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VCE Mathematical Methods ¾
Differentiation [0.9]
Workshop Solutions

Error Logbook:

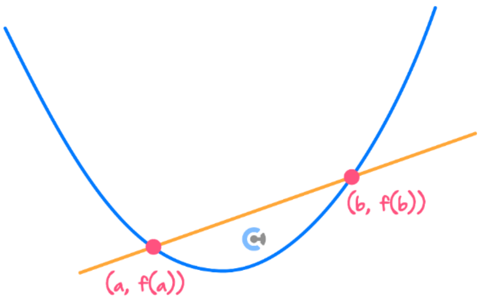
Mistake/Misconception #1		Mistake/Misconception #2	
Question #:	Page #:	Question #:	Page #:
Notes:		Notes:	

Mistake/Misconception #3		Mistake/Misconception #4	
Question #:	Page #:	Question #:	Page #:
Notes:		Notes:	

Section A: Recap

Average Rate of Change

Definition



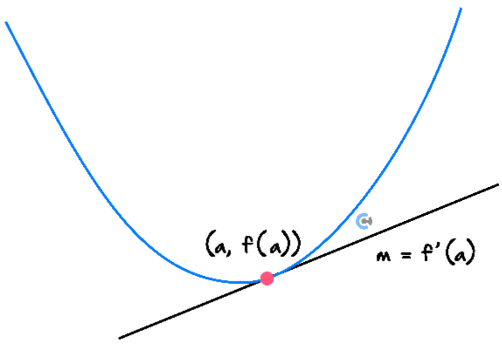
- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:

$$\textit{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$

- It is the gradient of the line joining the two points.

Instantaneous Rate of Change

Definition



- **Instantaneous rate of change** is a gradient of a graph at a single point/moment.

$$\textit{Instantaneous rate of change} = f'(x)$$

- **Differentiation** is the process of finding the derivative of a function.

- **First principles** derivative definition:

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Definition

Alternative Notation for Derivative

$$f'(x) = \frac{dy}{dx}$$

Definition

Derivatives of Functions

- The derivatives of many of the standard functions are in the summary table below:

$f(x)$	$f'(x)$
x^n	$n \times x^{n-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
e^x	e^x
$\log_e(x)$	$\frac{1}{x}$

Definition

The Product Rule

- The derivative of $h(x) = f(x) \times g(x)$ is given by:

$$h'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

- Or, in another form:

$$\frac{d}{dx}(u \cdot v) = u'v + v'u$$

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The Quotient Rule

- The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

- Or, written in another form:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Always differentiate the top function first.

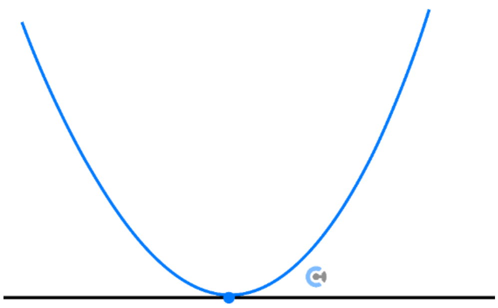
The Chain Rule

$$y = f\big(g(x)\big)$$

$$\frac{dy}{dx} = f'\big(g(x)\big)g'(x)$$

- The process for finding derivatives of **composite functions.**

Stationary Points



- The point where the gradient of the function is zero.

$$f'(x) = 0, \frac{dy}{dx} = 0$$

Calculator Commands: Finding Derivatives

- **Mathematica**

```
f'[x]
```

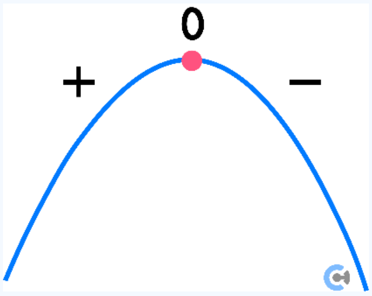
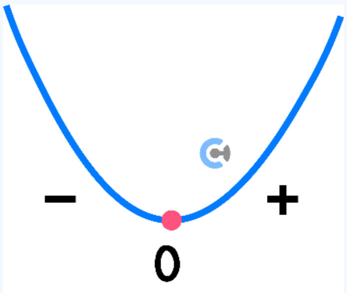
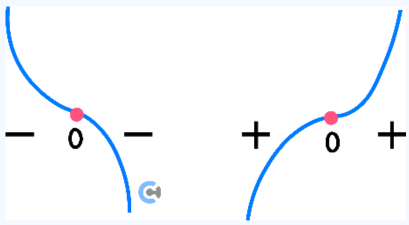
- **TI**
  - Shift Minus

$$\frac{d}{dx}(f(x))$$

- **Casio**
  - Math 2

$$\frac{d}{dx}(f(x))$$

Types of Stationary Points

Local Maximum	Local Minimum	Stationary Point of Inflection
		

- Sign test

- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighbouring points.

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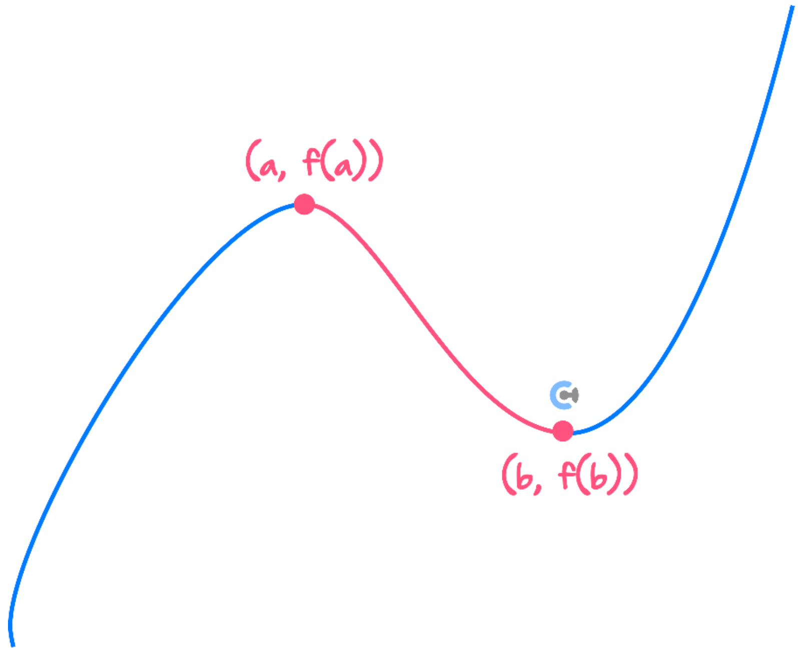
Strictly Increasing and Strictly Decreasing Functions



Strictly Increasing: $x \in (-\infty, a] \cup [b, \infty)$

Strictly Decreasing: $x \in [a, b]$

- Steps:

  1. Find the turning points.

  2. Consider the sign of the derivative between/outside the turning points.

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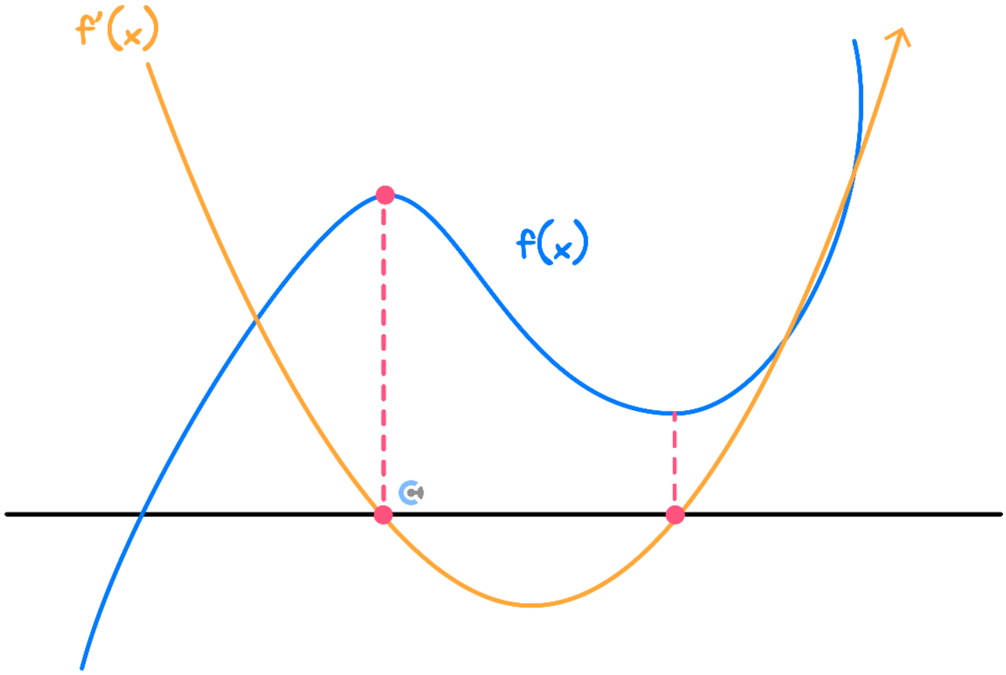
Graphs of the Derivative Function



$f(x)$	$f'(x)$
Stationary Point	x-intercepts
Increasing	Positive
Decreasing	Negative

$$y \text{ value of } f'(x) = \text{Gradient of } f(x)$$

- **Steps**

1. Plot x-intercept at the same x value as the stationary point of the original.

2. Consider the trend of the original function and sketch the derivative.

   - Original is increasing → Derivative is above the x-axis.

   - Original is decreasing → Derivative is below the x-axis.

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Section B: Warmup

Question 1

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 9x + 3$.

a. Find $f(1)$.

$$\boxed{f(1) = -2}$$

b. Find the average rate of change from $x = 0$ to $x = 2$.

$$\boxed{\frac{f(2) - f(0)}{2} = \frac{5 - 3}{2} = 1}$$

c.

i. Find $f'(x)$.

$$\boxed{f'(x) = 3x^2 + 6x - 9}$$

ii. Find $f'(-1)$.

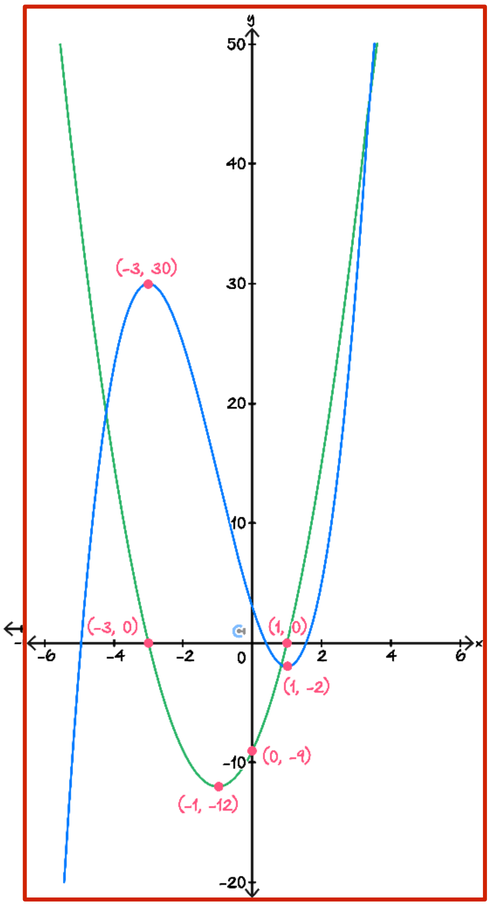
$$\boxed{f'(-1) = -12}$$

d. Determine the coordinates and nature of any stationary points.

$f'(x) = 0 \implies 3x^2 + 6x - 9 = 0 \implies x^2 + 2x - 3 = 0 \implies (x+3)(x-1) = 0.$ $f(-3) = 30$ and $f(1) = -2$. Stationary points $(-3, 30)$ and $(1, -2)$. Function is positive cubic so

$$(-3, 30) \text{ is a local maximum}$$
$$(1, -2) \text{ is a local minimum}$$

e. The graph of $y = f(x)$ is sketched on the axes below. Sketch the graph of $y = f'(x)$ on the same axes. Label all axial intercepts with coordinates.



f. Hence, state the values of x for which $f(x)$ is strictly decreasing.

$\boxed{-3 \leq x \leq 1}$

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Section C: Exam 1 Questions (17 Marks)

INSTRUCTION: 17 Marks. 25 Minutes Writing.

Question 2 (3 marks)

a. Let $y = \frac{\tan(2x)}{x^3}$. Find $\frac{dy}{dx}$. (1 mark)

```
In[1]:= f[x_] := Tan[2 x] / x^3

In[3]:= f'[x] // Together
```

Out[3]= $\frac{2 x \operatorname{Sec}[2 x]^2 - 3 \operatorname{Tan}[2 x]}{x^4}$

b. Let $f(x) = x^3 \tan(e^x)$. Evaluate $f'\left(\log_e\left(\frac{\pi}{6}\right)\right)$. (2 marks)

```
In[4]:= f[x_] := x^3 Tan[Exp[x]]

In[5]:= f'[Log[Pi / 6]]
```

Out[5]= $\sqrt{3}\ \operatorname{Log}\left[\frac{\pi}{6}\right]^2 + \frac{2}{9}\pi \operatorname{Log}\left[\frac{\pi}{6}\right]^3$

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Question 3 (3 marks)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = xe^{kx^2}$, where $k \in \mathbb{R}$.

a. Show that $f'(x) = (2kx^2 + 1)e^{kx^2}$. (1 mark)

$f'(x) = 1 \cdot e^{kx^2} + x \cdot e^{kx^2} \times 2kx$

$= e^{kx^2} + 2kx^2 e^{kx^2}$

$= (2kx^2 + 1)e^{kx^2}$

b. Find the value(s) of k for which the graph of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection. (2 marks)

$f(x) = f'(x)$

$x\cancel{e^{kx^2}} = (2kx^2+1)\cancel{e^{kx^2}}$.

$x = 2kx^2 + 1$

$0 = 2kx^2 - x + 1$

$\Delta = (-1)^2 - 4 \cdot 2k \times 1 = 0$

$1 - 8k = 0$

$\boxed{k = \frac{1}{8}.}$

$\boxed{k = 0.}$

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Question 4 (6 marks)

Consider the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = -x^2 + 5x - 4$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = e^x$.

a. State the rule of $g\big(f(x)\big)$. (1 mark)

$$\boxed{g(f(x)) = e^{-x^2+5x-4}}$$

b. Find the values of x for which $g\big(f(x)\big)$ is strictly decreasing. (2 marks)

$[g(f(x))]' = (5-2x)e^{-x^2+5x-4}$.

Note that $e^{-x^2+5x-4} > 0$ thus stationary point when $5 - 2x = 0 \implies x = \frac{5}{2}$.

Will be strictly decreasing for $x \geq \frac{5}{2}$.

c. Find the coordinates of the stationary point of the graph of $f\big(g(x)\big)$ and state its nature. (3 marks)

$f(g(x)) = -e^{2x} + 5e^x - 4$. Let $p(x) = f(g(x))$ then $P'(x) = -2e^{2x} + 5e^x$

$$e^x(5 - 2e^x) = 0 \implies e^x = \frac{5}{2} \implies x = \log_e\left(\frac{5}{2}\right).$$

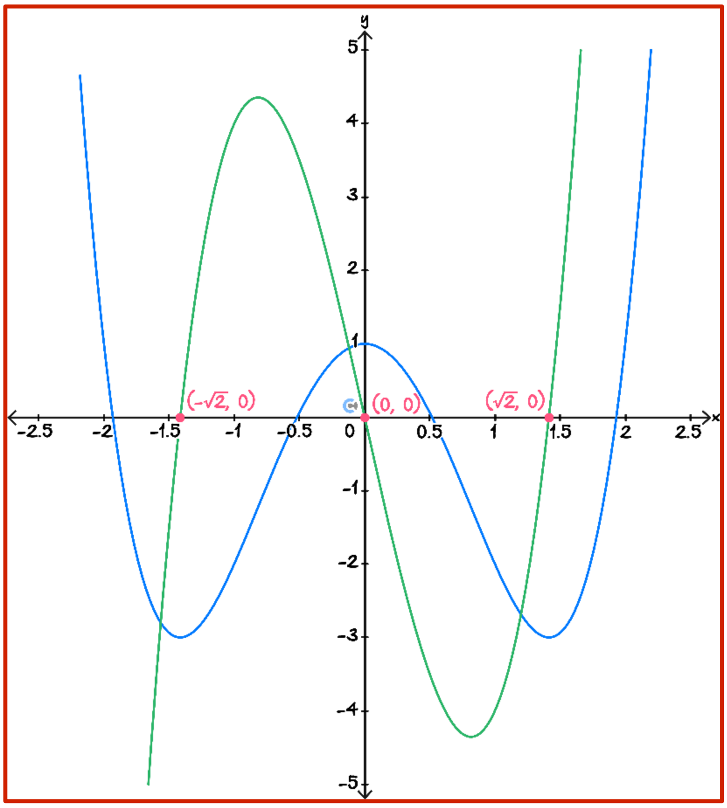
then $g\left(\log_e\left(\frac{5}{2}\right)\right) = \frac{5}{2}$ and $f\left(\frac{5}{2}\right) = -\frac{25}{4} + \frac{50}{4} - \frac{16}{4} = \frac{9}{4}$

If $a < \log_e\left(\frac{5}{2}\right)$ then $P'(a) > 0$ and if $b > \log_e\left(\frac{5}{2}\right)$ then $P'(b) < 0$.

One stationary point at $\left(\log_e\left(\frac{5}{2}\right), \frac{9}{4}\right)$ that is a local max.

Question 5 (5 marks)

The graph of $f(x) = x^4 - 4x^2 + 1$ is shown below.



a. Sketch the graph of $y = f'(x)$ on the axes above. Label all axes intercepts with coordinates. (3 marks)

$$f'(x) = 0 \implies 4x^3 - 8x = 0 \implies 4x(x^2 - 2) = 0 \implies x = 0, \pm\sqrt{2}$$

b. For both $f(x)$ and $f'(x)$ state whether they are an odd function, even function or neither. (1 mark)

f is even and f' is odd.

c. Let g be an even polynomial function of degree $n \geq 2$. Is it always true that g' is an odd function? (1 mark)

Yes. Because when we differentiate the power reduces by 1 so all the evens become odd.

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Section D: Tech Active Exam Skills

Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	done
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$\{x=1\}$
$f(1)$	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

Section E: Exam 2 Questions (20 Marks)

INSTRUCTION: 20 Marks. 25 Minutes Writing.

Question 6 (1 mark)

The average rate of change for the function with the rule $f(x) = y = -4e^{-\frac{2x}{5}}$ from $y = -3$ to $y = -1$ is closest to:

A. 1.02

B. -1.02

C. 1.37

D. 0.73

Question 7 (1 mark)

Let $f(x)$ and $g(x)$ be differentiable functions, with the following values given:

$$f(2) = 3,\ f'(4) = 5,\ g(2) = 4,\ g'(2) = 6.$$

Find the gradient of $f\left(g(x)\right)$ at $x = 2$.

A. 20

B. 15

C. 30

D. 24

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Question 8 (1 mark)

Given that $f(1) = 2, f'(1) = 3, g(1) = 4, g'(1) = 5$, find the gradient of $f(x)g(x)$ at $x = 1$.

A. 13

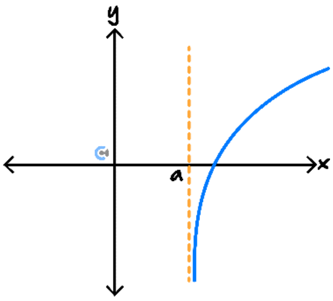
B. 22

C. 18

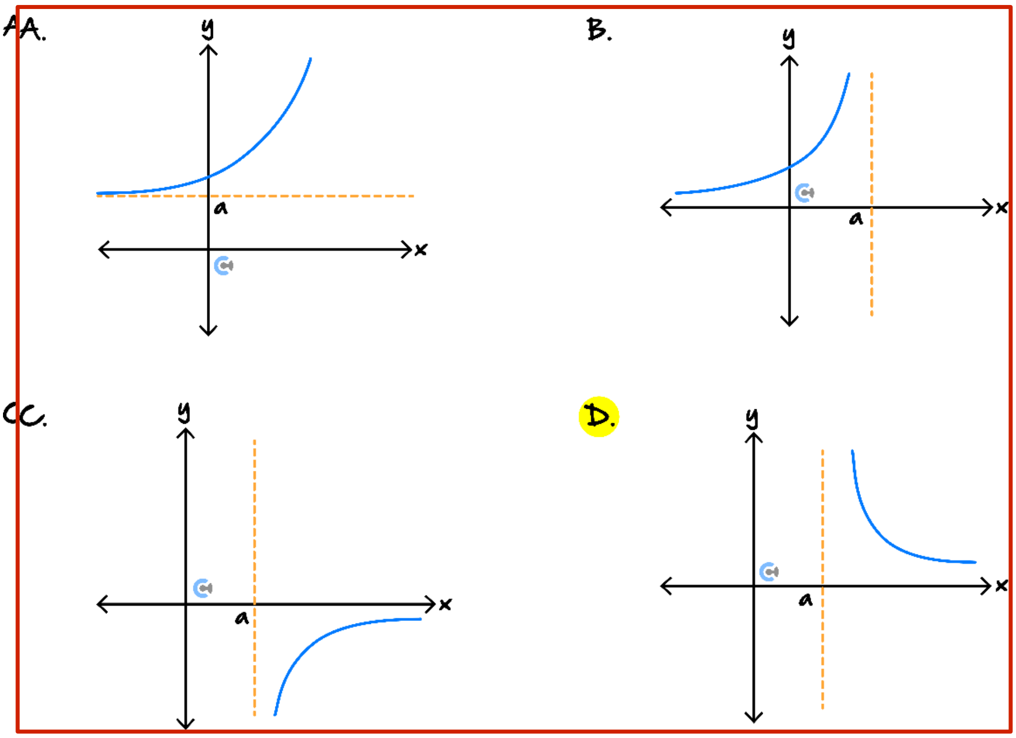
D. 20

Question 9 (1 mark)

The graph of the function f is shown below:



The graph corresponding to f' is:



Question 10 (1 mark)

Suppose $f(x)$ and $g(x)$ are differentiable, and the following values are given:

$$f(3) = 5,\ f'(3) = 4,\ g(3) = 2,\ g'(3) = 1.$$

Find the gradient of $\frac{f(x)}{g(x)}$ at $x = 3$.

A. $\frac{2}{5}$

B. $\frac{3}{4}$

C. $\frac{1}{2}$

D. $\frac{2}{3}$

Question 11 (1 mark)

Consider the graph of g with the rule $g(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has a y-intercept at $(0,10)$ and turning points when $x = -3$ and $x = 2$ and passes through $(3,64)$.

The rule of $g(x)$ is:

A. $g(x) = -4x^3 - 6x^2 + 72x + 10$

B. $g(x) = -3x^3 - 9x^2 + 27x + 10$

C. $g(x) = 3x^3 + 8x^2 + 10x + 10$

D. $g(x) = 2x^3 + 12x^2 + 6x + 10$

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Question 12 (1 mark)

The graph of $f(x) = ax^5 + bx^4 + x^3 - 3$, where a and b are real constants, will have three stationary points when:

A. $a > -\frac{4b^2}{15}$

B. $a \leq -\frac{4b^2}{15}$

C. $a < \frac{4b^2}{15}$

D. $a > \frac{4b^2}{15}$

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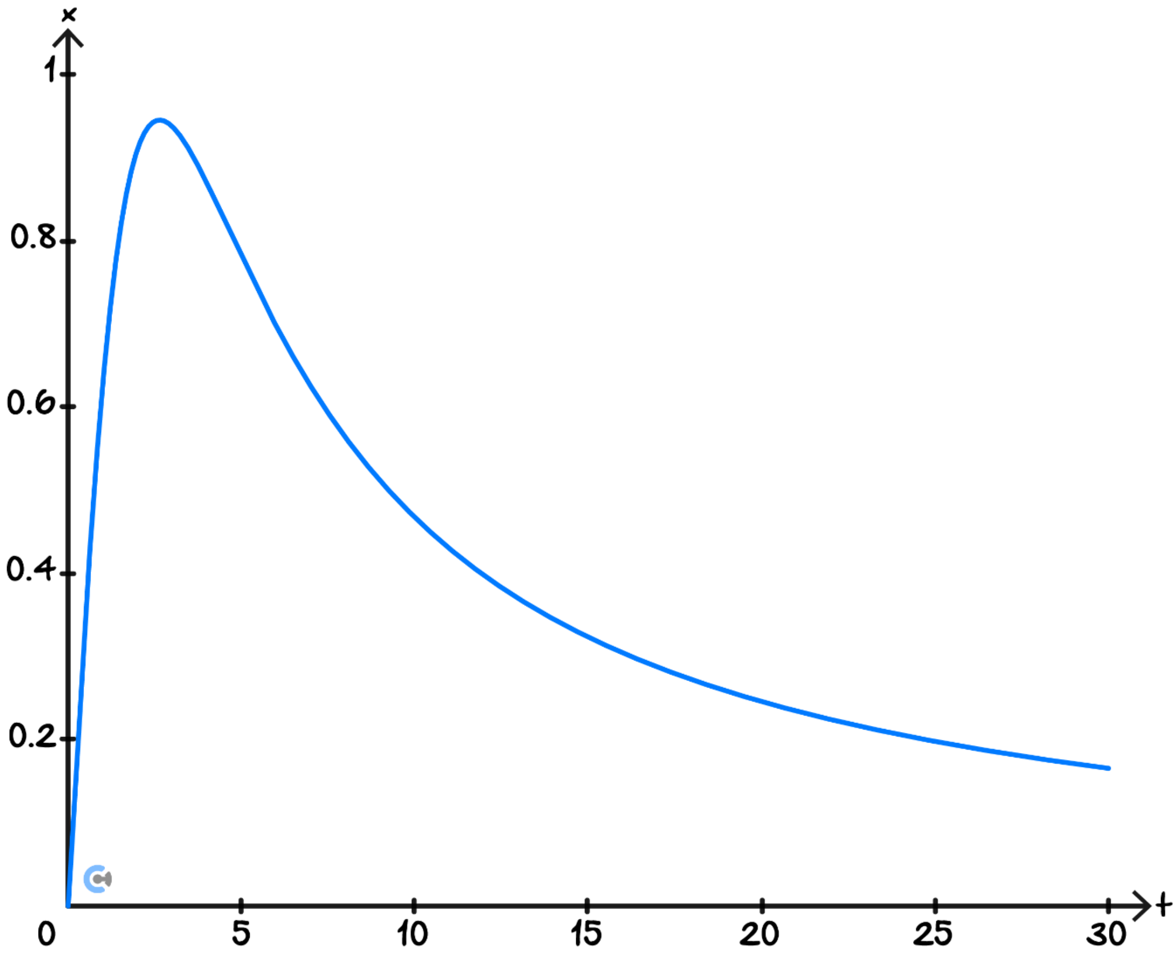
Question 13 (13 marks)

A tranquiliser is injected into a muscle from which it enters the bloodstream. The concentration, $x, mg/L$, of the tranquiliser in the bloodstream, may be modelled by the function:

$$x(t) = \frac{5t}{7+t^2}, t \geq 0$$

Where t is the number of hours after the injection is given. The graph of the equation is shown:



a. The tranquiliser is effective when the concentration is at least 0.5 mg/L. Find, correct to two decimal places, the length of time in hours for which the tranquiliser is effective according to this model. (2 marks)

Let $x(t) = \frac{5t}{7+t^2}$.
Solve $x(t) = 0.5 \implies t = 0.7573, 9.24264$
Therefore, total time 8.49 hours.

b.

i. Find the coordinates for the stationary point of $x(t)$. (2 marks)

Solve $f'(t) = 0 \implies t = \sqrt{7}$.
Stationary point $\left(\sqrt{7}, \frac{5}{2\sqrt{7}}\right)$

ii. State the nature of the stationary point from **part b.i.** (1 mark)

Local maximum.

iii. Hence, find the maximum concentration of tranquiliser in the bloodstream. Give your answer correct to three decimal places. (1 mark)

$\frac{5}{2\sqrt{7}} \approx 0.945$

c. For what times, is the concentration of transquiliser in the bloodstream strictly decreasing? (1 mark)

Decreasing for $t \geq \sqrt{7}$.

d. According to this model, the derivative of x with respect to t gives the measure of the rate of absorption of the tranquiliser in the bloodstream.

How many hours after the injection is the rate of absorption into the bloodstream 0.3 $mg/L/h$?

Give your answer correct to two decimal places. (1 mark)

$\boxed{\text{Solve } x'(t) = 0.3 \implies t = 1.44}$

So that the tranquiliser concentration does not get too low, the exact same tranquiliser is re-applied when $t = 8$. From this point onwards the concentration of tranquiliser in the bloodstream is measured by a new function $x_1(t)$.

e. State the domain for $x_1(t)$. (1 mark)

$\boxed{t \geq 8}$

f. $x_1(t)$ has the rule $x_1(t) = x(t) + x(t-a)$. State the value of a. (1 mark)

$\boxed{a = 8}$

g. Find the time it takes for the concentration of tranquiliser to double from its value at $t = 8$. Give your answer in hours correct to two decimal places. (2 marks)

$\boxed{\text{Solve } x_1(t) = 2x(8) \implies t = 8.977. \text{ Therefore } 0.98 \text{ hours.}}$

h. Determine the times, $t \geq 8$, when the concentration of tranquiliser in the bloodstream is strictly decreasing. (1 mark)

Solve $x_1'(t) = 0 \implies t = 10.40$. Looking at a graph $t \geq 10.40$.

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Section F: Extension Exam 1 (13 Marks)

INSTRUCTION: 13 Marks. 16 Minutes Writing.

Question 14 (3 marks)

For the function $f(x) = 3x^3 \tan(2x)$, $f'(x) = \frac{ax^2}{\cos(2x)}(b \sin(2x) + cx \sec(2x))$. Find the values of a, b and c.

$$\begin{aligned} f'(x) &= 9x^2 \tan(2x) + 6x^3 \sec^2(2x) \\ &= \frac{9x^2 \sin(2x)}{\cos(2x)} + \frac{6x^3}{\cos^2(2x)} \\ &= \frac{3x^2}{\cos(2x)}(3 \sin(2x) + 2x \sec(2x)) \end{aligned}$$

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Question 15 (7 marks)

Let $f : [0, \infty) \to \mathbb{R},\ f(x) = 5\sqrt{x} - x - 4$.

a. Find the coordinates of any stationary point of f and determine its nature. (3 marks)

$f'(x) = \frac{5}{2\sqrt{x}} - 1 = 0 \implies 5 = 2\sqrt{x} \implies x = \frac{25}{4}$.

Then $f(25/4) = 5 \times 5/2 - 25/4 - 4 = 25/4 - 16/4 = \frac{9}{4}$.

$f'(x) > 0$ if $x < 25/4$ and $f'(x) < 0$ if $x > 25/4$

Stationary point at $\left(\frac{25}{4}, \frac{9}{4}\right)$ and it is a local max.

Let A and B be the coordinates of the x-intercepts of the graph $y = f(x)$. Let C be any point on the graph of $y = f(x)$ that lies between the points A and B.

b. Determine the coordinates of A and B. (2 marks)

We must determine the x-intercepts. Solve $5\sqrt{x} - x - 4 = 0$. Let $a = \sqrt{x}$

$$-a^2 + 5a - 4 = 0$$
$$(4 - a)(a - 1) = 0$$

$a = 4$ or $a = 1$. Therefore coordinates of A and B are $(1, 0)$ and $(16, 0)$.

c. Hence, determine the maximum possible area of the triangle ABC. (2 marks)

Triangle area is $\frac{1}{2}$ base $\times$ height. The base is $16 - 1 = 15$. The height is the y-value of c.

The triangle will be maximum when C is the local max so $C = \left(\frac{25}{4}, \frac{9}{4}\right)$.

So triangle area $= \frac{15}{2} \times \frac{9}{4} = \frac{135}{8}$

Question 16 (3 marks)

Let $f: [0, 12\pi] \to \mathbb{R}, f(x) = 2\sin\left(\frac{x}{3}\right) - \frac{\pi}{2}$.

The rule for f' can be obtained from the rule of f under a transformation T given by:

$$T: \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to \left(x + \frac{3\pi}{2}, ay + b\right)$$

Find the values of a and b.

$a = -1/3$, $b = -\pi/6$

Section G: Extension Exam 2 (12 Marks)

INSTRUCTION: 12 Marks. 15 Minutes Writing.

Question 17 (1 mark)

Let f be a one-to-one differentiable function such that $f(3) = 7$, $f(7) = 8$, $f'(3) = 2$ and $f'(7) = 3$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. $g'(7)$ is equal to:

A. $\frac{1}{2}$

B. 2

C. $\frac{1}{6}$

D. $\frac{1}{3}$

Question 18 (1 mark)

Consider the function $f(x) = xg(x)$.

It is known that $g(0) = -3, g(2) = -2$ and $g(3) = 3$.

Also $g'(0) = -2,\ g'(2) = 1$ and $g'(3) = 5$ and that f has only one stationary point.

Which of the following options lists the x-coordinate and nature of the stationary point of f?

A. $x = 0$, local minimum.

B. $x = 2$, local maximum.

C. $x = 2$, local minimum.

D. $x = 3$, local minimum.

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Question 19 (1 mark)

Let f be a differentiable function. The derivative of $\log_e\left(f(x)\right)$ with respect to x is:

A. $\frac{f\prime(x)}{f(x)}$

B. $\frac{f(x)}{(f(x))^2}$

C. $\frac{f\prime(x)}{(f(x))^2}$

D. $f'(x)\log_e(f(x))$

Question 20 (1 mark)

Consider the differentiable function f. It is known that $f'(1) = 2, f'(2) = 4$ and $f'(6) = 1$.

The gradient of $3f(2x+1)+4$ when $x = \frac{1}{2}$ is:

A. 6

B. 24

C. 3

D. 5

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Question 21 (8 marks)

A train is moving along a railway. The train driver sees a large rock ahead and immediately applies the brakes when the front of the train is at a point P that is 5.5 km away from the rock at the point R.

The train's initial speed is w km/h and x km after the train passes the point P the train speed is given by:

$$v = k\log_e\left(\frac{x+1}{6}\right)$$

Assume that $w > 0$.

a. Find the value of k in terms of w. (1 mark)

$v = w$ when $x = 0$. Thus $w = k\log_e\left(\frac{1}{6}\right) \implies k = \frac{w}{\log_e(1/6)} = -\frac{w}{\log_e(6)}$

b. If $v = \frac{50\log_e(2)}{\log_e(6)}$ when $x = 2$, find the value of w. (2 marks)

Solve $v(2) = \frac{50\log_e(2)}{\log_e(6)} \implies k = -\frac{50}{\log_e(6)} \implies w = 50$

c. Show that the location where the train stops is independent of its initial speed w. (2 marks)

Train stops when $v = 0 \implies -\frac{w}{\log_e(6)}\log_e\left(\frac{x+1}{6}\right) = 0$. This expression is only zero when

$$\frac{\boxed{x+1}}{6} = 1 \implies x = 5$$

because $-\frac{w}{\log_e(6)} < 0$.
So train always stops when $x = 5$ independent of w.

d. Find a general formula for $\frac{d^n v}{dx^n}$, the n^{th} derivative of v, where $n \geq 1$. Leave your answer in terms of x, n and k. (3 marks)

$n = 1$: $\frac{dv}{dx} = \frac{k}{x+1}$

$n = 2$: $\frac{d^2v}{dx^2} = -\frac{k}{(x+1)^2}$

$n = 3$: $\frac{d^3v}{dx^3} = \frac{2k}{(x+1)^3}$

$n = 4$: $\frac{d^4v}{dx^4} = -\frac{1 \times 2 \times 3}{(x+1)^4}$

From this pattern deduce that

$$\frac{d^n v}{dx^n} = \frac{(-1)^{n-1}k(n-1)!}{(x+1)^n}$$

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