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VCE Mathematical Methods ¾
Differentiation [0.9]
Workshop

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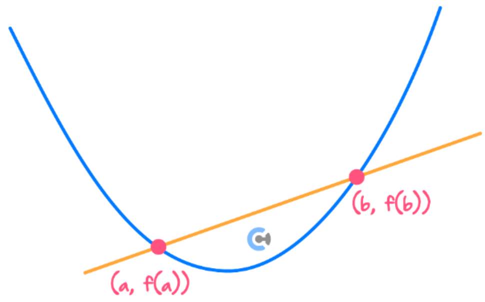
Section A: Recap

Average Rate of Change

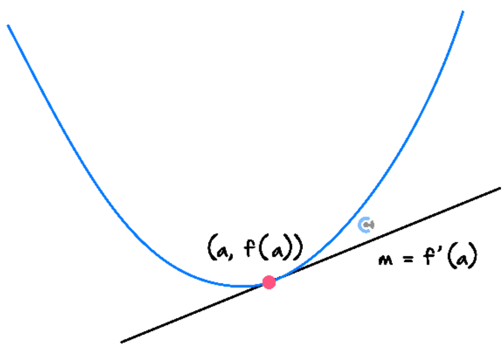


- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- It is the gradient of the line joining the two points.

Instantaneous Rate of Change



- **Instantaneous rate of change** is a gradient of a graph at a single point/moment.

$$\textit{Instantaneous rate of change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.
- **First principles** derivative definition:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Definition

Alternative Notation for Derivative

$$f'(x) = \frac{dy}{dx}$$

Definition

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \times x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$

Definition

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

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The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

  - Always differentiate the top function first.

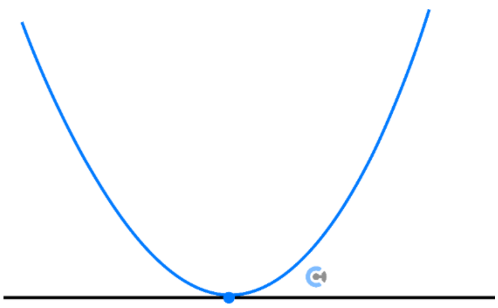
The Chain Rule

$$y = f\big(g(x)\big)$$

$$\frac{dy}{dx} = f'\big(g(x)\big)g'(x)$$

- The process for finding derivatives of **composite functions.**

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

Calculator Commands: Finding Derivatives

- **Mathematica**

  f'[x]

- **TI**

  Shift Minus

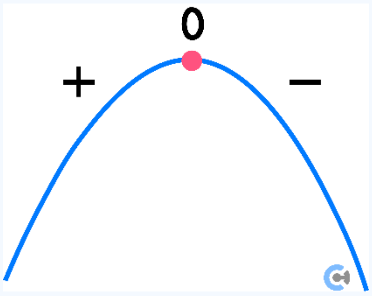
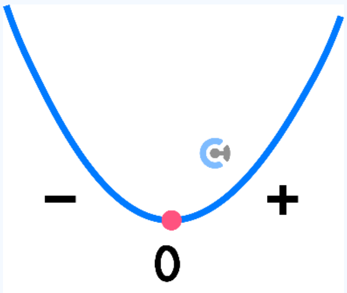
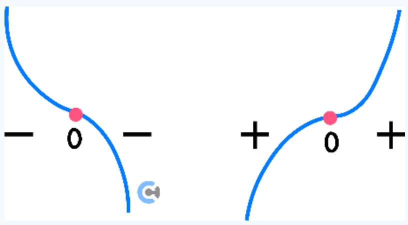
  $\frac{d}{dx}(f(x))$

- **Casio**

  Math 2

  $\frac{d}{dx}(f(x))$

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

Sign test

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighbouring points.

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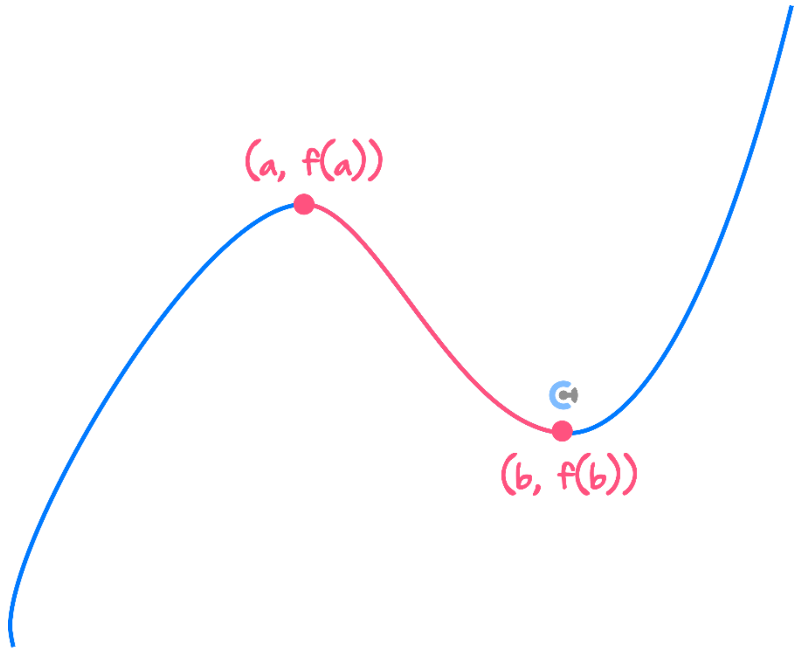
Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:

1. Find the turning points.
2. Consider the sign of the derivative between/outside the turning points.

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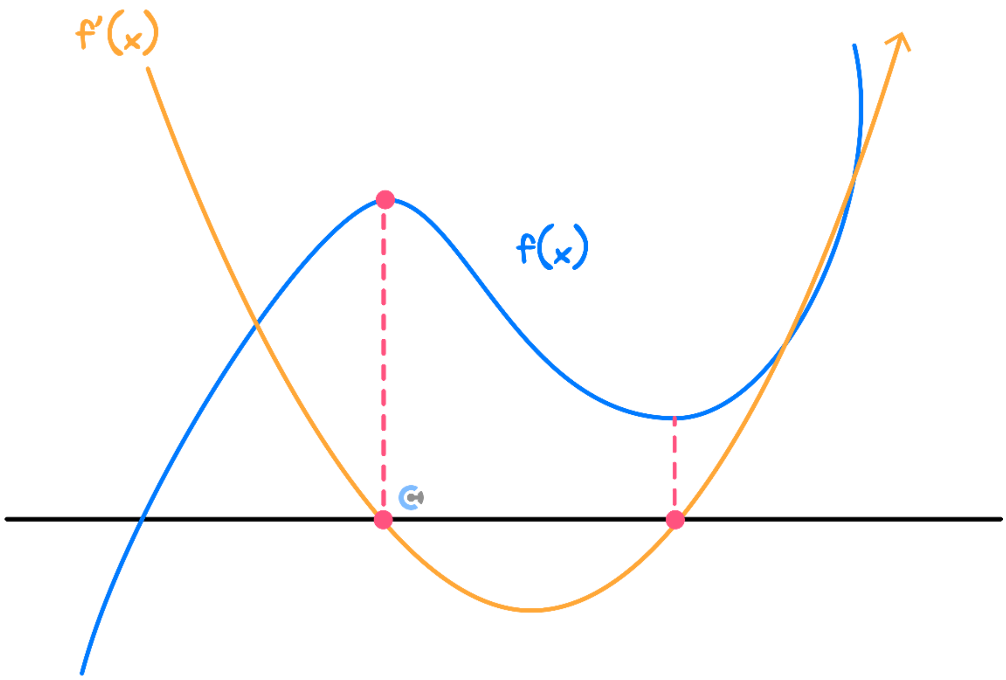
Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	x-intercepts
Increasing	Positive
Decreasing	Negative

$$y \text{ value of } f'(x) = \text{Gradient of } f(x)$$

- **Steps**

1. Plot x-intercept at the same x value as the stationary point of the original.
2. Consider the trend of the original function and sketch the derivative.
    - Original is increasing → Derivative is above the x-axis.
    - Original is decreasing → Derivative is below the x-axis.

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Section B: Warmup

Question 1

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 9x + 3$.

a. Find $f(1)$.

b. Find the average rate of change from $x = 0$ to $x = 2$.

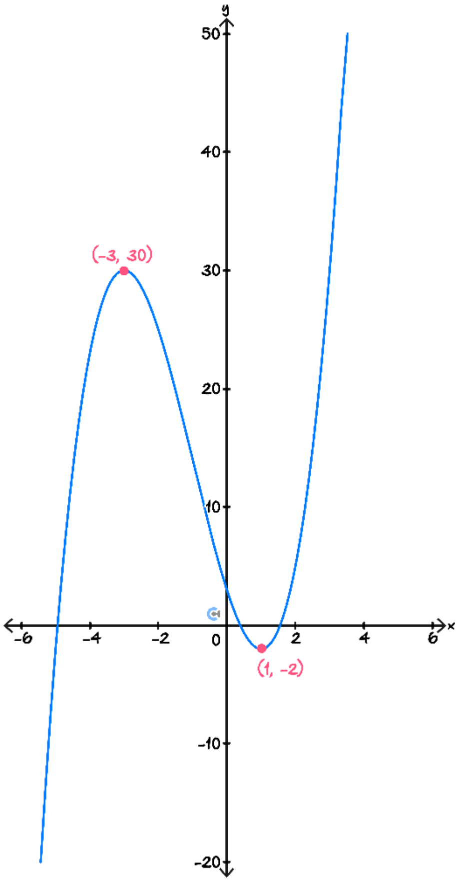
c.

i. Find $f'(x)$.

ii. Find $f'(-1)$.

d. Determine the coordinates and nature of any stationary points.

e. The graph of $y = f(x)$ is sketched on the axes below. Sketch the graph of $y = f'(x)$ on the same axes. Label all axial intercepts with coordinates.



f. Hence, state the values of x for which $f(x)$ is strictly decreasing.

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Section C: Exam 1 Questions (17 Marks)

INSTRUCTION: 17 Marks. 25 Minutes Writing.

Question 2 (3 marks)

a. Let $y = \frac{\tan(2x)}{x^3}$. Find $\frac{dy}{dx}$. (1 mark)

b. Let $f(x) = x^3 \tan(e^x)$. Evaluate $f'\left(\log_e\left(\frac{\pi}{6}\right)\right)$. (2 marks)

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Question 3 (3 marks)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = xe^{kx^2}$, where $k \in \mathbb{R}$.

a. Show that $f'(x) = (2kx^2 + 1)e^{kx^2}$. (1 mark)

b. Find the value(s) of k for which the graph of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. (2 marks)

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Question 4 (6 marks)

Consider the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = -x^2 + 5x - 4$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = e^x$.

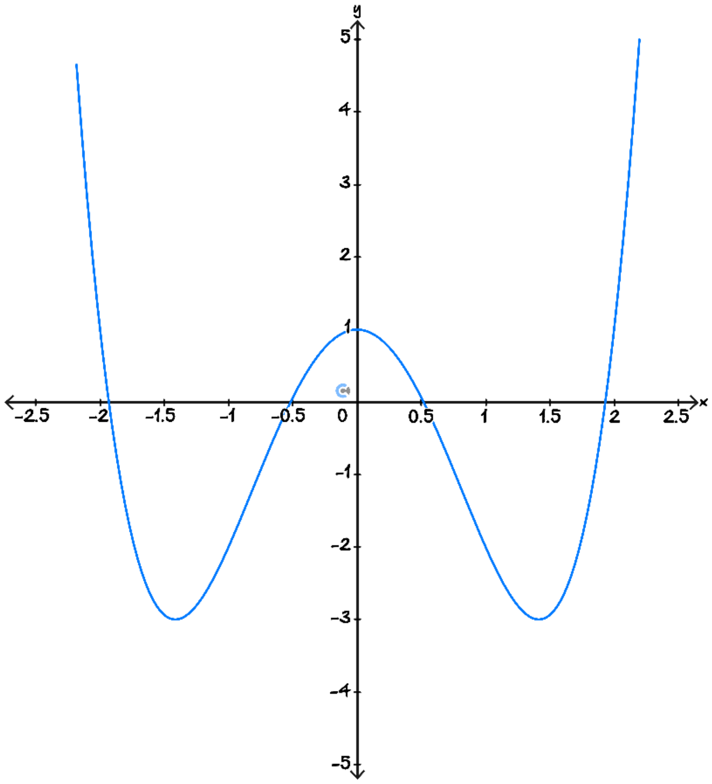
a. State the rule of $g(f(x))$. (1 mark)

b. Find the values of x for which $g(f(x))$ is strictly decreasing. (2 marks)

c. Find the coordinates of the stationary point of the graph of $f(g(x))$ and state its nature. (3 marks)

Question 5 (5 marks)

The graph of $f(x) = x^4 - 4x^2 + 1$ is shown below.



a. Sketch the graph of $y = f'(x)$ on the axes above. Label all axes intercepts with coordinates. (3 marks)

b. For both $f(x)$ and $f'(x)$ state whether they are an odd function, even function or neither. (1 mark)

c. Let g be an even polynomial function of degree $n \geq 2$. Is it always true that g' is an odd function? (1 mark)

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Section D: Tech Active Exam Skills

Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=\mathbf{e}^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	$\mathbf{e}$

- **Casio:**

define $f(x) = e^{-x^2+2x}$	
	done
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	
	$\{x=1\}$
$f(1)$	
	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

Section E: Exam 2 Questions (20 Marks)

INSTRUCTION: 20 Marks. 25 Minutes Writing.

Question 6 (1 mark)

The average rate of change for the function with the rule $f(x) = y = -4e^{-\frac{2x}{5}}$ from $y = -3$ to $y = -1$ is closest to:

A. 1.02

B. -1.02

C. 1.37

D. 0.73

Question 7 (1 mark)

Let $f(x)$ and $g(x)$ be differentiable functions, with the following values given:

$$f(2) = 3,\ f'(4) = 5,\ g(2) = 4,\ g'(2) = 6.$$

Find the gradient of $f\big(g(x)\big)$ at $x = 2$.

A. 20

B. 15

C. 30

D. 24

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Question 8 (1 mark)

Given that $f(1) = 2, f'(1) = 3, g(1) = 4, g'(1) = 5$, find the gradient of $f(x)g(x)$ at $x = 1$.

A. 13

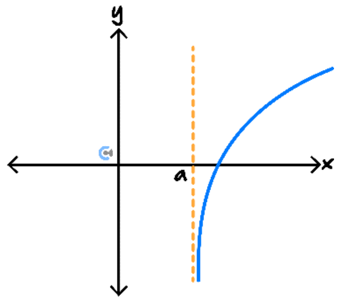
B. 22

C. 18

D. 20

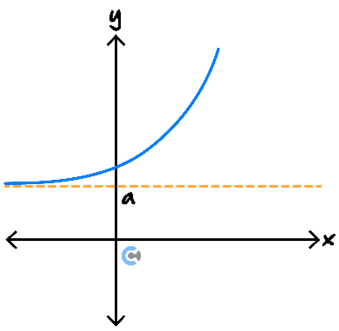
Question 9 (1 mark)

The graph of the function f is shown below:

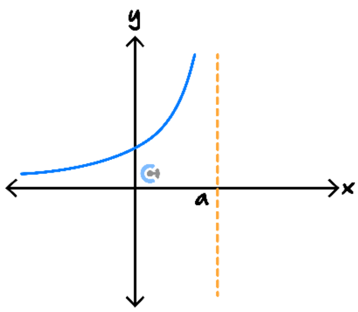


The graph corresponding to f' is:

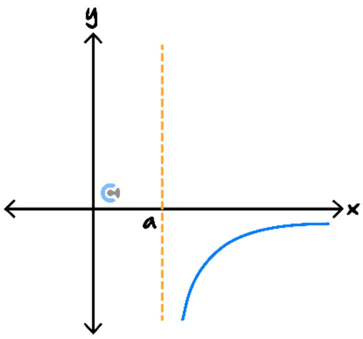
A.



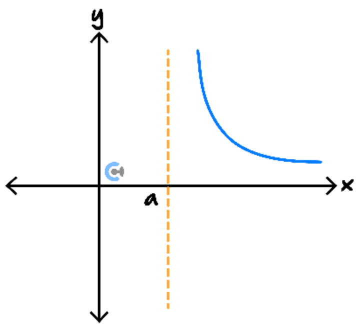
B.



C.



D.



Question 10 (1 mark)

Suppose $f(x)$ and $g(x)$ are differentiable, and the following values are given:

$$f(3) = 5,\ f'(3) = 4,\ g(3) = 2,\ g'(3) = 1.$$

Find the gradient of $\frac{f(x)}{g(x)}$ at $x = 3$.

A. $\frac{2}{5}$

B. $\frac{3}{4}$

C. $\frac{1}{2}$

D. $\frac{2}{3}$

Question 11 (1 mark)

Consider the graph of g with the rule $g(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has a y-intercept at $(0,10)$ and turning points when $x = -3$ and $x = 2$ and passes through $(3,64)$.

The rule of $g(x)$ is:

A. $g(x) = -4x^3 - 6x^2 + 72x + 10$

B. $g(x) = -3x^3 - 9x^2 + 27x + 10$

C. $g(x) = 3x^3 + 8x^2 + 10x + 10$

D. $g(x) = 2x^3 + 12x^2 + 6x + 10$

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Question 12 (1 mark)

The graph of $f(x) = ax^5 + bx^4 + x^3 - 3$, where a and b are real constants, will have three stationary points when:

A. $a > -\frac{4b^2}{15}$

B. $a \leq -\frac{4b^2}{15}$

C. $a < \frac{4b^2}{15}$

D. $a > \frac{4b^2}{15}$

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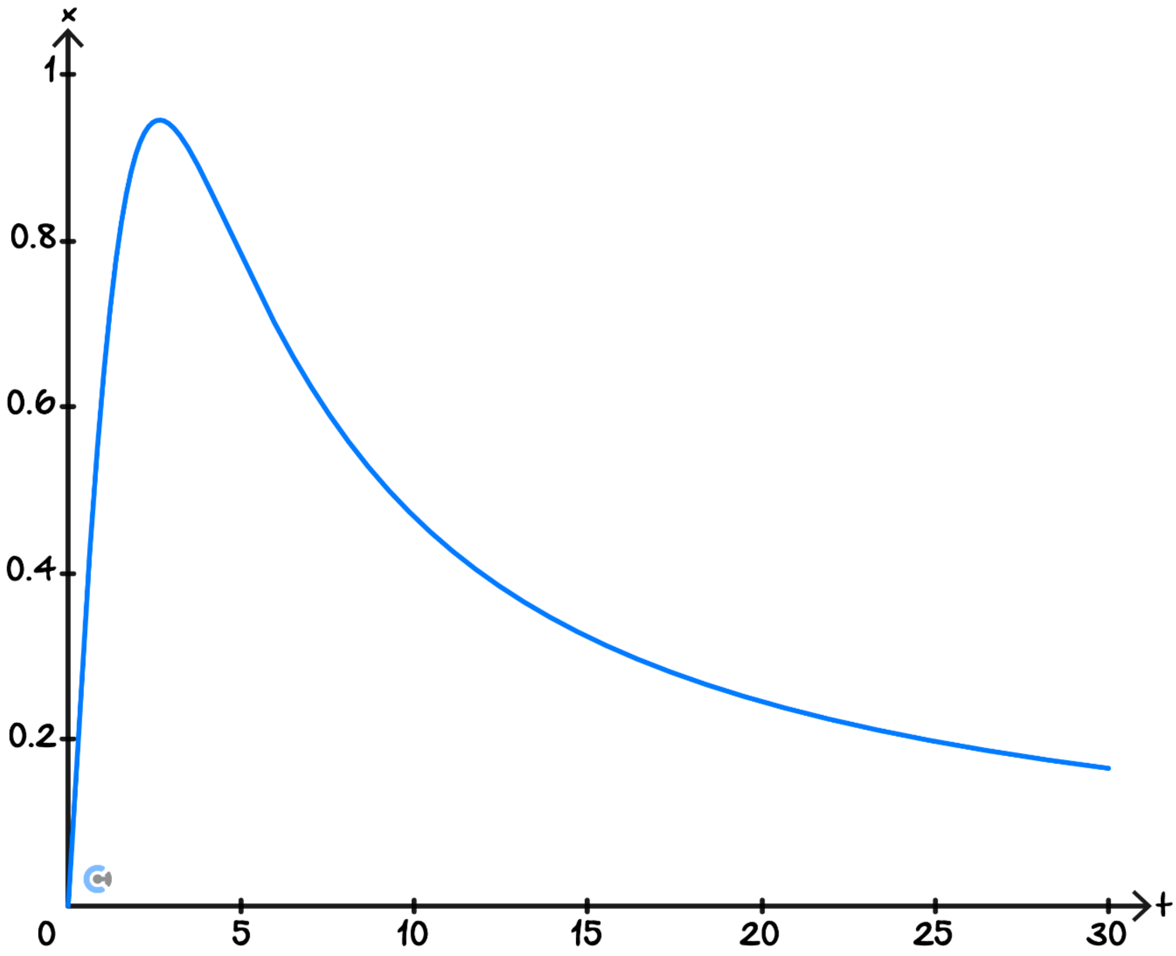
Question 13 (13 marks)

A tranquiliser is injected into a muscle from which it enters the bloodstream. The concentration, $x, mg/L$, of the tranquiliser in the bloodstream, may be modelled by the function:

$$x(t) = \frac{5t}{7 + t^2}, t \geq 0$$

Where t is the number of hours after the injection is given. The graph of the equation is shown:



a. The tranquiliser is effective when the concentration is at least 0.5 mg/L. Find, correct to two decimal places, the length of time in hours for which the tranquiliser is effective according to this model. (2 marks)

b.

i. Find the coordinates for the stationary point of $x(t)$. (2 marks)

ii. State the nature of the stationary point from **part b.i.** (1 mark)

iii. Hence, find the maximum concentration of tranquiliser in the bloodstream. Give your answer correct to three decimal places. (1 mark)

c. For what times, is the concentration of transquiliser in the bloodstream strictly decreasing? (1 mark)

d. According to this model, the derivative of x with respect to t gives the measure of the rate of absorption of the tranquiliser in the bloodstream.

How many hours after the injection is the rate of absorption into the bloodstream 0.3 $mg/L/h$?

Give your answer correct to two decimal places. (1 mark)

So that the tranquiliser concentration does not get too low, the exact same tranquiliser is re-applied when $t = 8$. From this point onwards the concentration of tranquiliser in the bloodstream is measured by a new function $x_1(t)$.

e. State the domain for $x_1(t)$. (1 mark)

f. $x_1(t)$ has the rule $x_1(t) = x(t) + x(t - a)$. State the value of a. (1 mark)

g. Find the time it takes for the concentration of tranquiliser to double from its value at $t = 8$. Give your answer in hours correct to two decimal places. (2 marks)

h. Determine the times, $t \geq 8$, when the concentration of tranquiliser in the bloodstream is strictly decreasing. (1 mark)

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Section F: Extension Exam 1 (13 Marks)

INSTRUCTION: 13 Marks. 16 Minutes Writing.

Question 14 (3 marks)

For the function $f(x) = 3x^3 \tan(2x)$, $f'(x) = \frac{ax^2}{\cos(2x)}(b\sin(2x) + cx\sec(2x))$. Find the values of a, b and c.

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Question 15 (7 marks)

Let $f : [0, \infty) \to \mathbb{R},\ f(x) = 5\sqrt{x} - x - 4$.

a. Find the coordinates of any stationary point of f and determine its nature. (3 marks)

Let A and B be the coordinates of the x-intercepts of the graph $y = f(x)$. Let C be any point on the graph of $y = f(x)$ that lies between the points A and B.

b. Determine the coordinates of A and B. (2 marks)

c. Hence, determine the maximum possible area of the triangle ABC. (2 marks)

Question 16 (3 marks)

Let $f: [0, 12\pi] \to \mathbb{R}, f(x) = 2\sin\left(\frac{x}{3}\right) - \frac{\pi}{2}$.

The rule for f' can be obtained from the rule of f under a transformation T given by:

$$T: \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to \left(x + \frac{3\pi}{2}, ay + b\right)$$

Find the values of a and b.

Section G: Extension Exam 2 (12 Marks)

INSTRUCTION: 12 Marks. 15 Minutes Writing.

Question 17 (1 mark)

Let f be a one-to-one differentiable function such that $f(3) = 7$, $f(7) = 8$, $f'(3) = 2$ and $f'(7) = 3$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. $g'(7)$ is equal to:

A. $\frac{1}{2}$

B. 2

C. $\frac{1}{6}$

D. $\frac{1}{3}$

Question 18 (1 mark)

Consider the function $f(x) = xg(x)$.

It is known that $g(0) = -3, g(2) = -2$ and $g(3) = 3$.

Also $g'(0) = -2,\ g'(2) = 1$ and $g'(3) = 5$ and that f has only one stationary point.

Which of the following options lists the x-coordinate and nature of the stationary point of f?

A. $x = 0$, local minimum.

B. $x = 2$, local maximum.

C. $x = 2$, local minimum.

D. $x = 3$, local minimum.

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Question 19 (1 mark)

Let f be a differentiable function. The derivative of $\log_e(f(x))$ with respect to x is:

A. $\frac{f\prime(x)}{f(x)}$

B. $\frac{f(x)}{(f(x))^2}$

C. $\frac{f\prime(x)}{(f(x))^2}$

D. $f'(x)\log_e(f(x))$

Question 20 (1 mark)

Consider the differentiable function f. It is known that $f'(1) = 2, f'(2) = 4$ and $f'(6) = 1$.

The gradient of $3f(2x+1)+4$ when $x = \frac{1}{2}$ is:

A. 6

B. 24

C. 3

D. 5

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Question 21 (8 marks)

A train is moving along a railway. The train driver sees a large rock ahead and immediately applies the brakes when the front of the train is at a point P that is 5.5 km away from the rock at the point R.

The train's initial speed is $w\ km/h$ and $x\ km$ after the train passes the point P the train speed is given by:

$$v = k\log_e\left(\frac{x+1}{6}\right)$$

Assume that $w > 0$.

a. Find the value of k in terms of w. (1 mark)

b. If $v = \frac{50\log_e(2)}{\log_e(6)}$ when $x = 2$, find the value of w. (2 marks)

c. Show that the location where the train stops is independent of its initial speed w. (2 marks)

d. Find a general formula for $\frac{d^n v}{dx^n}$, the n^{th} derivative of v, where $n \geq 1$. Leave your answer in terms of x, n and k. (3 marks)

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