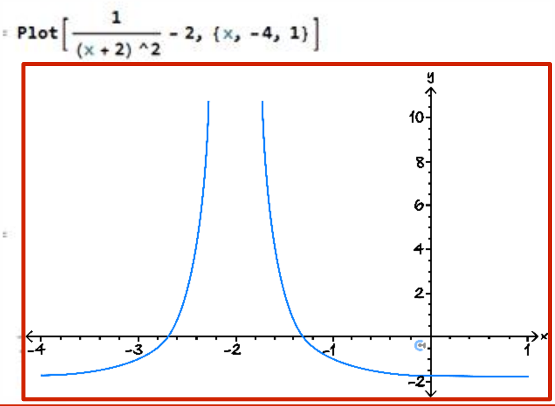
Question 10

For the function:

$$f:(b,-\infty)\to\mathbb{R}, f(x)=\frac{1}{(x+2)^2}-2$$

a. Find the largest value of b such that the inverse function exists.

$f(x)=(x+1)^2$. The function must be one-to-one for the inverse to exist. f is a truncus with an asymptote at $x=-2$. Therefore the smallest value of b is -2.



b. Fully define the inverse function.

Swap x and y. $x=\frac{1}{(y+2)^2}-2 \implies (y+2)^2=\frac{1}{x+2} \implies y=\pm\frac{1}{\sqrt{x+2}}-2.$
Since $\text{dom } f=(-2,\infty)=\text{ran } f^{-1}$ and $\text{ran } f^{-1}=\text{dom } f=(-2,\infty)$ we must have

$$f^{-1}:(-2,\infty)\to\mathbb{R},\ f^{-1}(x)=\frac{1}{\sqrt{x+2}}-2.$$

c. Find the range of the inverse function.

$\text{ran } f^{-1}=\text{dom } f=(-2,\infty)$

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Question 11

For the following functions:

$$f : (-\infty, k] \to \mathbb{R}, f(x) = 2x^2 - 8x + 4$$

a. Find the largest value of k such that the inverse function exists.

$f(x) = 2(x-2)^2 - 4$. Therefore f has a turning point at $(2, -4)$ so it is one-to-one for $x \in (-\infty, 2]$. Therefore $k = 2$.

b. Fully define the inverse function.

Swap x and y. $x = 2(y-2)^2 - 4 \implies \dfrac{x+4}{2} = (y-2)^2 \implies y = \pm\sqrt{\dfrac{x+4}{2}} + 2$. Now dom f^{-1} = ran $f = [-4, \infty)$ and ran f^{-1} = dom $f = (-\infty, 2]$. Therefore,

$$f^{-1} : [-4, \infty) \to \mathbb{R},\ f^{-1}(x) = 2 - \sqrt{\frac{x+4}{2}}.$$

c. Find the range of the inverse function.

ran f^{-1} = dom $f = (-\infty, 2]$.

d. Find the point of intersection between f and f^{-1}.

The functions intersect on the line $y = x$. Therefore solve

$$2x^2 - 8x + 4 = x$$
$$2x^2 - 7x + 4 = 0$$
$$(2x-1)(x-4) = 0$$

$x = \frac{1}{2}, 4$. But only $x = \frac{1}{2}$ is in the domain for both f and f^{-1}. Therefore intersection at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

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Question 12

Find the inverse function of:

$$f(x) = e^{2x} + 4e^{x} + 1$$

And determine whether f and f^{-1} have any points of intersection.

$f(x) = (e^x + 2)^2 - 3$. Swap x and y.

$$\begin{aligned} x &= (e^y + 2)^2 - 3 \\ e^y &= \sqrt{x+3} - 2 \\ y &= \log_e(-2 + \sqrt{x+3}) \end{aligned}$$

now dom $f^{-1} = \text{ran } f = (1, \infty)$

$$f^{-1} : (1, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_e(-2 + \sqrt{x+3})$$

A rough sketch of the functions will show that there is no intersection between f and f^{-1}.

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Plot[{Exp[2 x] + 4 Exp[x] + 1, Log[-2 + √(x + 3)]}, {x, -5, 5},
 PlotRange → {-20, 20}]
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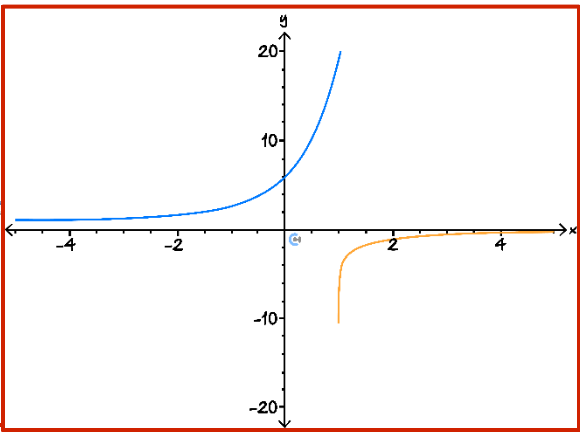

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Sub-Section [1.1.4]: Finding the Composition of Inverse Functions

Question 13

Let $f: (3, \infty) \to \mathbb{R}, f(x) = x^2 - 4x + 7$.

Find the rule and domain for $f^{-1}(f(x))$.

$f^{-1}(f(x)) = x$ for $x \in (3, \infty)$, since dom $f^{-1} \circ f =$ dom $f = (3, \infty)$.

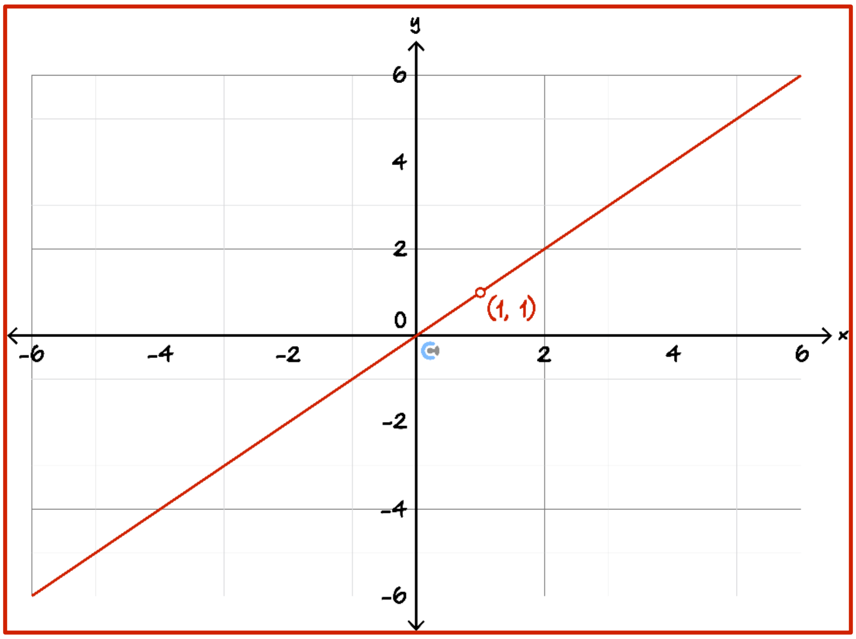
Question 14

Let $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \frac{5}{x-1} + 3$.

a. Find the rule and domain for $f^{-1}(f(x))$.

$f^{-1}(f(x)) = x$ for $x \in \mathbb{R} \setminus \{1\}$, since dom $f^{-1} \circ f =$ dom $f = \mathbb{R} \setminus \{1\}$.

b. Sketch the graph of $f^{-1}(f(x))$ on the axis below.



Question 15

Let $f(x) = x^2 - 2kx + 9$, where $x \geq 0$ and $k \geq 0$.

The function $f^{-1} \circ f$ is defined on its maximal domain.

Find the rule and domain for $f^{-1}(f(x))$.

$f(x) = (x-k)^2 + 9 - k^2$. We want f to be one-to-one on as large of a domain as possible. Therefore, the domain of f is $[k, \infty)$,

$$f^{-1}(f(x)) = x \quad \text{for } x \in [k, \infty), \text{ since dom } f^{-1} \circ f = \text{dom } f = [k, \infty).$$

Question 16

Let $f^{-1}: \left[\frac{\pi}{2}, \pi\right] \to \mathbb{R}, f^{-1}(x) = \sin(x)$.

Define the function f and find the rule and domain for $f^{-1}(f(x))$.

dom f = ran $f^{-1} = [0, 1]$ and ran f = dom $f^{-1} = \left[\frac{\pi}{2}, \pi\right]$

Now $f^{-1}(\pi) = 0 \implies f(0) = \pi$. Therefore,

$$f : [0, 1] \to \mathbb{R},\ f(x) = \pi - \sin^{-1}(x)$$

$$f^{-1}(f(x)) = x \text{ for } 0 \leq x \leq 1.$$

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Section B: [1.2] - Functions and Relationships Exam Skills (Checkpoints)

Sub-Section: [1.2.1] - Finding a New Domain to Fix Composite Functions

Question 17

Consider the functions the following functions defined over their maximal domains:

$$f(x) = \log_e(x) \text{ and } g(x) = e^x - 1$$

a. Show that $f(g(x))$ does not exist.

ran $g = (-1, \infty)$ and dom $g = (0, \infty)$
Therefore, $f(g(x))$ does not exist since ran $g \not\subseteq$ dom f.

b. Find the maximal domain of g such that $f(g(x))$ exists.

We require that for all x in the maximal domain of g, $g(x) > 0$. Hence

$$e^x - 1 > 0 \implies e^x > 1 \implies x > 0$$

Therefore, dom $g = (0, \infty)$

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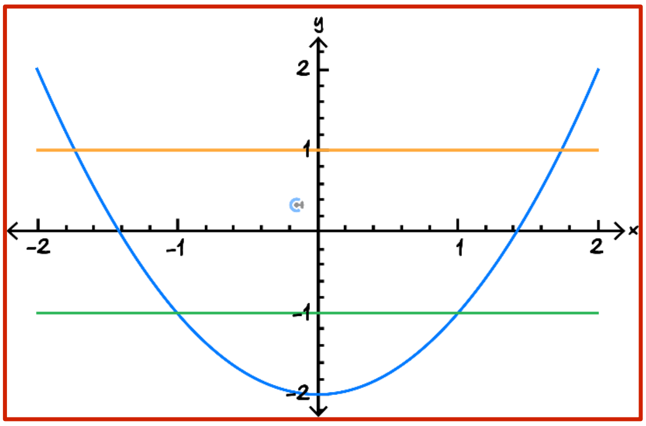
Question 18

Consider the following functions defined over their maximal domains:

$$f(x) = (x^2 - 2)^2 \text{ and } g(x) = \sqrt{x-1}$$

Find the maximal domain of f such that $g(f(x))$ exists.

Observe that the domain of $g(x)$ is $[1, \infty)$.
Thus for $g(f(x))$ to exist, we require $f(x) = (x^2-2)^2 \geq 1 \implies x^2 - 2 \geq 1$ or $x^2 - 2 \leq -1$.
We can solve for $x^2 - 2 = 1 \implies x = \pm\sqrt{3}$ and $x^2 - 2 = -1 \implies x = \pm 1$, and then sketch $y = x^2 - 2$ to solve our inequalities.



From this graph we see that $x^2 - 2 \leq -1$ if $x \in [-1, 1]$.
Similarly we see that $x^2 - 2 \geq 1$ if $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$
Hence the maximal domain of f for $g(f(x))$ to exist $x \in (-\infty, -\sqrt{3}] \cup [-1, 1] \cup [\sqrt{3}, \infty)$

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Question 19

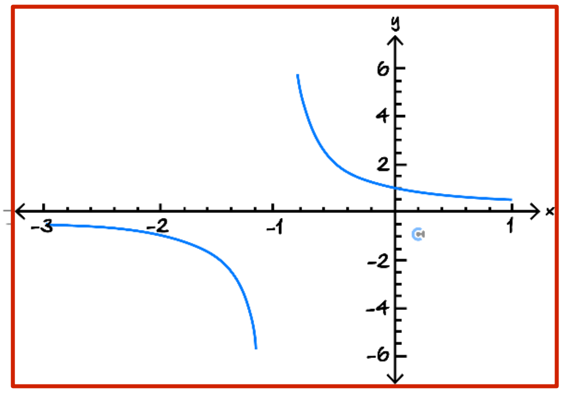
Consider the following functions defined over their maximal domains:

$$f(x) = \frac{1}{1+x} \text{ and } g(x) = \sqrt{16 - (x-1)^2}$$

Find the maximal domain of f such that $g(f(x))$ exists.

We require that ran $f \subseteq$ dom g.
Observe that if $x \in$ dom g, then $(x-1)^2 \leq 16$, which leaves us with $x \in [-3, 5]$.
Hence $f(x) = \dfrac{1}{1+x} \in [-3, 5]$. We proceed by solving $f(x) = -3, 5$ and sketching our graph to get our inequality.

$$\frac{1}{1+x} = -3 \implies -3x = 4 \implies x = -\frac{4}{3}$$
$$\frac{1}{1+x} = 5 \implies 5x = -4 \implies x = -\frac{4}{5}$$



From these three pieces of information, we see that the maximal domain of f such that $g(f(x))$ exists is $x \in \left(-\infty, -\frac{4}{3}\right] \cup \left[-\frac{4}{5}, \infty\right)$.

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Question 20

Consider the following functions:

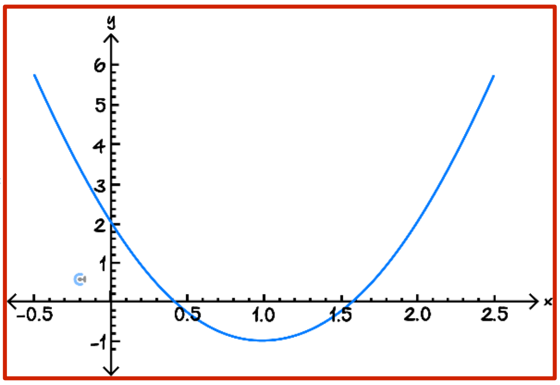
$$f: [0,2) \to \mathbb{R}, f(x) = \log_2(4 - x^2) \text{ and } g: (-\infty, 2) \to \mathbb{R}, g(x) = 3(x - 1)^2 - 1$$

Find the largest interval of x-values for which $f(g(x))$ and $g(f(x))$ both exist.

We need $f(x) = \log_2(4 - x^2) < 2 \to 4 - x^2 < 4 \to x^2 > 0.$
As $x \in \text{dom } f, x \in (0,2)$

In[141]= `Plot[3 (x - 1) ^2 - 1, {x, -0.5, 2.5}]`

Out[141]=



As $g\left(1 \pm \frac{1}{\sqrt{3}}\right) = 0$ and $g(0) = g(2) = 2$, from our graph we see that $3x^2 - 1 \in [0, 2)$ when

$$x \in \left(0, 1 - \frac{1}{\sqrt{3}}\right] \cup \left[1 + \frac{1}{\sqrt{3}}, 2\right).$$

Combining this restriction with our restriction for ran $f \subseteq$ dom g, we get that,

$$x \in \left[1 + \frac{1}{\sqrt{3}}, 2\right).$$

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Sub-Section: [1.2.2] - Finding the Range of Complex Composite Functions

Question 21

Find the range of $f(x) = e^{x^2+1}$.

The range of $g(x) = x^2 + 1$ is $[1, \infty)$.
The range of $h : [1, \infty) \to \mathbb{R}, h(x) = e^x$ is $[e, \infty)$.
As $f(x) = h(g(x))$, the range of f is $[e, \infty)$.

Question 22

Find the range of $f: [0, \infty) \to \mathbb{R}, f(x) = \log_3(3^x + 8)$.

The range of $g : [0, \infty) \to \mathbb{R}, g(x) = 3^x + 8$ is $[9, \infty)$.
The range of $h : [9, \infty) \to \mathbb{R}, h(x) = \log_3(x)$ is $[2, \infty)$.
As $f(x) = h(g(x))$, the range of f is $[2, \infty)$.

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Question 23

Find the range of $f(x) = \sqrt{\frac{x}{x+1}}$ where f is defined on its maximal domain.

The range of $g(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1}$ is $\mathbb{R}\backslash\{1\}$.
Thus the range of f is the range of $\sqrt{x}$ on the intersection of it's maximal domain, $[0, \infty)$ and the range of g. Specifically,

$$\mathbb{R}\backslash\{1\} \cap [0, \infty) = [0, \infty)\backslash\{1\}$$

The range of $\sqrt{x}$ on this domain is $[0, \infty)\backslash\{1\}$, hence the range of f is, $[0, \infty)\backslash\{1\}$.

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Question 24

Consider the following functions defined on all real numbers:

$$f(x) = \sin(x) \text{ and } g(x) = \log_3(4x^2 - 4x + 2)$$

Find the range of $g(f(x))$.

We observe that the range of $f(x)$ is $[-1, 1]$. Hence the range of $g(f(x))$ is the range of g restricted to $[-1, 1]$.
Now observing $g(x)$, we note that it is the composition of $\log_3(x)$ and $4x^2 - 4x + 2 = (2x - 1)^2 + 1$.
The range of $h(x) = (2x - 1)^2 + 1$ on the interval $[-1, 1]$ can be found by evaluating $h(-1), h(1)$ and the y-value of the turning point of h, which is 1.
As $h(-1) = (-3)^2 + 1 = 10$, and $h(-1) = 1^2 + 1 = 2$, we see that the range of h on the interval $[-1, 1]$ is $[1, 10]$.
Since $\log_3(x)$ is an increasing function, the range of $\log_3(x)$ on the interval $[1, 10]$ is $[0, \log_3(10)]$.
Hence the range of $g(f(x))$ is $[0, \log_3(10)]$

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Sub-Section: [1.2.3] - Finding the Gradient of Inverse Functions

Question 25

Consider the function $f: [0, \infty) \to \mathbb{R}, f(x) = x^2$.

The gradient of f at $x = a$ is $2a$.

Let g be the inverse function of f. Find the gradient of g when $x = 2$.

We observe that $g(x) = \sqrt{x}$, hence $g(2)$ is $\sqrt{2}$.
When $x = \sqrt{2}$, the gradient of f is $2\sqrt{2}$.
Hence the gradient of $g(x)$ when $x = 2$ is $\frac{1}{2\sqrt{2}}$.

Question 26

Consider the one-to-one function f with the following properties:

$$f(2) = 5, f(5) = 7, f'(2) = 3 \text{ and } f'(5) = 1$$

Let g be the inverse function of f. Find the gradient of g when $x = 5$.

We have that $g(5) = 2$ and that $f'(5) = 1$. Therefore

$$g'(5) = \frac{1}{1} = 1$$

Question 27

Consider the function $f(x)$, the gradient of f at $x = a$ is $2f(a) + 2a$, and $f(0) = 1$.

From this information, we can tell that the gradient of f^{-1} at $x = b$ is c. Find b and c.

If $f(a) = b$ and $f'(a) = \frac{1}{c}$, we know that the gradient of f^{-1} at $x = b$ is c.
The only a for which we know $f(a)$ and $f'(a)$ is $a = 0$.
Hence $b = f(0) = 1$, and $c = \frac{1}{f'(0)} = \frac{1}{2(1) + 2(0)} = \frac{1}{2}$.

Question 28

Consider the differentiable, one-to-one, function $f: (0, 1) \to \mathbb{R}$. It is known that:

1. $f'(x) = -[f(x)]^2$, for all $x \in (0, 1)$.
2. $\text{ran } f = (1, \infty)$.

If g is the inverse function of f, find the domain and range of $g'(x)$.

Hint: $g'(a)$ denotes the gradient of g at $x = a$.

The domain of $g'(x)$ is the domain of g which is the range of $f = (1, \infty)$.
The range of $g'(x)$ is the reciprocal of the range of $f'(x)$.
As $f'(x) = -[f(x)]^2$, the range of $f'(x)$ is $(-\infty, -1)$.
Hence the range of $g'(x)$ is $(-1, 0)$.

Sub-Section: Exam 1 Questions

Question 29

Let $f: [0, \infty) \to \mathbb{R}, f(x) = \sqrt{x+4}$.

a. State the range of f.

$[2, \infty)$

b. Let $g: (-\infty, c] \to \mathbb{R}, g(x) = x^2 + 6x + 7$, where $c < 0$.

Find the largest possible value of c such that the range of g is a subset of the domain of f.

We require $g(x) = x^2 + 6x + 7 \geq 0$.
We solve $g(x) = 0$ by completing the square, thus,

$$x^2 + 6x + 9 - 2 = (x+3)^2 - 2 = 0 \implies x + 3 = \pm\sqrt{2} \implies x = -3 \pm \sqrt{2}$$

As $g(x)$ is a positive parabola, for $g(x) \geq 0$ either, $x \leq -3 - \sqrt{2}$ or $x \geq -3 + \sqrt{2}$.
Hence $c = -3 - \sqrt{2}$.

c. For the value of c found in **part b.**, state the range of $f(g(x))$.

For the value of c found in part b, the range of g is $[0, \infty)$.
Hence the range of $f(g(x))$ is simply the range of $f = [2, \infty)$

d. Let $h: \mathbb{R} \to \mathbb{R}, h(x) = x^2 + 5$.

State the range of $f(h(x))$.

$f(h(x)) = \sqrt{x^2+5+4} = \sqrt{x^2+9}$. Hence the range of $f(h(x)) = [3, \infty)$.

Question 30

Let $f: (-2, \infty) \to \mathbb{R}, f(x) = 3 - \frac{4}{(x+2)^2}$.

State the rule and domain of f^{-1}.

We solve $x = f(y)$ for y, thus,

$$\begin{aligned} x &= 3 - \frac{4}{(y+2)^2} \\ \implies 3 - x &= \frac{4}{(y+2)^2} \\ \implies \frac{4}{3-x} &= (y+2)^2 \\ \implies \frac{2}{\sqrt{3-x}} &= y+2 \quad \text{Since dom } f = (-2,\infty) \\ \implies y &= \frac{2}{\sqrt{3-x}} - 2 \end{aligned}$$

The domain of f^{-1} is simply the range of f which is $(-\infty, 3)$. Hence the function f^{-1} is,

$$f^{-1}: (-\infty, 3) \to \mathbb{R}, f^{-1}(x) = \frac{2}{\sqrt{3-x}} - 2$$

Space for Personal Notes

Question 31

a. Let $f: \mathbb{R}\backslash\{3\} \to \mathbb{R}$, $f(x) = \frac{1}{x-3}$. Find the rule for f^{-1}.

We solve $f(y) = x$ for y. Thus,

$$x = \frac{1}{y-3} \implies \frac{1}{x} = y - 3 \implies y = \frac{1}{x} + 3$$

Hence the rule for f^{-1} is, $f^{-1}(x) = \frac{1}{x} + 3$

b. State the domain of f^{-1}.

The domain of f^{-1} is the range of f which is $\mathbb{R}\backslash\{0\}$.

c. Let $g(x) = f(x-c) + d$ for $c, d \in \mathbb{R}$.

Find the values of c and d, given that $g = f^{-1}$.

$g(x) = \frac{1}{x-c-3} + d = \frac{1}{x} + 3$. Hence $d = 3$ and $c = -3$.

d. Given that $f'(1) = -\frac{1}{4}$ and $f'(4) = -1$, find the value of $g'(1)$.

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(1+3)} = \frac{1}{f'(4)} = -1$$

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Question 32

Find the maximal domain of f, where $f(x) = \frac{1}{\sqrt{x^2-6x+5}}$.

For x to be in the maximal domain of f, we require that $x^2 - 6x + 5 > 0$.
We can factorise $x^2 - 6x + 5$ as $(x-5)(x-1)$ to see that it is equal to 0 when $x = 1, 5$.
As $x^2 - 6x + 5$ is an upwards parabola, we see that it is greater than 0 when $x < 1$ or $x > 5$.
Hence the maximal domain of f is $(-\infty, 1) \cup (5, \infty)$

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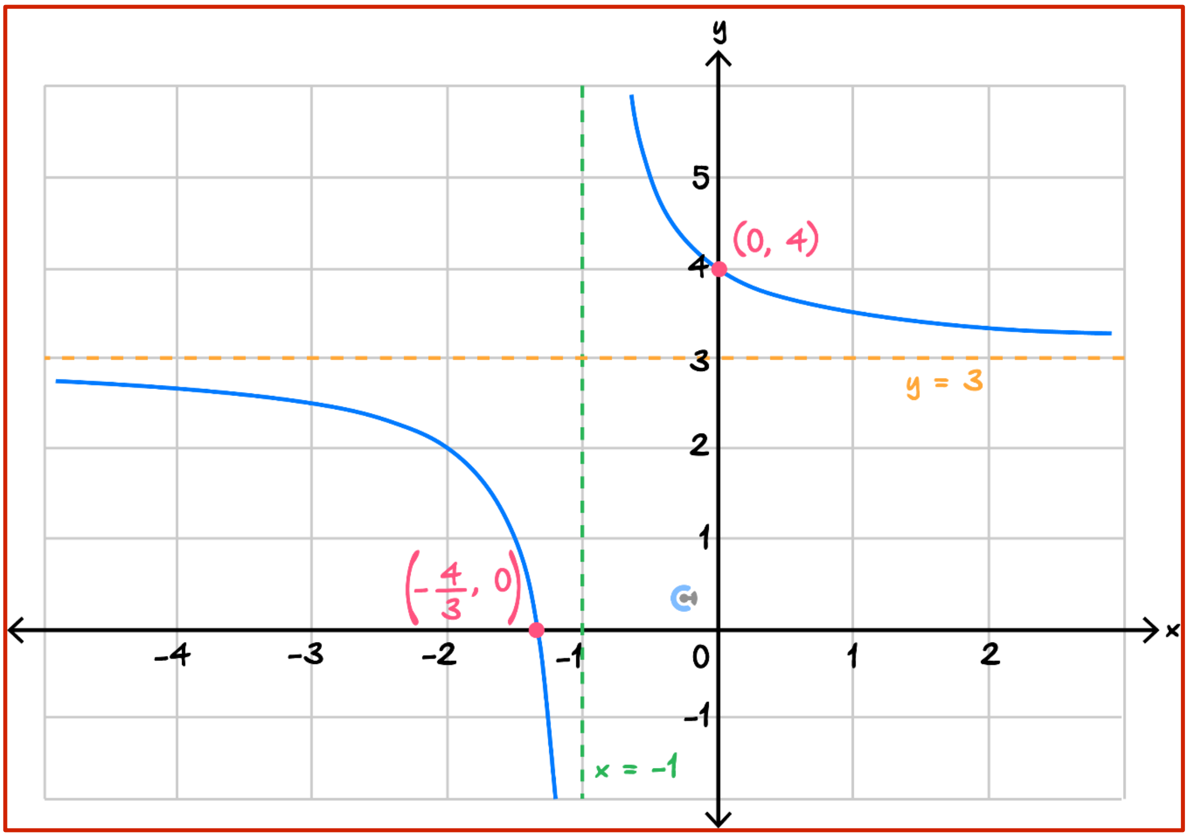
Question 33

a. Sketch the graph of $f(x) = 3 + \frac{1}{x+1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates.



b. Find the values of x for which $f(x) \in (2, 4)$.

We see that $f(x) = 4$ when $x = 0$ and $f(x) = 2$ when $x = -2$.
From the above graph we can see that $f(x) \in (2, 4)$ when $x \in (-\infty, -2) \cup (0, \infty)$.

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Sub-Section: Exam 2 Questions

Question 34

Which one of the following is the inverse function of $g: (-\infty, 2] \to \mathbb{R}, g(x) = 4(x-2)^2 + 3$?

A. $f: [3, \infty) \to \mathbb{R}, f(x) = 2 + \frac{\sqrt{x-3}}{2}$

B. $f: [3, \infty) \to \mathbb{R}, f(x) = 2 - \frac{\sqrt{x-3}}{2}$

C. $f: [3, \infty) \to \mathbb{R}, f(x) = 4 + \frac{\sqrt{x-3}}{4}$

D. $f: [3, \infty) \to \mathbb{R}, f(x) = 4 - \frac{\sqrt{x-3}}{4}$

Question 35

The maximal domain of the function f is $(-\infty, 1 - \sqrt{5}\,] \cup [1 + \sqrt{5}, \infty)$.

A possible rule of f is:

A. $f(x) = \sqrt{5 - (x-1)^2}$

B. $f(x) = \log_e(5 - (x-1)^2)$

C. $f(x) = \frac{1}{\sqrt{5 - (x-1)^2}}$

D. $f(x) = \frac{1}{\log_e(5 - (x-1)^2)}$

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Question 36

Let f be a one-to-one differentiable function and the following values are known:

$$f(-1) = 3, f(3) = 7, f'(-1) = 5 \text{ and } f'(3) = 2$$

Let $g(x) = f^{-1}(x)$, the value of $g'(3)$ is:

A. 5

B. 2

C. $\frac{1}{5}$

D. $\frac{1}{2}$

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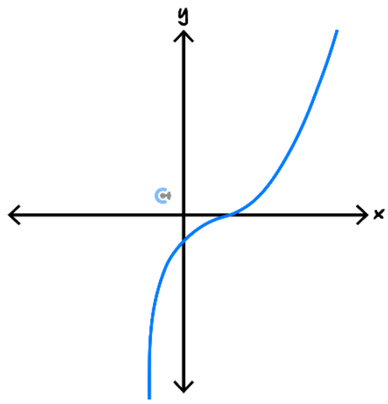
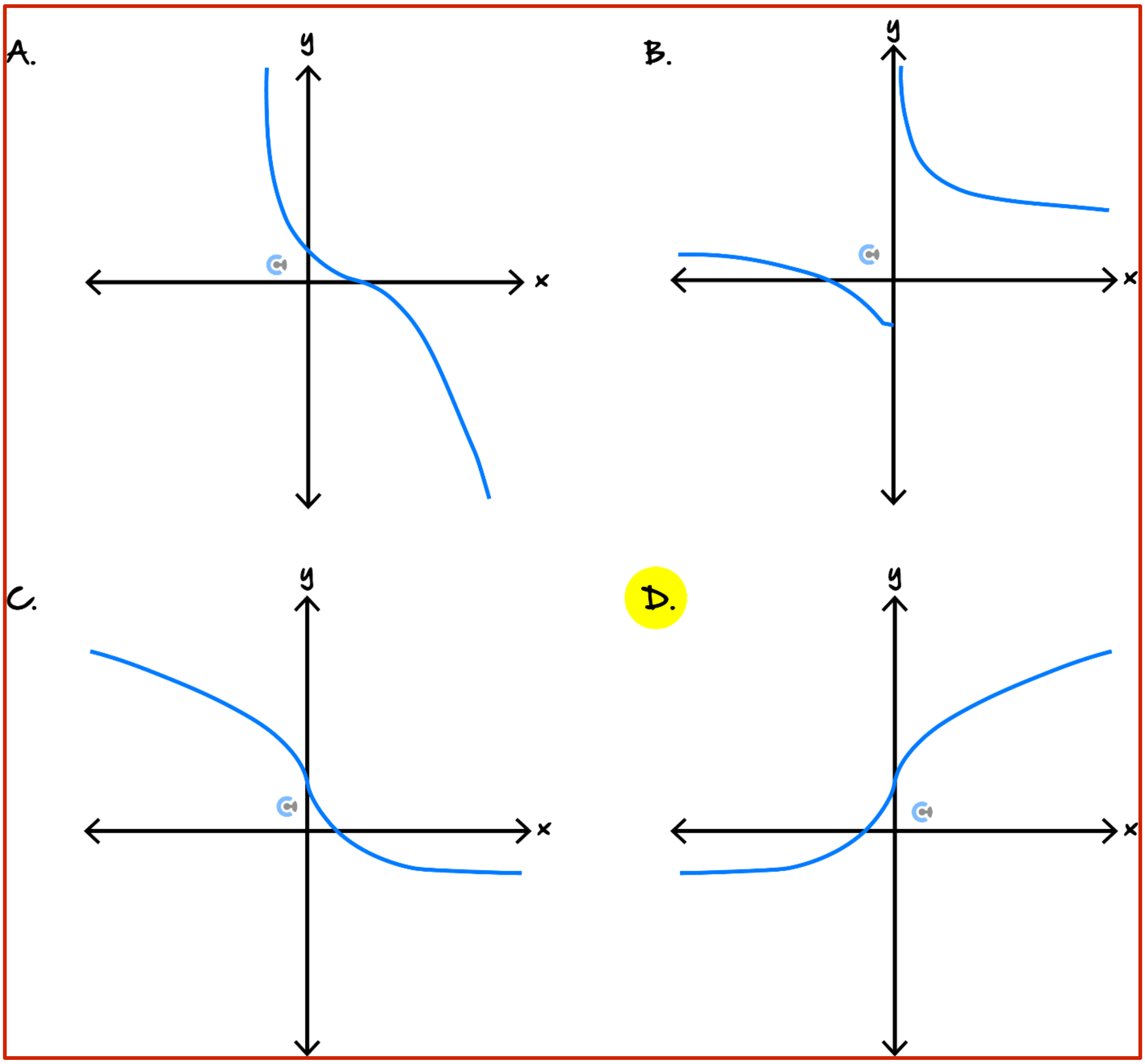
Question 37

Part of the graph of the function f is shown below. The same scale has been used on both axes.



The corresponding part of the graph of the inverse function f^{-1} is best represented by:



Question 38

Consider the following functions:

$$f : \left(-\frac{\sqrt{3}}{2}, \infty\right) \rightarrow \mathbb{R}, f(x) = \log_e\left(x + \frac{\sqrt{3}}{2}\right)$$

$$g : (-\infty, 3) \rightarrow \mathbb{R}, g(x) = \cos(x)$$

The largest interval of x-values for which $f(g(x))$ and $g(f(x))$ both exist is:

A. $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

B. $\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{6}\right)$

C. $\left(-\frac{5\pi}{6}, \frac{5\pi}{6}\right)$

D. $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

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Question 39

a. Express $\frac{3x+2}{x+3}$ in the form of $a + \frac{b}{x+2}$, where a and b are non-zero integers.

$$\frac{3x+2}{x+3} = \frac{3(x+3)-7}{x+3} = 3 + \frac{-7}{x+2}$$

Hence $a = 3$ and $b = -7$.

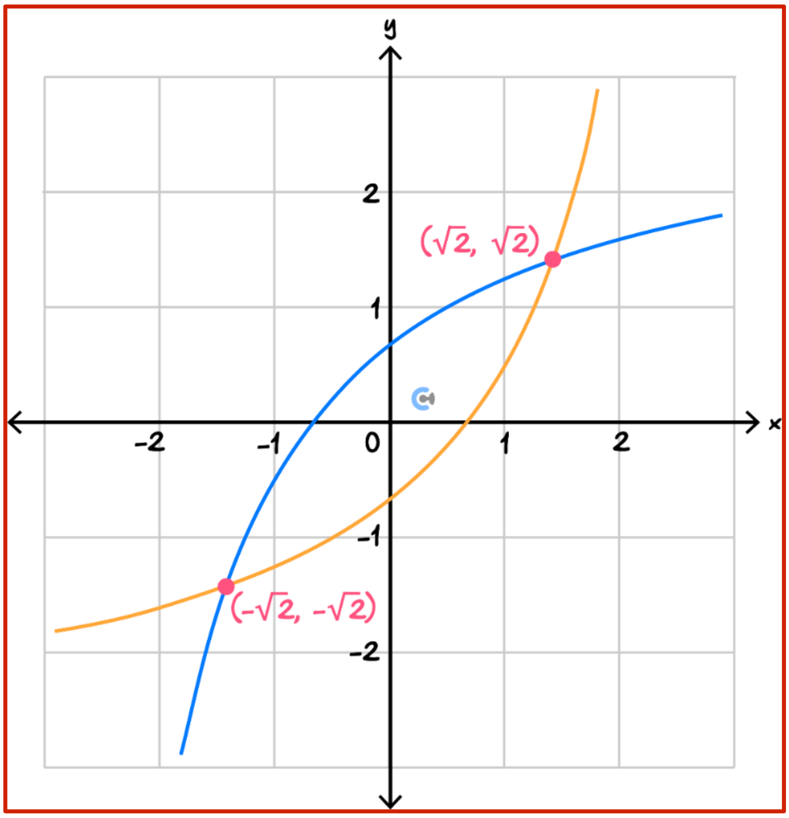
b. Let $f : \mathbb{R}\backslash\{-3\} \to \mathbb{R}, f(x) = \frac{3x+2}{x+3}$.

i. Find the rule and domain of f^{-1} and the inverse function of f.

We solve $f(y) = x$ for y to get the rule for f^{-1}.
Hence $f^{-1}(x) = \dfrac{2-3x}{x-3}$.
The domain of f^{-1} is the range of f which is $\mathbb{R}\backslash\{3\}$

ii. Part of the graph of f is shown in the diagram below.

Sketch the graph of $y = f^{-1}$, labelling all points of intersection with their coordinates.



c. Let $g(x) = -\sqrt{16 - x^2}$.

i. Show that both $f(g(x))$ and $g(f(x))$ do not exist.

The domain of f is $\mathbb{R}\backslash\{-3\}$.
The range of g is $[-4, 0]$.
Thus ran $g \not\subseteq$ dom f hence $f(g(x))$ does not exist.
The domain of g is $[-4, 4]$
The range of f is $\mathbb{R}\backslash\{3\}$.
Thus ran $f \not\subseteq$ dom g hence $g(f(x))$ does not exist.

ii. Find the largest interval on which both $f(g(x))$ and $g(f(x))$ are defined on.

We solve $f(x) = 4 \implies x = -2$.
We solve $f(x) = -4 \implies x = -10$.
From the graph of f we see that $f(x) \in [-4, 4]$ if $x \in (-\infty, -10] \cup [-2, \infty)$
Thus $g(f(x))$ is defined for $x \in (-\infty, -10] \cup [-2, \infty)$
We solve $g(x) = -3 \implies x = \pm\sqrt{7}$.
Thus $f(g(x))$ is defined for $x \in [-4, 4]\backslash\{\pm\sqrt{7}\}$.
Hence $f(g(x))$ and $g(f(x))$ are both defined on the intersection of the two sets, specifically $x \in [-2, 4]\backslash\{\sqrt{7}\}$.
This set contains two intervals, the bigger one being, $[-2, \sqrt{7})$.

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Question 40

Let $f(x) = 2^{-x}$ and $g(x) = 4x^2 - 4x + 3$.

a.

i. State the rule of $f(g(x))$.

$$f(g(x)) = 2^{-4x^2-4x+3}$$

ii. State the range of $f(g(x))$.

The range of $g(x) = (2x-1)^2 + 2$ is $[2, \infty)$.
As f is a decreasing function, which tends towards 0 as $x \to \infty$, the range of f is,

$$\left(0, \frac{1}{4}\right].$$

b. Let $h: [a, \infty) \to \mathbb{R}, h(x) = g(f(x))$. Find the smallest value of a such that h is a one-to-one function.

As $g(x) = (2x-1)^2 + 2$, the largest intervals for which it is a one-to-one function on are, $\left[\frac{1}{2}, \infty\right)$, or, $\left(\infty, \frac{1}{2}\right]$.
As $f(x)$ is a decreasing function, the range of f when restricted to $[a, \infty)$ is $(0, 2^a]$.
Since $(0, 2^{-a}] \not\subseteq \left[\frac{1}{2}, \infty\right)$ for all a, we must consider values of a for which

$$(0, 2^{-a}] \subseteq \left(\infty, \frac{1}{2}\right].$$

The smallest such value of a is $a = 1$.

c. For the value of a found in **part b.**, state the rule and domain for h^{-1}.

We can solve $h(y) = x$ for y. This implies that $g(f(y)) = x$, thus we will first solve $g(z) = x$ for z, restricting our attention to $z < \frac{1}{2}$ because of our work from part b. This yields,

$$z = \frac{1}{2}(1 - \sqrt{x-2})$$

Now we solve $f(y) = z$, we see that $y = -\log_2\left(\frac{1}{2}(1-\sqrt{x-2})\right) = 1 - \log_2(1 - \sqrt{x-2})$.

Hence the rule for h^{-1} is $h^{-1}(x) = 1 - \log_2(1 - \sqrt{x-2})$.

The domain for h^{-1} is the range of h. As the range of f restricted to $[1, \infty)$ is $\left(0, \frac{1}{2}\right]$, the range of h is simply the range of g restricted to $\left(0, \frac{1}{2}\right]$.

As g is one-to-one in this interval, the range of g restricted to $\left(0, \frac{1}{2}\right]$ is $[2, 3)$.

Hence the domain of h^{-1} is $[2, 3)$.

d. How many solutions does the equation $f(g(x)) + g(f(x)) = 0$ have?

We sketch the graph of $f(g(x)) + g(f(x))$.

As it is entirely above the x-axis, our equation has 0 solutions.

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Section C: [1.3] - Transformations (Checkpoints)

Sub-Section [1.3.1]: Applying Transformations to Points

Question 41

Consider the following transformations of the plane:

- S, a dilation by a factor of 2 from the y-axis, followed by a translation of 3 units up.
- T, a translation of 2 units left and 1 unit up.
- W, a reflection in the line $y = x$.

a. Find $S(x, y)$.

$$S(x, y) = (2x, y + 3)$$

b. Find $T(x, y) = (x', y')$. Express x and y in terms of x' and y'.

$T(x, y) = (x - 2, y + 1) = (x', y')$.

Hence $x' = x - 2 \implies x = x' + 2$, and $y' = y + 1 \implies y = y' - 1$.

c. Find $W(x, y)$.

$$W(x, y) = (y, x).$$

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Question 42

Consider the following transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (-2x + 4, 5(y + 3))$.

T can be described using the following sequence of transformations:

- A dilation by a factor of a from the x-axis, followed by,
- A dilation by a factor of b from the y-axis, followed by,
- A reflection in the y-axis, followed by,
- A translation c units in the positive direction of the x-axis, followed by,
- A translation of d units in the positive direction of the y-axis.

a. Find a, b, c, and d.

We need to turn x into $2x$ and y into $5y$ using our two dilations, since the reflection will take $2x$ to $-2x$ and then we can worry about the translations.

Hence $a = 5$ and $b = 2$.

Now we simply translate our sequence to the right point, meaning $c = 4$ and $d = 15$.

b. Describe T as a sequence of two translations, followed by two dilations, and a reflection.

We can rewrite our transformation as follows, $T(x, y) = (-2(x - 2), 5(y + 3))$. From here we see that we must get our translations to map (x, y) to $(x + 2, y - 1)$ before applying our dilations / reflections. Hence our sequence of transformations is as follows,

- A translation of 2 units in the negative direction of the x-axis, followed by,
- A translation of 3 units in the positive direction of the y-axis, followed by,
- A dilation by a factor of 2 from the y-axis, followed by,
- A dilation by a factor of 5 from the x-axis, followed by,
- A reflection in the y-axis.

c. The image of $(p, -5)$ under T is $(2, q)$. Find p and q.

We apply T to $(p, -5)$ getting, $(-2p + 4, -10) = (2, q)$.
Hence $q = -10$ and $-2p + 4 = 2 \implies p = 1$.

Question 43

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ described by the following sequence of transformations:

- A dilation by a factor of $\frac{1}{5}$ from the x-axis, followed by,
- A translation of 2 units in the positive direction of the x-axis, followed by,
- A reflection in the y-axis, followed by,
- A translation of 3 units in the positive direction of the x-axis, followed by,
- A translation of 5 units in the negative direction of the y-axis, followed by,
- A dilation by a factor of 5 from the x-axis, followed by,
- A reflection in the x-axis, followed by,
- A dilation by a factor of 3 from the y-axis.

a. Find (x', y'), the image of (x, y) under T.

In order, the transformations take the point (x, y) to,

$$(x, y) \mapsto \left(x, \frac{y}{5}\right) \mapsto \left(x + 2, \frac{y}{5}\right) \mapsto \left(-x - 2, \frac{y}{5}\right) \mapsto \left(-x + 1, \frac{y}{5}\right)$$
$$\mapsto \left(-x + 1, \frac{y}{5} - 5\right) \mapsto (-x + 1, y - 25) \mapsto (-x + 1, 25 - y) \mapsto (-3x + 3, 25 - y)$$

Thus $(x', y') = (-3x + 3, 25 - y)$

b. Express x in terms of x' and y in terms of y'.

As $x' = -3x + 3$ we get $x = \frac{x' - 3}{-3} = \frac{3 - x'}{3}$.
As $y' = 25 - y$ we get $y = 25 - y'$.

c. A transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ maps $T(x, y) = (x', y')$ to (x, y).

Describe S as a sequence of 2 translations followed by 2 reflections followed by a dilation.

- A translation of 3 units in the negative direction of the x-axis, followed by,
- A translation of 25 units in the negative direction of the y-axis, followed by,
- A reflection in the x-axis, followed by,
- A reflection in the y-axis, followed by,
- A dilation by a factor of $\frac{1}{3}$ from the y-axis.

Question 44

a. Describe a reflection in the line $y = x + b$ using elementary transformations.

- A translation of b - units in the negative direction of the y -axis, followed by,
- A reflection in the line $y = x$, followed by,
- A translation of b - units in the positive direciton of the y-axis.

A reflection in the line $y = ax$ can be described via the following transformation:

$$T(x, y) = \left(\frac{x(1-a^2)+2ay}{1+a^2}, \frac{y(a^2-1)+2ax}{1+a^2}\right).$$

b. Describe a reflection in the line $y = ax + b$ using elementary transformations and T.

- A translation of b - units in the negative direction of the y -axis, followed by,
- T, followed by,
- A translation of b - units in the positive direciton of the y-axis.

c. Find the image of the point $(2, 4)$ when it is reflected in the line $y = 3x + 5$.

We apply the transformations in b to our point, noting that $T(x, y) = \left(\frac{-8x + 6y}{10}, \frac{8y + 6x}{10}\right)$.
Hence in order, our transformations map $(2, 4)$ onto,

$$(2, -5) \mapsto (2, -1) \mapsto \left(\frac{-16 - 6}{10}, \frac{-8 + 12}{10}\right) = (-2.2, 0.4) \mapsto (-2.2, 5.4)$$

d. Show using coordinate geometry that T describes a reflection in the line $y = ax$.

Hint: Find the line going through a point (x_0, y_0) with a gradient $-\frac{1}{a}$.

Then, equate that line to $y = ax$ to get a point (x_1, y_1).

Then, (x_1, y_1) is the midpoint of (x_0, y_0) and $(x_0', y_0') = T(x_0, y_0)$.

We follow the hint.

A line going through the point (x_0, y_0) with a gradient $-\frac{1}{a}$ has equation

$$y = \frac{-1}{a}(x - x_0) + y_0$$

There point of intersection (x_1, y_1) lies on both that line and the line $y = ax$, hence,

$$ax_1 = \frac{-1}{a}(x_1 - x_0) + y_0 \implies a^2x_1 + x_1 = x_0 + ay_0 \implies x_1 = \frac{ay_0 + x_0}{a^2 + 1}$$

and $y_1 = ax_1 = \dfrac{a^2y_0 + ax_0}{a^2 + 1}$

Since (x_1, y_1) is the midpoint of (x_0, y_0) and (x_0', y_0') we see that,

$$(x_0', y_0') + (x_0, y_0) = 2(x_1, y_1) \implies (x_0', y_0') = 2(x_1, y_1) - (x_0, y_0)$$

Hence

$$x_0' = 2\frac{ay_0 + x_0}{a^2 + 1} - x_0 = \frac{2ay_0 + 2x_0 - a^2x_0 - x_0}{a^2 + 1} = \frac{x_0(1 - a^2) + 2ay_0}{1 + a^2}$$

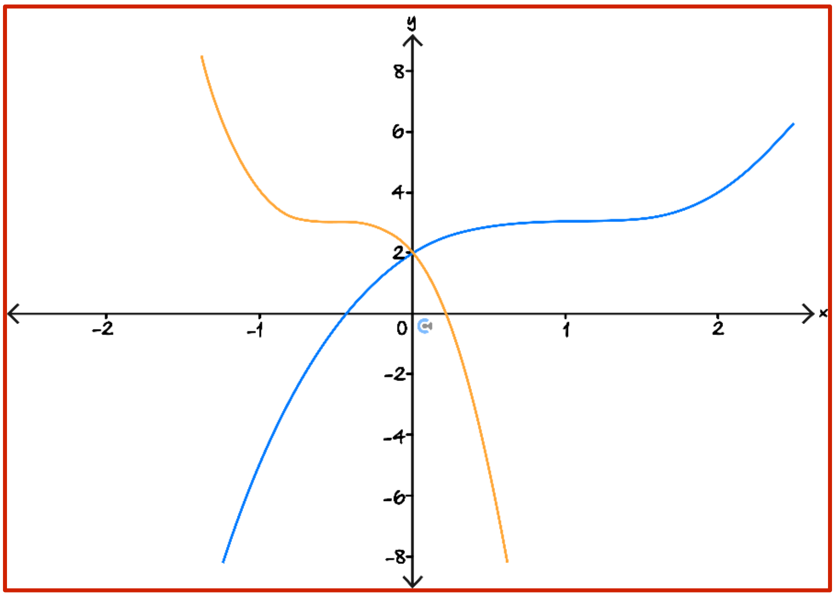
and $$y_0' = 2\frac{a^2y_0 + ax_0}{a^2 + 1} - y_0 = \frac{2a^2y_0 + 2ax_0 - a^2y_0 - y_0}{a^2 + 1} = \frac{y_0(a^2 - 1) + 2ax_0}{a^2 + 1}$$

This transformation sends the point (x_0, y_0) to $T(x_0, y_0)$. Hence as (x_0, y_0) is arbitrary, T describes a reflection in the line $y = ax$.

Sub-Section [1.3.2]: Transforming Graphs of Functions

Question 45

a. The graph of $f(x)$ is shown below.



On the same axes, sketch the graph of $g(x) = f(-2x)$.

b. Let $f(x) = e^x$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x - 1, y + 2)$ maps the graph of $f(x)$ onto the graph of $g(x)$. Find the rule for $g(x)$.

Consider some points (x', y') on the graph of $g(x)$.

We observe that $(x', y') = T(x, y) = (x - 1, y + 2)$ for some point (x, y) on the graph of $f(x)$. To relate x' with y' we express x in terms of x' and y in terms of y', specifically,

$$x' = x - 1 \implies x = x' + 1 \quad \text{and} \quad y' = y + 2 \implies y = y' - 2$$

We substitute the above two into $y = e^x$ to relate x' with y'. Hence

$$y' - 2 = e^{x+1} \implies y' = e^{x+1} + 2$$

Thus the rule for $g(x)$ is, $g(x) = e^{x+1} + 2$

c. Find the rule for the image of the graph of $y = \cos(x)$ under the transformation,

$$S = \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(-3x, \frac{1}{2}y\right).$$

We apply the same logic as in part *b*.

Observe that $x' = -3x \implies x = -\frac{x'}{3}x$, and $y' = \frac{1}{2}y \implies y = 2y'$. Substituting these into $y = \sin(x)$ yields,

$$2y' = \cos\left(\frac{-x'}{3}\right) \implies y' = \frac{1}{2}\cos\left(\frac{x'}{3}\right)$$

Thus the rule for the image of the graph of $y = \cos(x)$ under S is, $y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

Question 46

a. Let $f(x) = 5\sqrt{x} - 3$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (4x, 3 - y)$ maps the graph of $f(x)$ onto the graph of $g(x)$. Find the rule for $g(x)$.

Consider some points (x', y') on the graph of $g(x)$.

We observe that $(x', y') = T(x, y) = (4x, 3 - y)$ for some point (x, y) on the graph of $f(x)$. To relate x' with y' we express x in terms of x' and y in terms of y', specifically,

$$x' = 4x \implies x = \frac{x'}{4} \quad \text{and} \quad y' = 3 - y \implies y = 3 - y'$$

We substitute the above two into f to relate x' with y'. Hence

$$3 - y' = 5\sqrt{\frac{x'}{4}} - 3 \implies y' = 6 - \frac{5}{2}\sqrt{x'}$$

Thus the rule for $g(x)$ is, $g(x) = 6 - \frac{5}{2}\sqrt{x}$

b. Find the rule for the image of the graph of $y = e^{x+2} - \log_e(-2x)$ under the transformation,

$$S : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (-2x - 1, y + 3).$$

We apply the same logic as in part a.

Observe that $x' = -2x - 1 \implies x = -\frac{x'+1}{2}$ and $y' = y + 3 \implies y = y' - 3$. We substitute the following two values into $y = -e^{x+2} - \log_e(-2x)$ to get,

$$y' - 3 = e^{2-\frac{x'+1}{2}} - \log_e(x'+1) \implies y' = e^{2-\frac{x'+1}{2}} - \log_e(x'+1) + 3$$

Thus the rule for the image of the graph of $y = e^{x+2} - \log_e(-2x)$ under S is, $e^{2-\frac{x'+1}{2}} - \log_e(x'+1) + 3$

c. Let $f(x) = (x - 1)(x + 2)(x - 3)$, and $g(x) = 4f(2 - x) + 5$.

Solve $g(x) = 5$.

$g(x) = 5 \implies 4f(2 - x) + 5 = 5 \implies 4f(2 - x) = 0 \implies f(2 - x) = 0.$

Hence $2 - x = -2, 1, 3 \implies x = -1, 1, 4.$

Space for Personal Notes

Question 47

a. Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ which can be described by the following sequence of transformations:

- A translation is 3 units up and 2 units left, followed by,
- A dilation by a factor of 3 from the x-axis and $\frac{1}{2}$ from the y-axis followed by,
- A reflection in the x-axis.

T maps the graph of $f(x)$ onto the graph of $g(x) = \log_e(x)$. Find the rule of $f(x)$.

We see that under T,

$$(x, y) \mapsto (x-2, y+3) \mapsto \left(\frac{x-2}{2}, 3(y+3)\right) \mapsto \left(\frac{x-2}{2}, -3(y+3)\right) = (x', y')$$

For any point (x, y) on the graph of $y = f(x)$, we know that $y' = g(x') = \log_e(x')$. Substituting $x' = \frac{x-2}{2}$ and $y' = -3(y+3)$ into this equation yields,

$$-3(y+3) = \log_e\left(\frac{x-2}{2}\right) \implies y = -\frac{1}{3}\log_e\left(\frac{x-2}{2}\right) - 3$$

Hence, $f(x) = -\frac{1}{3}\log_e\left(\frac{x-2}{2}\right) - 3$.

b. Consider the transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$, which the following sequence of transformations can describe:

- A dilation by a factor of 2 from the x-axis and 5 from the y-axis, followed by,
- A translation 1 unit down and 4 units right.

Find the rule for the image of the graph of $y = 25x^2 + 5x - 1$ under S.

We see that under S,

$$(x, y) \mapsto (5x, 2y) \mapsto (5x + 4, 2y - 1) = (x', y')$$

Hence $x = \frac{x' - 4}{5}$ and $y = \frac{y' + 1}{2}$.

Substituting these equations into $y = 25x^2 + 5x - 1$ yields

$$\frac{y' + 1}{2} = (x' - 4)^2 + (x' - 4) - 1 = x'^2 - 7x' + 11 \implies y' = 2x'^2 - 14x' + 21$$

Thus the rule for the image of the graph of $y = 25x^2 + 5x - 1$ under S is,

$$y = 2x^2 - 14x + 21$$

c. A transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$, $U(x, y) = (2x + 5, 3 - 2y)$ maps the graph of $y = af(x) + b$ onto the graph of $y = f(cx + d)$. Find the values of a, b, c, and d.

As $x' = 2x + 5$ we see that $x = \frac{x' - 5}{2}$, and as $y' = 3 - 2y$ we see that $y = \frac{3 - y'}{2}$.

We note that if a pair (x, y) lies on the graph $y = af(x) + b$, then their image under U, (x', y') lies on the graph of $y = f(cx + d)$. Hence,

$$\frac{3 - y'}{2} = af\left(\frac{x' - 5}{2}\right) + b \implies y' = -2af\left(\frac{x' - 5}{2}\right) + 3 - 2b = f(cx' + d)$$

Equation coefficients yields,

$$-2a = 1 \implies a = -\frac{1}{2} \quad \text{and} \quad 3 - 2b = 0 \implies b = \frac{3}{2} \quad \text{and} \quad c = \frac{1}{2} \quad \text{and} \quad d = -\frac{5}{2}$$

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Question 48

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which is described by the following sequence of transformations:

- A translation of 3 units upwards and 5 units left, followed by,
- A reflection in the line $y = x$, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the x-axis and $\frac{1}{4}$ from the y-axis, followed by,
- A reflection in the x-axis.

T maps the graph of $f : (-\infty, 2], f(x) = 3x^2 + 12x + 5$ onto the graph of g.

Find the rule of g.

We see that under T,

$$(x, y) \mapsto (x-5, y+3) \mapsto (y+3, x-5) \mapsto \left(\frac{y+3}{4}, \frac{x-5}{2}\right) \mapsto \left(\frac{y+3}{4}, -\frac{x-5}{2}\right) = (x', y')$$

Hence $x' = \dfrac{y+3}{4} \implies y = 4x' - 3$ and $y' = -\dfrac{x-5}{2} \implies x = -2y' + 5$

Since our transformation will be inverting f, let us express x as a function of y.

$$\begin{aligned} & y = 3x^2 + 12x + 5 \\ \implies & y = 3(x+2)^2 - 7 \\ \implies & \frac{y+7}{3} = (x+2)^2 \\ \implies & x = -\sqrt{\frac{y+7}{3}} - 2 \end{aligned}$$

Since $x \le -2$. Now we substitute x' and y' into our equation to get.

$$\begin{aligned} & -2y' + 5 = -\sqrt{\frac{4x'+4}{3}} - 2 \\ \implies & -2y' = -\frac{2\sqrt{x'+1}}{\sqrt{3}} - 7 \\ \implies & y' = \sqrt{\frac{x'+1}{3}} + \frac{7}{2} \end{aligned}$$

Hence the rule for g is, $g(x) = \sqrt{\dfrac{x+1}{3}} + \dfrac{7}{2}$

Sub-Section [1.3.3]: Find Transformations from Transformed Function

Question 49

a. Let $f(x) = x^2$ and $g(x) = 3x^2 - 2$.

Describe a transformation that maps the graph of f onto the graph of g.

Choose some point (x', y') on the graph of $g(x)$. Then $\frac{y'+2}{3} = f(x')$, hence there is some point (x, y) on the graph of $f(x)$ such that,

$$\left(x', \frac{y'+2}{3}\right) = (x, y) \implies (x', y') = (x, 3y - 2)$$

This gives us our transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x, 3y - 2)$

b. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (ax + b, cx + d)$ maps the graph of $y = \log_e(x)$ to the graph of $y = 5 - \log_e(2x + 3)$.

Find the values of a, b, c, and d.

We first apply T to the graph of $y = \log_e(x)$.

Let (x', y') be a point on the image of $y = \log_e(x)$ under T. Then there is some pair (x, y) on the graph of $y = \log_e(x)$ such that,

$$(x', y') = (ax + b, cx + d) \implies (x, y) = \left(\frac{x'-b}{a}, \frac{y'-d}{c}\right).$$

We substitute this into the equation $y = \log_e(x)$ to get,

$$y' = c\log_e\left(\frac{x'-b}{a}\right) + d = 5 - \log_e(2x' + 3)$$

By comparing coefficients, we see that, $a = \frac{1}{2}, b = -\frac{3}{2}, c = -1$ and $d = 5$.

c. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ maps the graph of $f(x) = \sqrt{x}$ onto the graph of $g(x) = 3\sqrt{x-1} + 5$.

- A dilation by a factor of 3 from the x-axis, followed by
- A translation of 1 unit(s) in the positive direction of the x-axis, followed by
- A translation of 5 units in the positive direction of the y-axis.

- A translation of ______ unit(s) in the positive direction of the x-axis, followed by
- A translation of ______

Fill in the blanks.

We observe that $g(x) = 3f(x-1) + 5$. Thus any pair (x', y') on the graph of $g(x)$ satisfies,

$$\frac{y'-5}{3} = f(x'-1)$$

Hence we can relate some pair (x, y) on the graph of $f(x)$, to (x', y') by,

$$(x, y) = \left(x' - 1, \frac{y'-5}{3}\right) \implies (x', y') = (x + 1, 3y + 5)$$

We can then describe our transformation as above.

Question 50

a. Let $f(x) = 4(x-5)^2$.

The transformations:

$$S : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x + b, ay),$$

and

$$T : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (cx + d, y).$$

Both map the graph of $y = x^2$ onto the graph of f.

Find the values of a, b, c, and d.

We first apply S onto the graph of $y = x^2$, this yields the graph of,

$$y = a(x-b)^2.$$

Comparing coefficients to $f(x) = 4(x-5)^2$ we see that $a = 4$ and $b = 5$.

Now we apply T onto the graph of $y = x^2$, this yields the graph of,

$$y = \left(\frac{x-d}{c}\right)^2.$$

Comparing coefficients to $f(x) = 4(x-5)^2$ we see that $c = \frac{1}{2}$ and $d = 5$.

b. Consider a function $f : [0, \infty) \to \mathbb{R}, f(x) = 100 - 4x$.

A different function g has the property, that g decreases at half the rate of f at any point in time and that $g(0) = f(0)$. State a single transformation that maps the graph of f onto the graph of g.

Since g decreases at half the rate of f at any point in time and $g(0) = f(0)$, we know that $g(x) = \frac{f(x)}{2} + 50 = 100 - 2x = f\left(\frac{x}{2}\right)$.

Hence a dilation by a factor of 2 from the y-axis will transform the graph of f onto the graph of g.

c. Let $g(x) = -\frac{f(4x+12)}{5} - 20$.

Fill in the blank lines to make the f... graph of $g(x)$.

- A dilation by a factor of $\frac{1}{5}$ from the x-axis, followed by,
- A translation of -12 units in the positive direction of the x-axis, followed by,
- A translation of 20 units in the positive direction of the y-axis, followed by,
- A dilation by a factor of $\frac{1}{4}$ from the y-axis, followed by,

- A dilation by a factor
- A translation of ______
- A translation of ______
- A dilation by a factor
- A reflection in the x-axis.

Choose a point (x', y') on the graph of g. We observe that,

$$-5(y' + 20) = f(4x' + 12)$$

Hence there is some point (x, y) on the graph of f such that,

$$(x, y) = (4x' + 12, -5y' - 100)) \implies (x', y') = \left(\frac{x-12}{4}, -\frac{y+100}{5}\right) = \left(\frac{x-12}{4}, -\left(20 + \frac{y}{5}\right)\right)$$

The last equation is useful for us since we are first dilating then translating then reflecting y, but we are first translating then dilating x. Hence we can read off the required transformations from the last equation.

Question 51

a. Describe a sequence of three transformations that map the graph of $f(x) = \sqrt{7 - 6x - x^2}$ onto the graph of $g(x) = \sqrt{4 - x^2}$.

By completing the square, we observe that $f(x) = \sqrt{4(x-2)^2}$.

For $x \geq 2$ this can be simplified down to $f(x) = 2(x-2)$.

As $f\left(\frac{x-b}{a}\right) = x$ we observe that $a = 2$ and $b = -4$.

b. Let $f : [2, \infty) \to \mathbb{R}$, $f(x) = \sqrt{4x^2 - 16x + 16}$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (ax + b, y)$ maps the graph of $f(x)$ onto the graph of $g : [0, \infty) \to \mathbb{R}, g(x) = x$.

Find the values of a and b.

By completing the square, we observe that $f(x) = \sqrt{4(x-2)^2}$.

For $x \geq 2$ this can be simplified down to $f(x) = 2(x-2)$.

As $f\left(\frac{x-b}{a}\right) = x$ we observe that $a = 2$ and $b = -2$.

c. A function f has its only stationary point at $(2, 3)$ and its only x-axis intercept at $(-5, 0)$.

A function g has its only stationary point at $(6, -2)$ and only x-axis intercept at $(-8, 0)$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (ax + b, cy)$ maps the graph of f onto the graph of g.

Find a, b, and c.

We know that T maps stationary points to stationary points, hence $T(2, 3) = (2a + b, 3c) = (6, -2)$. This implies that $c = \frac{-2}{3}$

Since T has no vertical translations, it maps x-axis intercepts to x-axis intercepts. Hence $T(-5, 0) = (-5a + b, 0) = (-8, 0)$.

We solve $-5a + b = -8$ and $2a + b = 6$ simultaneously to find a and b.

Subtracting the first equation from the second yields $7a = 14 \implies a = 2$. Substituting this back into the first equation implies that $b = 5a - 8 = 10 - 8 = 2$.

Question 52 Tech-Active.

Let $f(x) = x^4 + x^3 + x^2 + x + 1$ and $g(x) = x^4 + 2x^3 + 4x^2 + 8x + 11$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (ax + b, cx + d)$ maps the graph of f onto the graph of g.

Find a, b, c, d and show that they are unique.

We observe that the rule of g is,

$$g(x) = cf\left(\frac{x-b}{a}\right) + d = x^4 + 2x^3 + 4x^2 + 8x + 11$$

We expand out f and compare the coefficients to get the following simultaneous equations.

$$\begin{aligned} c - \frac{bc}{a} + \frac{b^2c}{a^2} - \frac{b^3c}{a^3} + \frac{b^4c}{a^4} + d &= 11 \\ \frac{c}{a} - \frac{2bc}{a^2} + \frac{3b^2c}{a^3} - \frac{4b^3c}{a^4} &= 8 \\ \frac{c}{a^2} - \frac{3bc}{a^3} + \frac{6b^2c}{a^4} &= 4 \\ \frac{c}{a^3} - \frac{4bc}{a^4} &= 2 \\ \frac{c}{a^4} &= 1 \end{aligned}$$

We solve these equations to get $a = 2, b = 0, c = 16$ and $d = -5$.

Any other such values of a, b, c, d must satisfy those simultaneous equations. As those equations only have one solution, our values of a, b, c and d are unique.

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Section D: [1.4] - Transformations Exam Skills (Checkpoints)

Sub-Section [1.4.1]: Apply Quick Method to Find Transformations

Question 53

Find the image of the graph of $y = x^2$ under the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (1 - 2x, y + 5)$.

Apply the transformation $x \mapsto 1 - 2x$ in an opposite manner, so we replace x with $\frac{x-1}{2}$ Thus, (applying the y-axis transformations as well) we get,

$$y = \left(\frac{x-1}{2}\right)^2 + 5$$

Question 54

Describe a sequence of transformations that maps the graph of $y = x^3$ onto the graph of $y = 2(3x + 2)^3 - 3$.

In our equation we replace x with $3x + 2$, thus we apply those transformations in reverse including the order.

- A translation of 2 units left, followed by,
- A dilation by a factor of $\frac{1}{3}$ from the y-axis, followed by,

Then we apply the y-axis transformations as normal.

- A dilation by a factor of 2 from the x-axis, followed by,
- A translation of 3 units down.

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Question 55

Find the image of the graph of $y = 2\log_2(x) - 3$ under the following sequence of transformations:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units left and 3 units up, followed by,
- A reflection in the y-axis, followed by,
- A dilation by a factor of 5 from the y-axis.

We observe that the last 3 transformations apply to x, thus applying them in reverse (including the order) yields,

$$x \to \frac{1}{5}x \to -\frac{1}{5}x \to -\frac{1}{5}x + 2$$

Applying the y-axis transformations in order yields,

$$y \to 3y + 3$$

Thus, the rule for the image of our graph under the transformations is,

$$y = 3(2\log_2\left(-\frac{1}{5}x + 2\right) - 3) + 3 = 6\log_2\left(-\frac{1}{5}x + 2\right) - 6$$

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Question 56

Consider four linear functions, $p_1(x)$, $p_2(x)$, $q_1(x)$, and $q_2(x)$.

A transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x', y')$$

maps the graph of $y = f(x)$ onto the graph of $y = (p_1 \circ p_2 \circ f \circ q_2 \circ q_1)(x)$. Express x' in terms of x and y' in terms of y.

By the quick method we apply the reverse of the x-axis transformations in the reverse order, thus $x' = (q_1^{-1} \circ q_2^{-1})(x)$.
We apply the y-axis transformations in the correct order, this yields $y' = (p_1 \circ p_2)(y)$.

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Sub-Section [1.4.2]: Apply Transformations of Functions to Find its Domain and Range

Question 57

The function $f : \mathbb{R} \to \mathbb{R}$ has a range of $[2, \infty)$.

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (5 - 2x, 3 + y)$ maps the graph of f onto the graph of g. State the domain and range of g.

We simply apply T to both our domain and range.
As x is a real number $5 - 2x$ can be any real number.
As $y \geq 2$, we know that $y + 3 \geq 5$.
From here we see that the domain of g is R and the range of g is $[5, \infty)$.

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Question 58

The function $f : (-\infty, -1] \to \mathbb{R}$ has a range of $[-2, \infty)$.

Describe a sequence of transformations that maps the graph of f onto a graph of a function with a domain of $[0, \infty)$ and a range of $(-\infty, 2]$.

Since our domain and ranges both swap the signs of the ∞, we require reflections about both axes.

- A reflection about the x-axis, followed by,
- A reflection about the y-axis.

After applying these transformations, we have a domain of $[1, \infty)$ and a range of $(-\infty, 2]$.

- We just need a translation to fix the domain.
- A translation of 1 unit to the left.

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Question 59

Consider the function, $f : \mathbb{R}\backslash\{-2\} \to \mathbb{R}, f(x) = \frac{3}{(x+2)^2} - 5$.

The following sequence of transformations maps the graph of f onto the graph of g:

- A reflection in the x-axis, followed by,
- A dilation by a factor of 3 from the x-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ from the y-axis, followed by,
- A translation of 3 units up and 2 units left.

State the domain and range of g.

Recall that the domain of f is $\mathbb{R}\backslash\{-2\}$ and the range of f is $(-5, \infty)$. Under our transformations,

$$(x, y) \mapsto (x, -y) \mapsto \left(\frac{1}{2}x, -3y\right) \mapsto \left(\frac{1}{2}x - 2, 3 - 3y\right)$$

Now we just apply these transformations to our domain and range.

If $x \neq -2$, then $\frac{1}{2}x - 2 \neq -3$ and if $y > -5$, then $3 - 3y < 18$.

Hence the domain of g is $\mathbb{R}\backslash\{-3\}$ and the range of g is $(-\infty, 18)$.

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Question 60

Let $f : (-2, 1] \to \mathbb{R}, f(x) = 2(x + 1)^2 - 3$.

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, cy + d)$ where a and c are both non-zero.

The transformation T maps the graph of f onto the graph of g.

a. Explain why the range of g will always be of the form $[p, q]$ for some real $p < q$.

The range of f is $[-3, 5]$.
Let $y' = cy + d$. We note that y' is in the range of g if and only if y is in the range of f.
As we know that $-3 \leq y \leq 5$, we see that $-3c + d \leq y' \leq 5c + d$ if $c > 0$ or, $-3c + d \geq y' \geq 5c + d$ if $c < 0$.
As $c \neq 0$, in both cases these restrictions create an interval with square brackets.

b. Explain why the domain of g will always be of the form $(p, q]$ or $[p, q)$ for some real $p < q$.

The domain of f is $(-2, 1]$.
Let $x' = ax + b$. We note that x' is in the domain of g if and only if x is in the domain of f.
As we know that $-2 < x \leq 1$, we see that, $-2a + b < x \leq a + b$ if $a > 0$ or, $-2a + b > x \geq a + b$ if $a < 0$.
The first restriction produces a range of the form $(p, q]$ whilst the second produces a range of the form $(q, p]$

c. For what values of a, is the domain of g of the form $(p, q]$?

$a > 0$

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Sub-Section [1.4.3]: Apply Transformations of Functions to Find Transformed Points and Tangents

Question 61

The equation of the tangent to the graph of $f(x)$ at the point $(1, 3)$ is $y = 2x + 1$.

The transformation, $T(x, y) = \left(x, \frac{y}{3} + 1\right)$ maps the graph of f onto the graph of g.

Find the equation of the tangent to the graph of g at the point $(1, 2)$.

As the image of the point $(1, 3)$ under T is $(1, 2)$, we simply apply T to our tangent line. Thus, our tangent to the graph of g at $(1, 2)$ is,

$$y = \frac{1}{3}(2x + 1) + 1 = \frac{2x + 4}{3}$$

Question 62

The points $(2, 4)$ and $(4, 7)$ lie on the graph of $f(x)$.

Evaluate $g(2)$, where $g(x) = 3f(6 - x) + 5$.

$$g(2) = 3f(6 - 2) + 5 = 3f(4) + 5.$$

As the point $(4, 7)$ lies on the graph of $y = f(x)$, we see that $f(4) = 7$, hence, $g(2) = 21 + 5 = 26$.

Question 63

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ described by the following sequence of transformations:

- A dilation by a factor of 2 from the x-axis, followed by,
- A translation by a factor of 4 in the negative direction of the x-axis, followed by,
- A dilation by a factor of $\frac{1}{3}$ from the y-axis, followed by,
- A translation by a factor of 5 in the positive direction of the y-axis.

The image of $A(u, v)$ under T is $(3, 7)$. Find the values of u and v.

Under T we observe that,
$(x, y) \to (x, 2y) \to (x - 4, 2y) \to x - 4$
$3, 2y \to x - 4$
Applying this transformation to A yields,
$$T(A) = \left(\frac{u-4}{3}, 2v + 5\right) = (3, 7).$$
Thus, $u = 13$ and $v = 1$.

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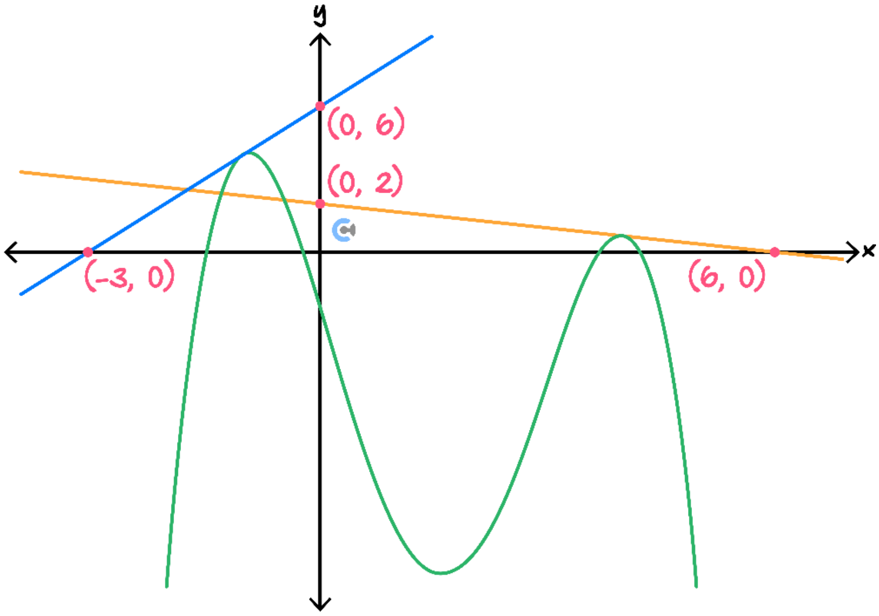
Question 64

The graph of $y = f(x)$ is drawn below along with two tangents at $x = 4$ and at $x = -1$.



Find the equation of the tangent to the graph of $g(x) = 1 - 3f(2 - 2x)$ when $x = -1$.

A possible transformation that maps the graph of f onto the graph of g is,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(1 - \frac{1}{2}x, 1 - 3y\right)$$

Thus, the pre-image of $(-1, g(-1))$ under T is $\big(4, f(4)\big)$, thus T maps the tangent to f at $x = 4$ onto the tangent to g at $x = -1$.

This tangent has the equation, $y = 2 - \frac{1}{3}x$.

Applying T to this tangent yields the equation, $y = 1 - 3\left(2 - \frac{2-2x}{3}\right) = -3 - 2x$

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Sub-Section [1.4.4]: Find Transformations with Constraints

Question 65

Consider the transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the following sequence of transformations:

- A dilation by a factor of a from the x-axis.
- A translation by a factor of b in the positive direction of the y-axis.

T maps the graph of $f(x) = \sqrt{x}$ onto the graph of $g(x) = \sqrt{9x} + 6$.

Find the values of a and b.

Under T we see that $(x, y) \to (x, ay + b)$.
Thus, the image of the graph of f under T has a rule of, $y = a\sqrt{x} + b = g(x)$.
We take the 9 out of the square root in the rule of g to get $g(x) = 3\sqrt{x} + 6$.
Now we can compare coefficients to get $a = 3$ and $b = 6$

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Question 66

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$ maps the graph of $y = 2^x$ onto the graph of $y = 8 \times 2^{3x-1} - 5$.

Find the values of a, b, and c.

The rule for the image of the graph of $y = 2x$ under T is,

$$y = 2^{\frac{x-b}{a}} + c = 8 \times 23x - 1 - 5$$

As we do not have a dilation from the x-axis, we take the 8 into the exponential to get

$$y = 2^{3x+2} - 5.$$

From here, we can compare coefficients to get $c = -5$, $a = \frac{1}{3}$ and $b = -6$.

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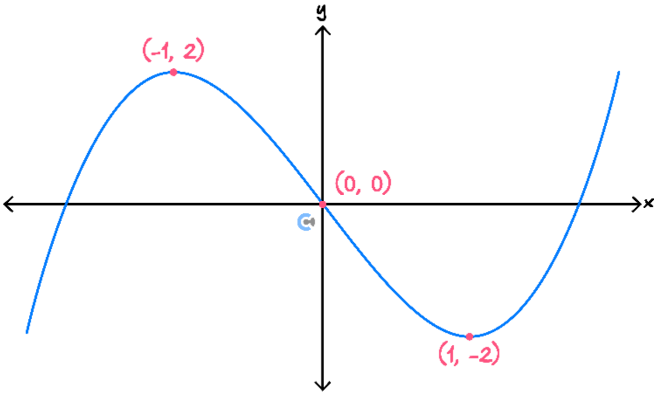
Question 67

The graph of $y = x^3 - 3x$ is drawn below.



The transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (a - x, b - y)$$

Maps the graph of $y = x^3 - 3x$ onto the graph of $y = (x - 1)^3 - 3x + 5$.

Find the values of a and b.

The image of the graph of $y = x^3 - 3x$ under T is,
$y = b - (a - x)^3 + 3(a - x) = b + (x - a)^3 - 3x + 3a$
Thus, $a = 1$ and $b + 3a = 5 \implies b = 2$.

Question 68

Consider the functions:

$$f:[-1,\infty)\to\mathbb{R}, f(x)=x^2+2x+2$$

$$g:(-\infty,1]\to\mathbb{R}, g(x)=4(2x-1)^2+3$$

Describe a sequence of a dilation followed by two translations and lastly a reflection that maps the graph of f onto the graph of g.

Looking at the domain of f and g, our reflection must be in the y-axis.
We now complete the square for the rule of f to get $f(x)=(x+1)^2+1$.
Since we have 1 dilation, we can bring the 4 into the square in the rule of g to get $g(x)=(4x-2)2+3$, and from here we see that we need to apply,

- A dilation by a factor of $\frac{1}{4}$ from the y-axis.

This maps the rule of f onto the rule of $y=(4x+1)^2+1$. As our last transformation will be a reflection in the y-axis, we need to use our two translations to map the graph of $y=(4x+1)^2+1$ onto the graph of $y=(4x+2)^2+3$.
Hence our two translations are,

- A translation of $\frac{1}{4}$ units left, followed by,

- A translation of 2 units up.

Lastly, we apply our reflection in the y-axis.

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Sub-Section [1.4.5]: Find Transformations of the Inverse Functions

Question 69

Consider the function, $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}, f(x) = \frac{2}{x-1} + 4$.

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + a, y + b)$ maps the graph of f onto the graph of its inverse function. Find the values of a and b.

The horizontal asymptote of f is $y = 4$, whilst the horizontal asymptote of f^{-1} is $y = 1$.
Thus, we need to translate the graph of f 3 units down, i.e. $b = -3$.
The vertical asymptote of f is $x = 1$, whilst the vertical asymptote of f^{-1} is $x = 4$.
Thus, we need to translate the graph of f 3 units to the right, i.e. $a = 3$.

Question 70

Consider the one-to-one functions, $f(x)$ and $g(x)$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (3 - x, 2y + 7)$ maps the graph of f onto the graph of g.

Describe a sequence of transformations that maps the graph of f^{-1} onto the graph of g^{-1}.

We can swap x and y in the equation of T to get a transformation $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (2x + 7, 3 - y)$ that maps the graph of f^{-1} onto the graph of g^{-1}.
From here we can read off a sequence of transformations.

- A dilation by a factor of 2 from the y-axis, followed by,
- A reflection in the x-axis, followed by,
- A translation of 7 units to the right and 3 units up.

Question 71

Let $f : (-\infty, 2] \to \mathbb{R}$, $f(x) = 3x^2 - 12x + 11$ and $g : [-3, \infty) \to \mathbb{R}$, $g(x) = 2\sqrt{x+3} + 4$.

a. Describe a sequence of transformations that maps the graph of f onto the graph of g^{-1} .

We first find the rule for g^{-1} by solving $g(y) = x$ for y. Thus,

$$2\sqrt{y+3} + 4 = x \implies \frac{x-4}{2} = \sqrt{y+3} \implies y = \frac{(x-4)^2}{4} - 3$$

Furthermore the domain of g^{-1} is the range of g which is $[4, \infty)$. Similarly, the range of g^{-1} is $[-3, \infty)$.
From here we see that our transformation needs,

- A reflection in the y-axis

to align our domains (Now the image of f after this reflection has a domain of $[-2, \infty)$, which a simple translation can map to the domain of g^{-1}).
The rule for the image of the graph of f after applying that reflection is

$$y = 3x^2 + 12x + 11 = 3(x+2)^2 - 1.$$

Now we apply the following transformations to map the graph of f onto the graph of g^{-1}.

- A dilation by a factor of $\frac{1}{12}$ from the x-axis, followed by,
- A translation of 6 units to the right and 2 units up

b. Hence, or otherwise, describe a sequence of transformations that maps the graph of g onto the graph of f^{-1} .

To map the graph of f^{-1} onto the graph of g we would just swap x and y in the transformation in **part a.** However, we are mapping the graph of g onto the graph of f^{-1}, thus we also need to swap our transformations and their order. Thus, our transformations are,

- *A* translation of 6 units down and 2 units to the left, followed by,
- *A* dilation by a factor of 12 from the y-axis, followed by,
- *A* reflection in the x-axis.

Question 72

Consider the function f which has the property that $f(x-3)-3=f^{-1}(x)$.

The transformation, $T:\mathbb{R}^2\to\mathbb{R}^2, T(x,y)=(4x+1,2-y)$ maps the graph of f onto the graph of g.

Describe a sequence of basic transformations (translations, dilations, and reflections in the x- and y-axis only) that maps the graph of g onto the graph of g^{-1}.

We will approach this problem by mapping the graph of g onto the graph of f, then onto the graph of f^{-1}, and finally onto the graph of g^{-1}.

Observe that the transformation $S:\mathbb{R}^2\to\mathbb{R}^2, S(x,y)=\left(\frac{x-1}{4},2-y\right)$ undoes T, hence maps the graph of g onto the graph of f.

Then we apply the transformation $R:\mathbb{R}^2\to\mathbb{R}^2, R(x,y)=(x+3,y-3)$ to map the graph of f onto the graph of f^{-1}.

We can swap x and y in the rule for T to create a transformation $Q:\mathbb{R}^2\to\mathbb{R}^2, Q(x,y)=(2-x,4y+1)$ that maps the graph of f^{-1} onto the graph of g^{-1}.

We compose these 3 transformations to create a transformation

$$\begin{aligned}U:\mathbb{R}^2\to\mathbb{R}^2, U(x,y)&=Q(R(S(x,y)))\\&=Q\left(R\left(\frac{x-1}{4},2-y\right)\right)\\&=Q\left(\frac{x-1}{4}+3,2-y-3\right)\\&=\left(2-\frac{x-1}{4}-3,4(-y-1)+1\right)=\left(-\frac{x}{4}-\frac{3}{4},-4y-3\right)\end{aligned}$$

Hence our transformation Q can be described with the following sequence of transformations,

- A reflection in both the x-axis and the y-axis, followed by,
- A dilation by a factor of $\frac{1}{4}$ from the y-axis and a dilation by a factor of 4 from the x-axis, followed by,
- A translation of $\frac{3}{4}$ units left and 3 units down.

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Sub-Section [1.4.6]: Find Opposite Transformations

Question 73

Describe a sequence of transformations that maps the graph of $y = 3e^{2x+1} - 4$ onto the graph of $y = e^x$.

Observe that $\frac{1}{3}(3e^{2x+1} - 4) + \frac{4}{3} = e^{2x+1}$.
Thus we can undo the "y" transformations with,

- A dilation by a factor of $\frac{1}{3}$ from the x-axis, followed by,
- A translation of $\frac{4}{3}$ units up.

Since $2x + 1 = x'$ we can undo the "x" transformations with,

- A dilation by a factor of 2 from the y-axis, followed by,
- A translation of 1 unit right.

Question 74

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = \left(2x + 3, \frac{1}{3}y - 4\right)$ maps the graph of $y = f(x)$ onto the graph of $y = x^3$.

Find the rule of f.

Choose a point, (x, y) on the graph of $y = f(x)$. Let (x', y') be the image of that point under T.
We can substitute $x' = 2x + 3$ and $y' = \frac{1}{3}y - 4$ into the equation $y' = (x')^3$ to get,

$$\frac{1}{3}y - 4 = (2x + 3)^3 \implies y = f(x) = 3(2x + 3)^3 + 12$$

Question 75

The following sequence of transformations maps the graph of f onto the graph of $y = \sqrt{x}$, for $x \in (2, \infty)$:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units left and 4 units up, followed by,
- A reflection in both the x-axis and the y-axis.

State the rule and domain of f.

Under our transformation we see that,

$$(x, y) \mapsto (x, 3y) \mapsto (x - 2, 3y + 4) \mapsto (2 - x, -3y - 4)$$

Choose a point (x, y) on the graph of x, and let $(x', y') = (2 - x, -3y - 4)$.
We see that $y' = \sqrt{x'}$, thus substituting the above values into this equation yields the rule for $f(x)$, specifically,

$$-3y - 4 = \sqrt{2 - x} \implies y = f(x) = -\frac{\sqrt{2 - x}}{3} - \frac{4}{3}.$$

Now choose some x in the domain of f. Then $2 - x = x'$ is in the domain of the image of f under T, hence $2 - x > 2 \implies x < 0$.
Hence the domain of f is $(-\infty, 0)$

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Question 76

Describe a transformation different from $(x, y) \mapsto (x, y)$, that maps the graph of $y = a(x-k)^5 + b(x-k)^3 + h$ onto itself.

We first map our graph onto the graph of $y = ax^5 + bx^3$. Then the transformation $(x, y) \mapsto (-x, -y)$ will map the graph of $y = ax^5 + bx^3$ onto itself, after which we can undo our first transformation to get back to our original graph.
The transformation to map the graph of $y = a(x-k)^5 + b(x-k)^3 + h$ onto the graph of $y = ax^5 + bx^3$ is $(x, y) \mapsto (x-k, y-h)$, and we can undo that transformation with the transformation, $(x, y) \mapsto (x+k, y+h)$. Now we combine these 3 transformations to get,

$$(x, y) \mapsto (x-k, y-h) \mapsto (k-x, h-y) \mapsto (2k-x, 2h-y)$$

Hence $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (2k-x, 2h-y)$ is our desired transformation.

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Sub-Section: Exam 1 Questions

Question 77

The following sequence of transformations maps the graph of $y = f(x)$ onto the graph of $y = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x\right)$:

- A translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis, followed by,
- A dilation by a factor of $\frac{1}{2}$ in from the y-axis, followed by,
- A dilation by a factor of 2 from the x-axis.

Find the rule of f.

Under our transformations,

$$(x, y) \mapsto \left(x + \frac{\pi}{6}, y\right) \mapsto \left(\frac{1}{2}x + \frac{\pi}{12}, 2y\right) = (x', y')$$

If a point (x, y) sits on the graph of $y = f(x)$, then (x', y') sits on the graph of $y' = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x'\right)$.
We simply substitute x and y into this equation to get

$$2y = \frac{1}{2}\cos\left(\frac{\pi}{3} - x - \frac{\pi}{6}\right) \implies y = \frac{1}{4}\cos\left(\frac{\pi}{6} - x\right)$$

Hence $f(x) = \frac{1}{4}\cos\left(\frac{\pi}{6} - x\right)$.

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Question 78

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 2 - \frac{1}{2}x^3$, and let $g : \mathbb{R} \to \mathbb{R}, g(x) = 6 - 2x$.

a.

i. Find $(g \circ f)(x)$.

$$(g \circ f)(x) = 6 - (4 - x^3) = 2 + x^3.$$

ii. Find $(f \circ g)(x)$ and express it in the form $k + m(x - h)^3$, where m, k and h are integers.

$$(f \circ g)(x) = 2 - \frac{1}{2}(6 - 2x)^3 = 2 - \frac{1}{2} \times (-2)^3 \times (x - 3)^3 = 2 + 4(x - 3)^3$$

b. The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + b, ay + c)$, where a, b and c are integers, maps the graph of $y = (f \circ g)(x)$ onto the graph of $y = (g \circ f)(x)$.

Find the values of a, b, and c.

Looking at the x-transformations we need to turn $(x-3)^3$ into x^3, hence we will map $x \mapsto x + 3$.
Looking at the y-transformations, we observe that $\frac{1}{4}(2 + 4x^3) + \frac{3}{2} = 2 + x^3$, thus we must map $y \mapsto \frac{1}{4}y + \frac{3}{2}$.
Hence $b = 3, a = \frac{1}{4}$ and $c = \frac{3}{2}$.

Question 79

Let $f : [1, \infty) \to \mathbb{R}, f(x) = 4(x-1)^2 - 3$ and let $g : [2, \infty) \to \mathbb{R}, g(x) = 1 - \sqrt{x-2}$.

a. Let g^{-1} be the inverse function of g.

i. State the domain and range of g^{-1}.

Dom g^{-1} = Ran $g = (-\infty, 1]$.
Ran g^{-1} = Dom $g = (-\infty, 2]$.

ii. Find the rule of g^{-1}.

We solve $g(y) = x$ for y, thus,

$$\begin{aligned} x &= 1 - \sqrt{y-2} \\ \implies (1-x)^2 &= y - 2 \\ \implies y &= 2 + (x-1)^2 \end{aligned}$$

b. The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + b, y + c)$ maps the graph of f onto the graph of g^{-1}.

Find the values of a, b, and c.

Due to the domain of g we know that we need a reflection in the y-axis, hence $a < 0$. The rule for the image of the graph of f under T is,

$$y = 4\left(\frac{x-b}{a} - 1\right)^2 - 3 + c$$

Since $-3 + c = 2$ we see that $c = 5$.
After bringing the 4 into the quadratic we see that $a = -2$, thus we require that,

$$2\left(\frac{x-b}{-2} - 1\right) = b - x - 2 = 1 - x \implies b = 3$$

Question 80

Let $f : \mathbb{R}\backslash\{a\} \to \mathbb{R}, f(x) = \frac{1}{x-a} + b$.

a. Find the rule and domain for the graph of f^{-1} in terms of a and b.

We solve $f(y) = x$ for y, thus

$$x = \frac{1}{y-a} + b \implies y - a = \frac{1}{x-b} \implies y = \frac{1}{x-b} + a$$

Hence the rule for f^{-1} is $f^{-1}(x) = \frac{1}{x-b} + a$, and the domain for f^{-1} is the range of f which is $\mathbb{R}\backslash\{b\}$.

b. The following sequence of transformations maps the graph of f to the graph of f^{-1}:

- A translation of 4 units in the positive direction of the x-axis, followed by,
- A translation of 4 units in the negative direction of the y-axis.

Find the value of a in terms of b.

Under those transformations we know that $(x, y) \mapsto (x+4, y-4)$, hence the image of the graph of f under that transformation is,

$$y = \frac{1}{x-4-a} + b - 4$$

Since this is equal to $\frac{1}{x-b} + a$ we see that $4 + a = b$ and $b - 4 = a$.
Both of these conditions imply that $a = b - 4$.

c. Let $g(x) = \frac{1}{x-c} + d$. A transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x+h, y+k)$$

maps the graph of g onto the graph of g^{-1}.

What restrictions are there on the values of h and k?

Under T the rule for the image of the graph of g is,

$$y = \frac{1}{x-h-c} + d + k$$

Since this is equal to $g^{-1}(x) = \frac{1}{x-d} + c$, we see that $h + c = d$ and $d + k = c$.
As $h = d - c$ and $k = c - d$ we see that $h = -k$.

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Sub-Section: Exam 2 Questions

Question 81

The graph of the function f passes through the point $(2, -3)$.

If $h(x) = 3f(x-2)$, then the graph of the function h must pass through the point:

A. $(0, 1)$

B. $(4, -9)$

C. $(0, -9)$

D. $(4, -1)$

Question 82

The graph of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x - 1$, is reflected in the y-axis and then translated 2 units to the left and then 3 units up.

Which one of the following is the rule of the transformed graph?

A. $y = 2^{2-x} + 2$

B. $y = 2^{2+x} + 2$

C. $y = \left(\frac{1}{2}\right)^{-2-x} + 2$

D. $y = \frac{1}{4}\left(\frac{1}{2}\right)^{x} + 2$

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Question 83

The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps the graph of $y = 4 - \log_e\left(\frac{x-1}{2}\right)$ onto the graph of $y = \log_e(x)$, has the rule:

A. $T(x, y) = \left(\frac{x-1}{2}, 4 - y\right)$

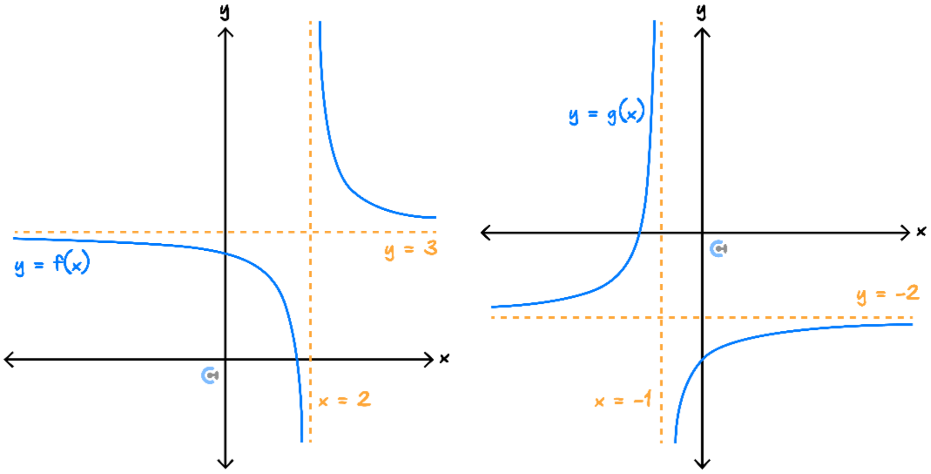
B. $T(x, y) = (2x + 1, -y - 4)$

C. $T(x, y) = (2x + 1, 4 - y)$

D. $T(x, y) = \left(\frac{x-1}{2}, -y - 4\right)$

Question 84

Consider the graph of f and g below, which have the same scale:



If T transforms the graph of f onto the graph of g, then:

A. $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (1 - x, y - 5)$

B. $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x - 3, y - 5)$

C. $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x - 3, 5 - y)$

D. $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (1 - x, 2 - y)$

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Question 85

The graph of the function g is obtained from the graph of the function $f: [-2, 3] \to \mathbb{R}, f(x) = 2x^2 - 4x + 5$, by a dilation of factor 2 from the y-axis, followed by a dilation of factor $\frac{1}{3}$, from the x-axis, followed by a reflection in the y-axis, and finally, followed by a translation of 1 unit in the negative direction of the y-axis.

The domain and range of g are respectively:

A. $[-6, 4]$ and $\left[\frac{8}{3}, 6\right]$

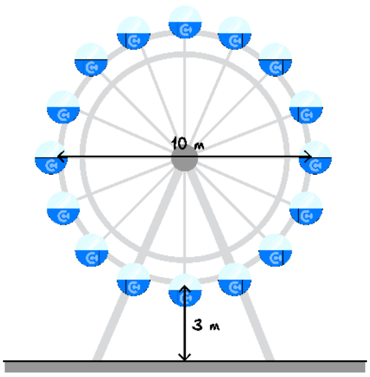
B. $\left[-1, \frac{2}{3}\right]$ and $[21, 41]$

C. $[-6, 4]$ and $\left[\frac{2}{3}, \frac{17}{3}\right]$

D. $[-6, 4]$ and $[0, 6]$

Question 86

The Contour Ferris Wheel pictured below takes 30 minutes to complete a trip.



Thus, the height of the bottom of a carriage t minutes after the start of a trip is given by,

$$h(t) = 8 - 5\cos\left(\frac{\pi t}{15}\right)$$

a. Describe a sequence of transformations that maps the graph of $\sin(t)$ onto the graph of h.

Observe that $\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$.

Thus we first translate our graph $\frac{\pi}{2}$ units left and dilate by a factor of $\frac{15}{\pi}$ from the y-axis.

This gives us $y = \cos\left(\frac{\pi t}{15}\right)$.

To get this into our desired form we now, simply reflect our graph in the t-axis, then dilate it by a factor of 5 from the t-axis and translate it 8 units up.

b. The horizontal displacement, d from the bottom of the carriage to the centre of the roller coaster t minutes after the start of a trip is,

$$d(t) = 5\sin\left(\frac{\pi t}{15}\right)$$

The transformation, $T(t, y) = (t + a, y + b)$ maps the graph of h onto the graph of d.

i. Find b.

$$b = 8$$

ii. Find the possible value of a.

We require $5\sin\left(\frac{\pi(t-a)}{15}\right) = -5\cos\left(\frac{\pi t}{15}\right)$

Since $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$ we simply need $-\frac{\pi a}{15} = -\frac{\pi}{2} \implies a = \frac{15}{2}$

c. 15 minutes into a trip on the Ferris Wheel, Caitlin crashes her car into the Ferris Wheel. This causes the Ferris Wheel to stop for 5 minutes before starting again at half speed.

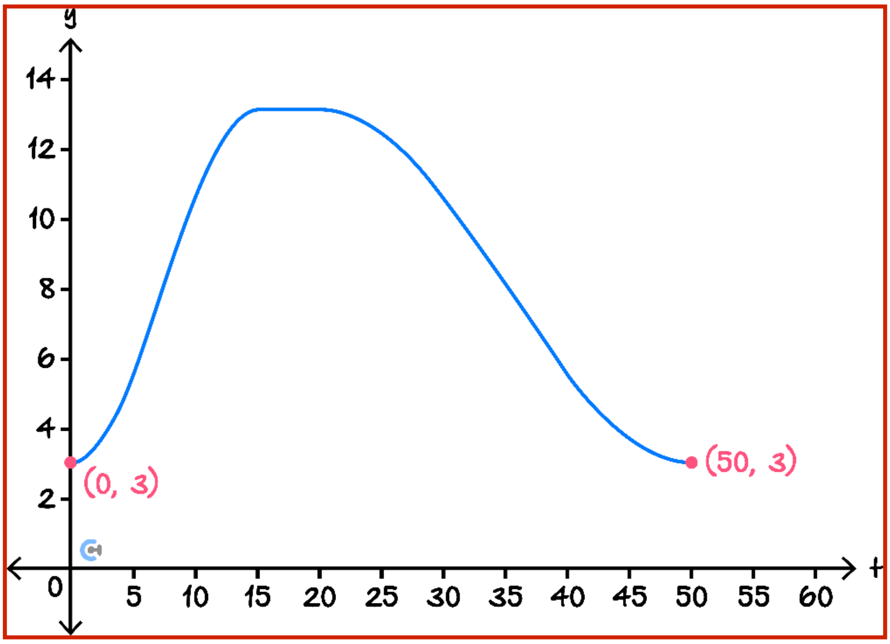
The height of the Ferris wheel in this trip, $h_1 : [0, r] \to \mathbb{R}$ is given by the following function:

$$h_1(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(pt + q) & 20 \leq t \leq r \end{cases}$$

Find a set of possible values of p, q, k, and r.

We know that $k = h(15) = 8 - 5\cos(\pi) = 13$.
Since it would take 15 minutes to finish the trip before Caitlin crashed her car into the Ferris wheel, it will now take 30 minutes in double time.
Hence $r - 20 = 30 \implies r = 50$.
Since we are going at half speed, after the crash we see that $p = \frac{1}{2}$. Now we simply require that $h\left(\frac{1}{2} \times 20 + q\right) = 13 \implies 10 + q = 15 \implies q = 5$

d. Part of the graph of h_1 is drawn on the axis below. Draw the rest of the graph of h_1 labelling endpoints with their coordinates.



Question 87

Consider the function, $f : (-1, 1) \to \mathbb{R}, f(x) = (2x - 1)^2 (x + 1)$.

a. State the range of f.

From the graph of f we see that the range is $[0, 2]$.

b. The following sequence of transformations, T, maps the graph of f onto the graph of g:

- A dilation by a factor of 3 from the x-axis, followed by,
- A translation of 2 units down and 5 units left, followed by,
- A reflection in the y-axis.

i. State the rule of g.

Under T we see that $(x, y) \mapsto (x, 3y) \mapsto (x - 5, 3y - 2) \mapsto (5 - x, 3y - 2) = (x', y')$. From the quick method, as $x = 5 - x'$ we see that $g(x) = 3f(5 - x) - 2 = 3(x - 6)(2x - 9)^2 - 2$

ii. State the domain of g.

We apply the transformation $x \mapsto 5 - x$ onto the interval $(-1, 1)$ to get the domain of g.
Thus the domain of g is $(4, 6)$.

iii. State the range of g.

We apply the transformation $y \mapsto 3y - 2$ onto the interval $[0, 2]$ to get the range of g.
Thus the range of g is $[-2, 4]$

c. The tangent to the graph of f at the point $A\left(-\frac{1}{4}, \frac{27}{16}\right)$ is given by the equation:

$$y = \frac{9}{8} - \frac{9x}{4}$$

i. Find B, the image of A under T.

$$B = \left(5 - \left(-\frac{1}{4}\right), 3\left(\frac{27}{16}\right) - 2\right) = \left(\frac{21}{4}, \frac{49}{16}\right)$$

ii. Find the equation of the tangent to the graph of g at point B.

We simply apply out transformation to the line to get,

$$y = 3\left(\frac{9}{8} - \frac{9(5-x)}{4}\right) - 2 = \frac{27x}{4} - \frac{259}{8}$$

d. A transformation, $S: \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (-x, a - y)$ maps the graph of f onto itself.

i. State the value of a.

The rule for the image of the graph of f under S is $y = a - f(-x)$.
As this is meant to equal $f(x)$, we see that $a - f(0) = f(0) \implies a = 2f(0) = 2$

ii. Hence, or otherwise, describe a sequence of transformations in terms of S and T as required, that maps the graph of g to itself, but does not map A to itself.

Let $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ undo the transformation T. Specifically,

$$T^{-1}(x, y) = \left(5 - x, \frac{y+2}{3}\right)$$

To map the graph of g onto itself, we can first apply T^{-1} to map the graph of g onto the graph of f, then apply S to map the graph of f onto itself, and then apply T to map the graph of f onto the graph of g.

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