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Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Polynomials [0.7]
Workshop Solutions

Error Logbook:

Mistake/Misconception #1		Mistake/Misconception #2	
Question #:	Page #:	Question #:	Page #:
Notes:		Notes:	

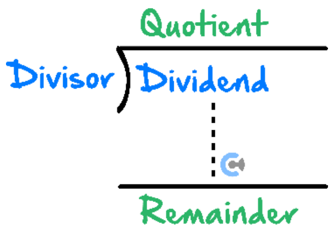
Mistake/Misconception #3		Mistake/Misconception #4	
Question #:	Page #:	Question #:	Page #:
Notes:		Notes:	

Section A: Recap

Roots of Polynomial Functions

$$\textbf{Roots} = x\textbf{-intercepts}$$

Polynomial Long Division



$$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{Divisor}}$$

TIPS:

- Always focus on the highest degree term first.
- Always remember to fill in any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Remainder Theorem

- **Definition**:
  - Find the remainder of the long division without needing long division.

$$\textbf{When } P(x) \textbf{ is divided by } (x - a)\textbf{, the remainder is } P(a)\textbf{.}$$

- **Steps**:
  1. Find x-values which make the divisor equal to 0.
  2. Substitute it into the dividend function.

Definition

Factor Theorem

- For every x-intercept, there is a factor.

$$\boldsymbol{If\ P(\alpha) = 0, then\ (x - \alpha)\ is\ a\ factor\ of\ P(x).}$$

Definition

Factorising Cubic Polynomials

- Steps:

1. Find a single root by trial and error.
   - (Factor theorem: Substitute into the function and see if we get zero.)
2. Use long division to find the quadratic factor.
3. Factorise the remaining factor.

Definition

Rational Root Theorem

- Rational root theorem **narrows down** the possible roots.

$$\boldsymbol{Potential\ root = \pm \frac{Factors\ of\ constant\ term\ a_0}{Factors\ of\ leading\ coefficient\ a_n}}$$

- If the roots are rational numbers, the roots can only be $\pm \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$.

Definition

Sum and Difference of Cubes

$$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

$$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

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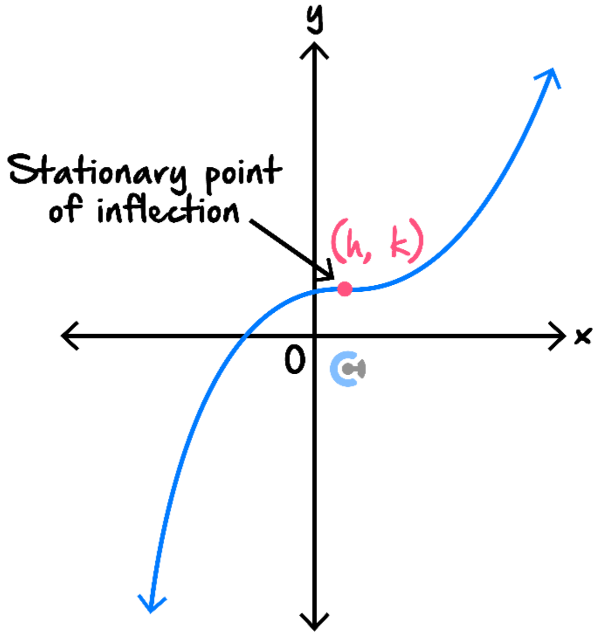
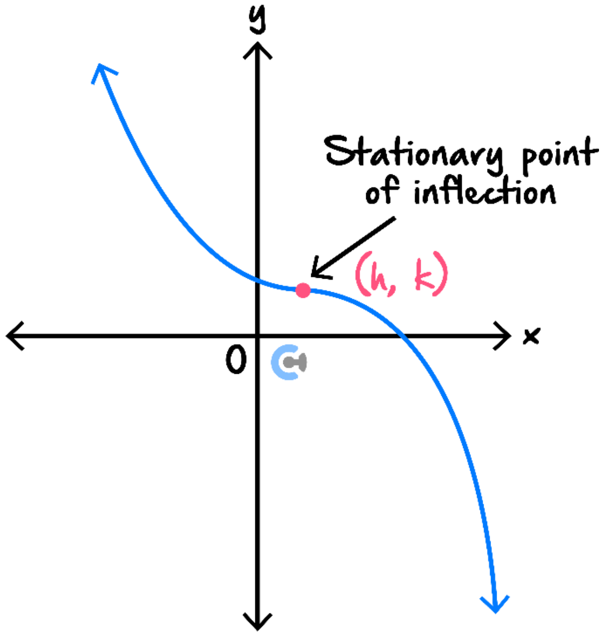
Definition

Graphs of $a(x-h)^n+k$, where n is Odd and Positive

- All graphs look like a "cubic".



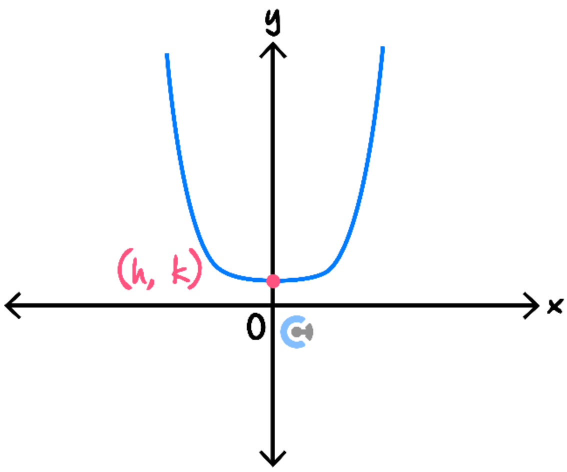


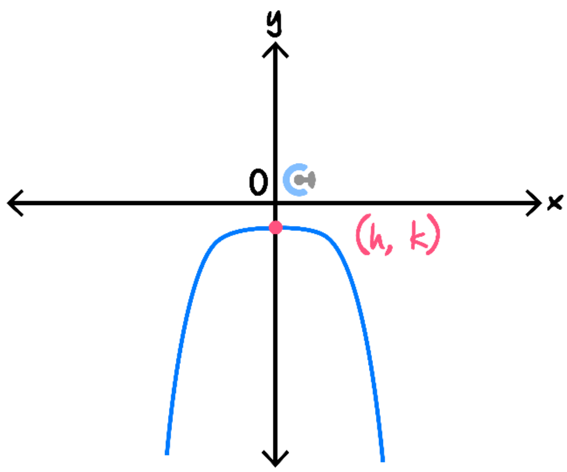
- The point (h,k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

Definition

Graphs of $a(x-h)^n+k$, where n is Even and Positive

- All graphs look like a "quadratic".





- The point (h,k) gives us the turning point.

Definition

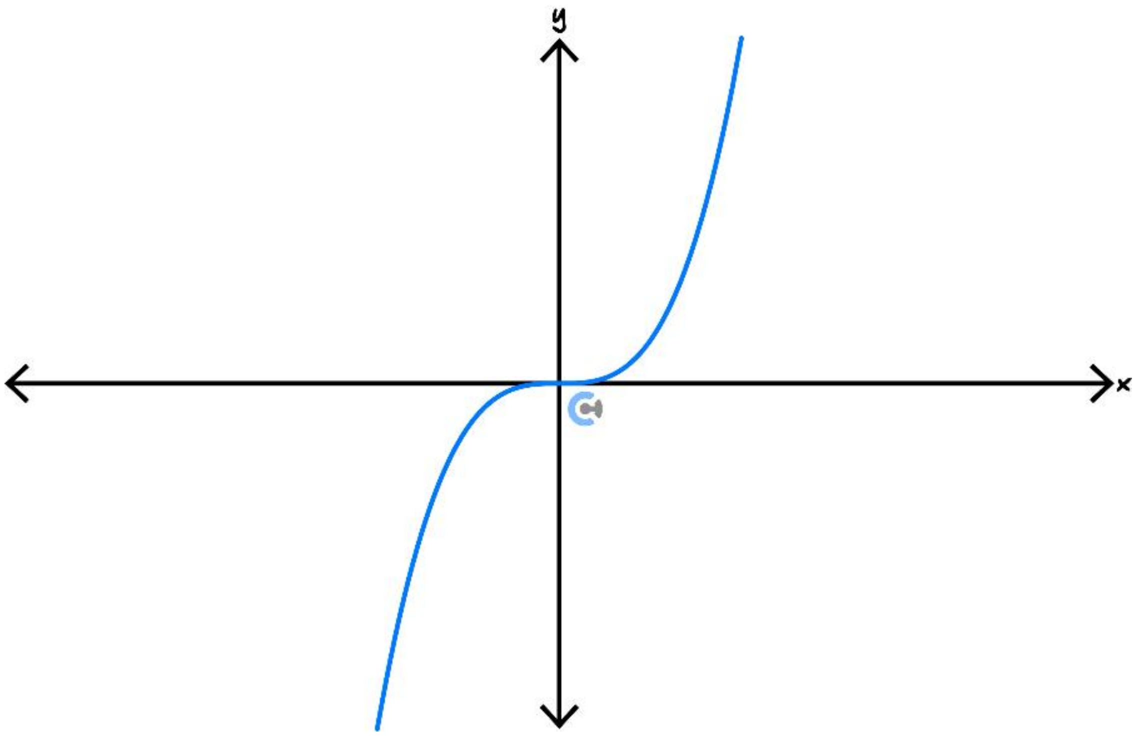
Graphs of Factorised Polynomials

- **Steps:**

  1. Plot x-intercepts.
  2. Determine whether the polynomial is positive or negative.
  3. Use the repeated factors to deduce the shape:
     - Non-repeated: Only x-intercept.
     - Even repeated: x-intercept and a turning point.
     - Odd repeated: x-intercept and a stationary point of inflection.

Definition

Odd Functions



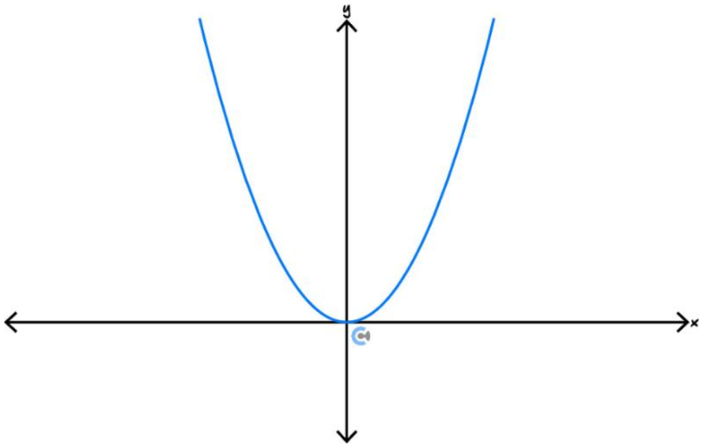
- E.g., $x^3, x^5, x^7 - x^3$. They are all odd powers.

$$f(-x) = -f(x)$$

- Property: Reflecting around the y-axis is the same as reflecting around the x-axis.

Definition

Even Functions



- E.g., $x^2, x^4, -x^{10}, x^4 - 4$. They are all even powers.

$$f(-x) = f(x)$$

- Property: It is symmetrical around the y-axis.

Definition

Power Functions

$$y = x^{\frac{n}{m}}$$

- m: Dictates the number of **tails**.
  - **Odd m = Two tails.**
  - **Even m = One tail.**
- n: Dictates the **range**.
  - Odd n: The range could be all real.
  - Even n: The range must be non-negative.
- $\frac{n}{m}$ (*Power*):
  - Power > 1: Looks like a polynomial function.
  - Power < 1: Looks like a root function.

Section B: Warm Up

Question 1

a. Find the remainder of the division $\frac{f(x)}{g(x)}$ where $f(x) = x^3 + 3x^2 + 2$ and $g(x) = x - 1$.

$$f(1) = 6$$

b. Use polynomial long division to write $f(x) = \frac{x^3+2x^2+3x+2}{x+2}$ in the form $f(x) = Q(x) + \frac{a}{x+2}$ for quadratic function Q and integer a.

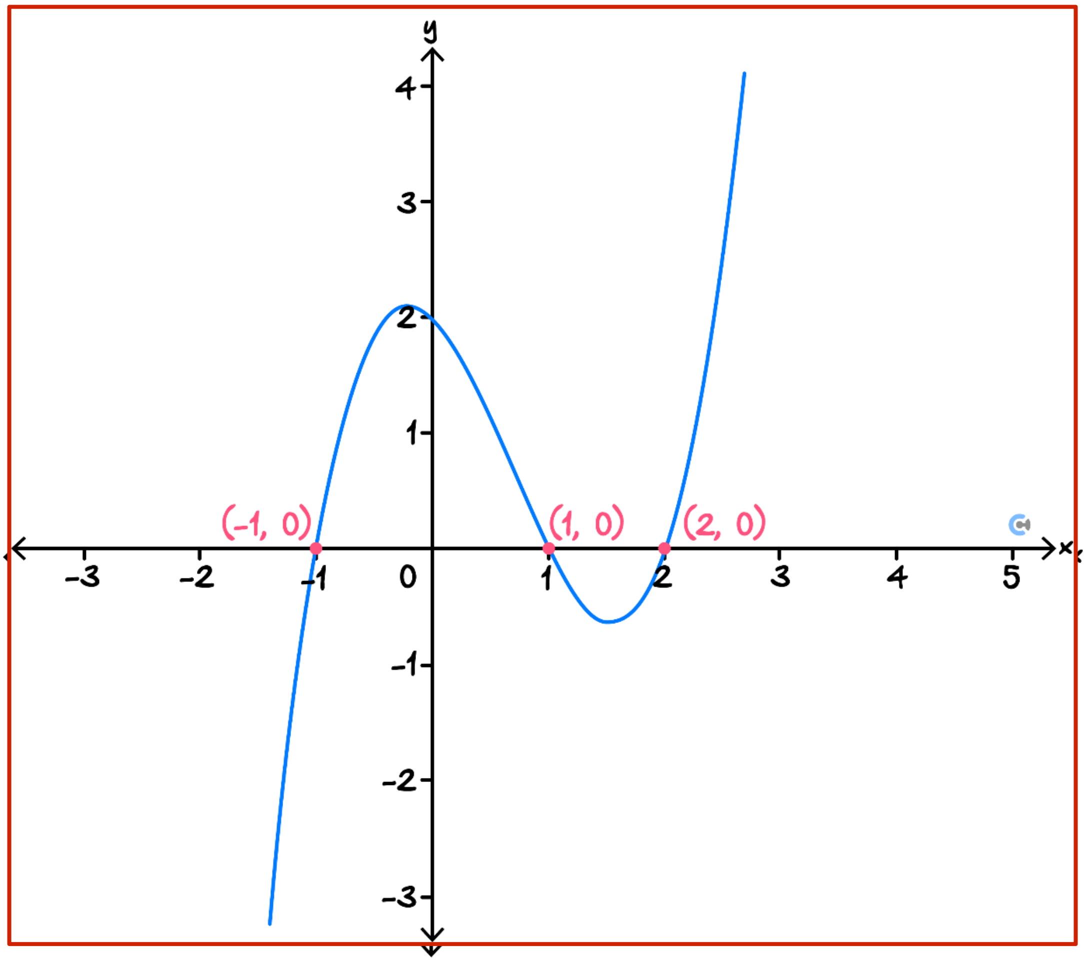
$$f(x) = x^2 + 3 - \frac{4}{x+2}$$

c.

i. Find all the roots of $f(x) = x^3 - 2x^2 - x + 2$.

$f(x) = (x-1)(x+1)(x-2)$ so roots are $x = \pm 1, 2$

ii. Sketch the graph of $y = f(x)$ on the axes below. Turning points occur at approximately $(-0.2, 2.1)$ and $(1.5, -0.6)$.



d. Factorise the expression $x^3 + 27$.

$$x^3 + 27 = (x+3)(x^2 - 3x + 9)$$

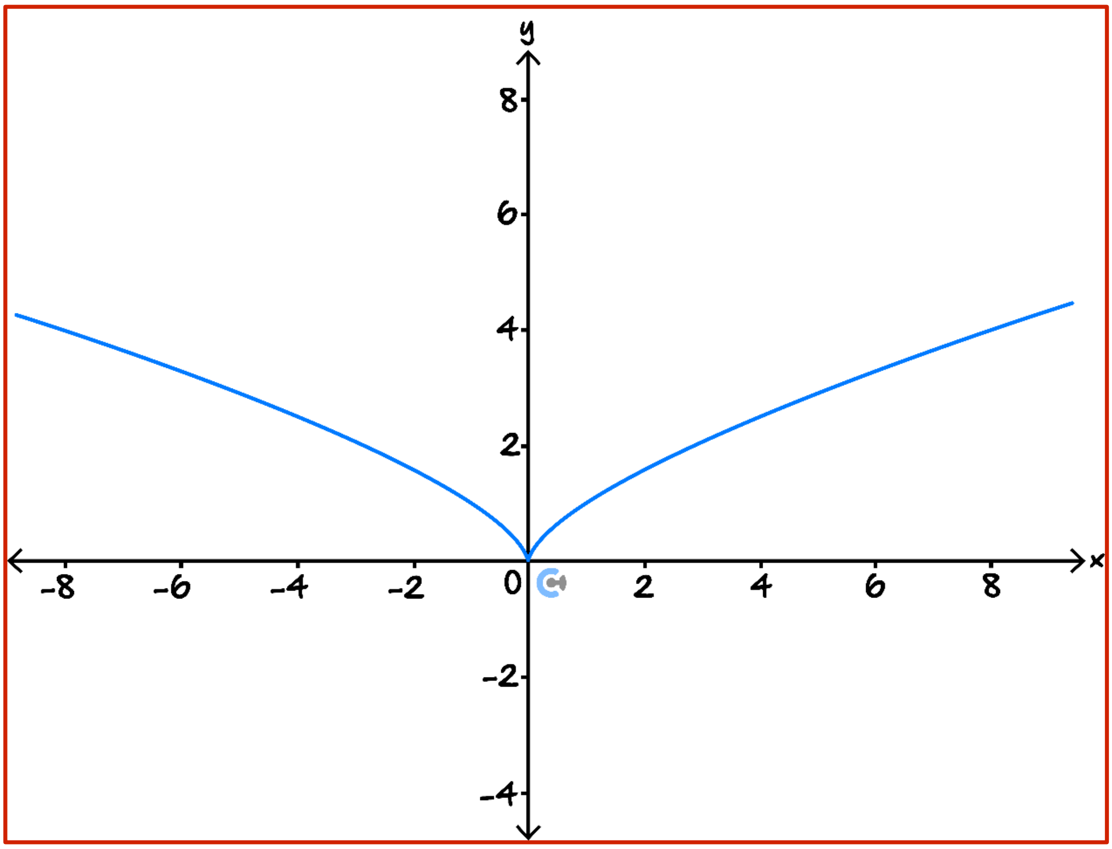
e. Expand the expression $(x-2)^3$.

$$x^3 - 6x^2 + 12x - 8$$

f. Show that the function $f(x) = x^3 - 3x$ is odd.

$$\begin{aligned} f(-x) &= (-x)^3 - 3(-x) \\ &= -x^3 + 3x \\ &= -(x^3 - 3x) \\ &= -f(x) \end{aligned}$$

g. Sketch the graph of $y = x^{\frac{2}{3}}$ on the axes below.



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Section C: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 25 Minutes Writing.

Question 2 (3 marks)

Let $f(x) = x^3 - ax^2 - 5x + b$, where a and b are constants. When $f(x)$ is divided by $x - 2$, the remainder is -4 and when $f(x)$ is divided by $x + 1$, the remainder is 8. Find the value of a and b.

$a = 2$ and $b = 6$.

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Question 3 (5 marks)

Consider the function given by $f(x) = 2x^3 - 3x^2 - 11x + 6$.

a. Find all x-intercepts of $f(x)$. (3 marks)

```
Solve[(2 x - 1) * (x - 3) * (x + 2) == 0, x]
[풀이 함수
{{x → -2}, {x → 1/2}, {x → 3}}
```

b. Hence, find all x-intercepts for $f\big(g(x)\big)$ where $g(x) = x^2 + 1$. (2 marks)

```
Solve[(2 x - 1) * (x - 3) * (x + 2) == 0, x]
[풀이 함수
{{x → -2}, {x → 1/2}, {x → 3}}
Solve[x^2 + 1 == 3, x]
[풀이 함수
{{x → -√2}, {x → √2}}
```

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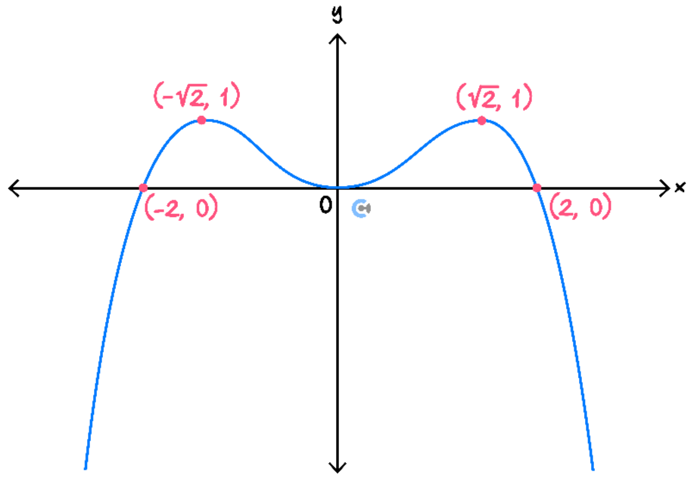
Question 4 (3 marks)

The function $f: R \to R, f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown below.

The graph of f touches the x-axis at the origin.



a. Find the rule of f. (2 marks)

$$f(x) = -\frac{1}{4}x^2(x-2)(x+2)$$

b. Find the values of c for which $f(x) + c = 0$, where $c \in \mathbb{R}$, has an even number of real solutions. (1 mark)

$c \in \mathbb{R} \setminus \{0\}$

NOTE: 0 is an even number.
$c \in R - \{0\}$

Question 5 (3 marks)

Consider the function given by $f(x) = x^4 + x^2 + 2$. (2 marks)

a. Show that $f(x)$ is an even function.

$$f(-x) = (-x)^4 + (-x)^2 + 2$$
$$= x^4 + x^2 + 2$$
$$= f(x)$$

b. The gradient of f when $x = 3$ is 114. State the gradient of f when $x = -3$. (1 mark)

-114

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Question 6 (6 marks)

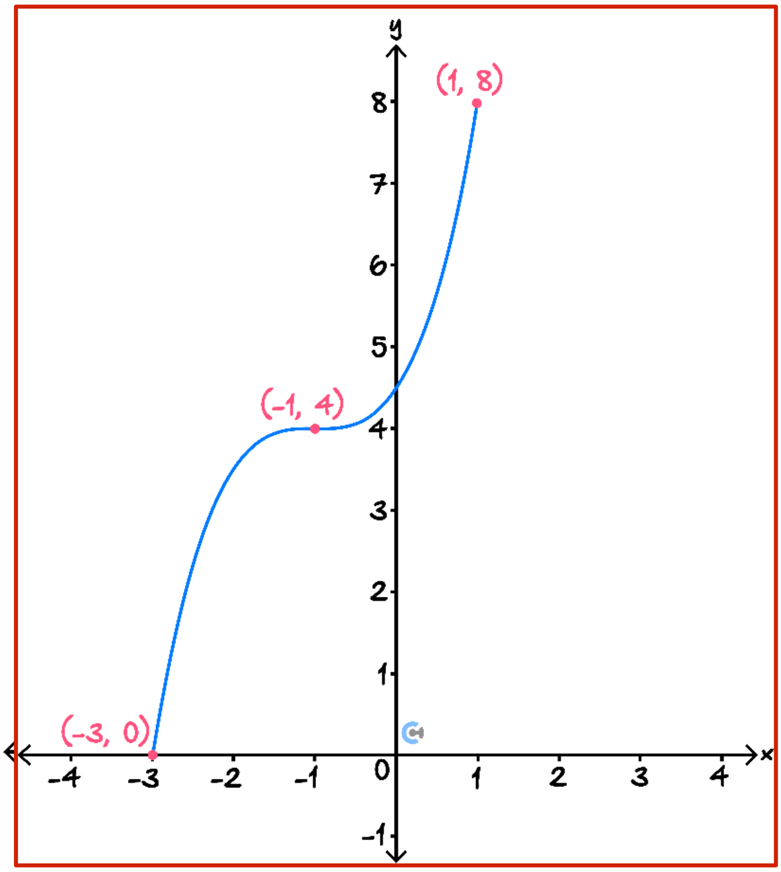
Consider the function $h: [-3,1] \to \mathbb{R}, h(x) = \frac{1}{2}x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{9}{2}$.

a. Given that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, write $h(x)$ in the form of $a(x+b)^3 + c$. (3 marks)

$a = \frac{1}{2}, b = 1$ and $c = 4$.

$$h(x) = \frac{1}{2}(x+1)^3 + 4$$

b. Sketch $y = h(x)$ on the axes below. Label any endpoints, axes, intercepts, and stationary points. (2 marks)



c. How many solution(s) will $h(x) = k$ always have for $k \in [0,8]$? (1 mark)

1 solution.

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Section D: Tech Active Exam Skills

Calculator Commands: Factor and Expand

- All the technologies have a "factor" and "expand" function.
- The general syntax is $\boldsymbol{factor}(\boldsymbol{expr})$, or $\boldsymbol{expand}(\boldsymbol{expr})$.
- **Example:** Factorise the expression $2x^3 - 23x^2 + 33x + 108$ and expand the expression $(x + 1)^5(x - 2)^3$.
- **TI:**

$\text{factor}(2\cdot x^3-23\cdot x^2+33\cdot x+108)$	$(x-9)\cdot(x-4)\cdot(2\cdot x+3)$
$\text{expand}((x+1)^5\cdot(x-2)^3)$	$x^8-x^7-8\cdot x^6+2\cdot x^5+25\cdot x^4+11\cdot x^3-26\cdot x^2-28\cdot x-8$

- **Casio:**

$\text{factor}(2x^3-23x^2+33x + 108)$

$(x-4)\cdot(x-9)\cdot(2\cdot x+3)$

$\text{expand}((x+1)^5(x-2)^3)$

$x^8-x^7-8\cdot x^6+2\cdot x^5+25\cdot x^4+11\cdot x^3-26\cdot x^2-28\cdot x-8$

- **Mathematica:**

```
In[7]:= Factor[108 + 33 x - 23 x^2 + 2 x^3]
Out[7]= (-9 + x) (-4 + x) (3 + 2 x)

In[8]:= Expand[(x + 1)^5 (x - 2)^3]
Out[8]= -8 - 28 x - 26 x^2 + 11 x^3 + 25 x^4 + 2 x^5 - 8 x^6 - x^7 + x^8
```

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Calculator Commands: Turning Point

- ALWAYS sketch the graph first to get an idea of the nature of the turning point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	done
$\text{solve}\left(\frac{d}{dx}(f(x))=0, x\right)$	$\{x=1\}$
$f(1)$	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

- **TI UDF:** We can use the analyse function.

- **Analyse a Function**

$$analysed\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

▸ Start Point: $\begin{bmatrix}-5 & \frac{262}{77}\end{bmatrix}$

▸ End Point: $\begin{bmatrix}5 & \frac{-316}{529}\end{bmatrix}$

▸ Maximal Domain:
$x\neq-1.68469$ and
$x\neq-0.629579$ and
$x\neq0.314273$ and
$-5\leq x\leq5$

▸ Asymptotes: (4)
$x=-1.68469$ (Vertical)
$x=-0.629579$ (Vertical)
$x=0.314273$ (Vertical)
$y=\frac{4}{3}-\frac{x}{3}$ (Oblique)

▸ x –Intercepts: (4)
$[-1.3772 \quad 0]$, $[-0.273891 \quad 0]$,
$[1 \quad 0]$, $[2.65109 \quad 0]$

▸ Vertical Intercept: $[0 \quad 1]$

▸ Derivative:

$$\frac{-\left(3\cdot x^6+12\cdot x^5-26\cdot x^3-24\cdot x^2-6\cdot x-4\right)}{\left(3\cdot x^3+6\cdot x^2+x-1\right)^2}$$

▸ Inflection Points: (2)
$[-1.11377 \quad 1.48672]$ (Increasing)
$[-0.11198 \quad 0.604642]$ (Increasing)

▸ Stationary Points: (2)
$[-3.45719 \quad 3.17894]$ (Local min.)
$[1.6173 \quad 0.124612]$ (Local max.)

Done

- **Overview:**

- This program will find for a given function:
  - Coordinates of endpoints.
  - The maximal domain.
  - The equations of straight-line asymptotes.
  - The rule of the derivative.
  - Inflection points and their nature.
  - Stationary points and their nature.

- **There are two analysis programs:**
  - Analyse which analyses a function over the domain R or the maximal domain.
  - Analysed which analyses over a domain with specified start and end points.
  - Both are found in the methods_func library. You can switch between the two on the calculator page by adding/removing the 'd' to reference the appropriate program.

- **Input:**

$$analyse(<function>,<variable>)$$

$$analysed(<function>,<variable>,<lower\ bound>,<upper\ bound>)$$

- **Other notes:**
  - It is recommended to use the analysed program when working with trigonometric functions.
  - Be careful when using functions with parameters since some parts of the programs may not be able to give a solution. :/
  - If at least one of the bounds is "?", the asymptote finder will be disabled and the program will analyse over the maximal domain.

Calculator Commands: Using Sliders/Manipulate on CAS

- **Mathematica**

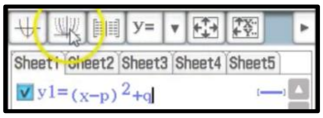
```
Manipulate[Plot[function, {x, xmin, xmax}],

   {unknown, lowerbound, upperbound}]
```

  - **NOTE:** The function **must** be typed out instead of using its saved name.

- **TI-Nspire**

- **Casio Classpad**



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Calculator Commands: Finding the equation of a polynomial that passes through points.

- Given n points, we can find a degree $n-1$ polynomial that passes through all these points.
- **Example:** Find the equation of the quadratic function that passes through the points $(0,6)$, $(2,2)$, and $(3,3)$.
- **TI:**

Define $f(x)=a\cdot x^2+b\cdot x+c$	*Done*
solve$(f(0)=6$ and $f(2)=2$ and $f(3)=3,a,b,c)$	$a=1$ and $b=^-4$ and $c=6$
$f(x)\|a=1$ and $b=^-4$ and $c=6$	$x^2-4\cdot x+6$

- **Casio:**

```
define f(x) = a*x^2 + b*x + c
                                        done
{f(0)=6
{f(2)=2
{f(3)=3 |a, b, c
                                        {a=1, b=-4, c=6}
f(x)|{a=1, b=-4, c=6}
                                        x^2-4·x+6
```

- **Mathematica:**

```
In[9]:= f[x_] := a x^2 + b x + c
In[10]:= Solve[f[0] == 6 && f[2] == 2 && f[3] == 3]
Out[10]= {{a → 1, b → -4, c → 6}}
In[11]:= f[x] /. {a → 1, b → -4, c → 6}
Out[11]= 6 - 4 x + x^2
```

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Section E: Exam 2 (27 Marks)

INSTRUCTION: 27 Marks. 34 Minutes Writing.

Question 7 (1 mark)

Let $f(x) = x^3 + ax^2 + bx + 2$. It is known that $\frac{f(x)}{4-x}$ has a remainder of 20, and f has a factor of $3x - 1$. Find the values of a and b.

A. $a = -\frac{97}{66}$ and $b = -\frac{371}{66}$.

B. $a = -\frac{17}{6}$ and $b = -\frac{1}{6}$.

C. $a = \frac{7}{2}$ and $b = -\frac{13}{2}$.

D. $a = \frac{259}{78}$ and $b = -\frac{563}{78}$.

Question 8 (1 mark)

Consider the following quadratic $y = (x+1)^2(x^2 + 2kx + 10)$. It is known that the quadratic has three distinct x-intercepts. What are the possible value(s) of k?

A. $k < -2\sqrt{10} \cup k > 2\sqrt{10}$.

B. $k = \pm 2\sqrt{10}$.

C. $k = \pm\sqrt{10}$.

D. $k < -\sqrt{10} \cup k > \sqrt{10}$.

Question 9 (1 mark)

The function $f(x) = x^5 + (k-1)x^4 + 3x^3 + x$ is an odd function when:

A. $k \in R$.

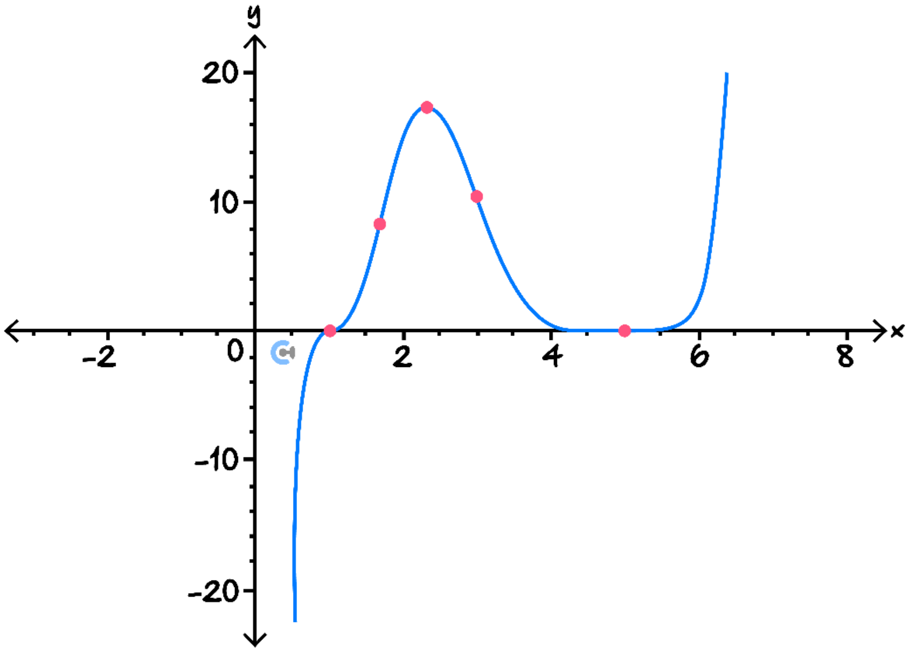
B. $k = -1$.

C. $k = 1$.

D. $k \leq 0$.

Question 10 (1 mark)

Given that $a < 0$, which one of the following equations can correspond to the given graph?



A. $y = -a(x+1)^3(x-5)^2$.

B. $y = a(5-x)(x-1)$.

C. $y = a(5-x)^4(x-1)$.

D. $y = -a(5-x)^6(x-1)^3$.

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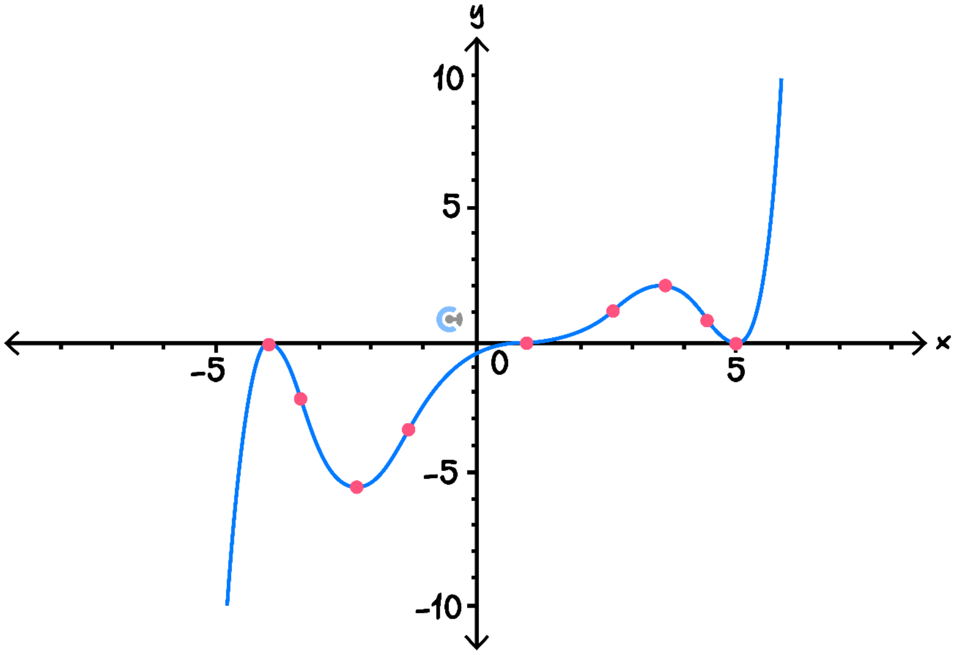
Question 11 (1 mark)

What is the minimum degree of the following polynomial?



A. 5.

B. 6.

C. 7.

D. 8.

Question 12 (1 mark)

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when:

A. $-7 < w < 25$.

B. $w \leq -7$.

C. $w \geq 25$.

D. $w < -7$ or $w > 25$.

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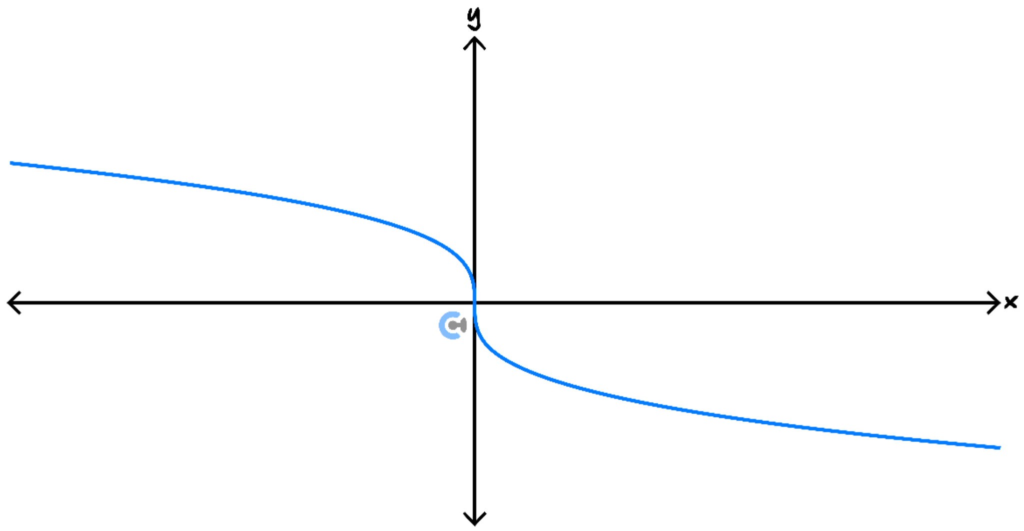
Question 13 (1 mark)

The following graph could have a rule:



A. $y = x^{1/3}$.

B. $y = x^{2/3}$.

C. $y = -x^{2/3}$.

D. $y = -x^{1/3}$.

Question 14 (1 mark)

Consider the function $f: R^+ \to R, f(x) = x^{\frac{p}{q}}$ and $g: R^+ \to R, g(x) = x^{\frac{m}{n}}$, where p, q, m, and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

If $\{x: f(x) > g(x)\} = (0, 1)$ and $\{x: g(x) > f(x)\} = (1, \infty)$, which of the following must be **false**?

A. $m > p$ and $q = n$.

B. $pn < qm$.

C. $f'(c) = g'(c)$ for some $c \in (0,1)$.

D. $f'(d) = g'(d)$ for some $d \in (1, \infty)$.

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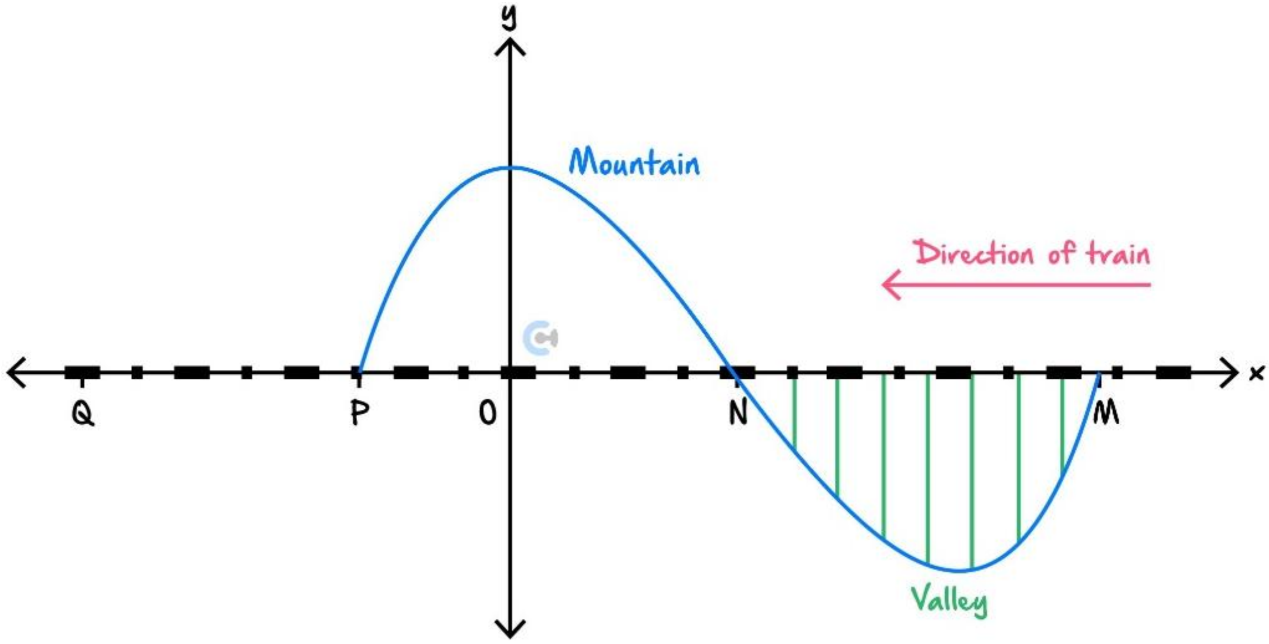
Question 15 (8 marks)



A train is travelling along a straight-level track from M towards Q.

The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.

Section NP passes through a tunnel in a mountain.

Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph:

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b, \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a. The curve defined from M to P passes through $N(2,0)$. The gradient of the curve at N is -0.06 and the the curve has a turning point at $x = 4$.

From this information, write down three simultaneous equations in $a, b,$ and $c,$ and hence, show that $a = 1, b = -6,$ and $c = 16$. (4 marks)

$f(2) = 0$ $\quad 0 = \frac{1}{200}(8a + 4b + c)$

$f'(2) = -0.06$ $\quad -0.06 = \frac{1}{200}(12a + 4b)$

$f'(4) = 0$ $\quad 0 = \frac{1}{200}(48a + 8b)$

b. Find giving exact values:

i. The coordinates of M and P. (2 marks)

$f(x)'=0$

$x = 2 \pm 2\sqrt{3}$.

$M: (2+2\sqrt{3}, 0)$

$P: (2-2\sqrt{3}, 0)$

ii. The length of the tunnel. (1 mark)

$2-(2-2\sqrt{3})$

$2\sqrt{3}$ km.

iii. The maximum depth of the valley below the train track. (1 mark)

$$f(x) = \frac{1}{200}(x^3 - 6x^2 + 16)$$
$$f'(x) = \frac{1}{200}(3x^2 - 12x)$$
$$Turning\ points\ at\ f'(x) = 0$$
$$(3x^2 - 12x) = 0$$
$$x = 0,4$$
$$From\ the\ graph\ we\ see\ that\ maximum\ at\ x = 0$$
$$Hence\ minimum\ at\ x = 4$$
$$f(4) = \frac{1}{200}(4^3 - 6 \times (4)^2 + 16) = -0.08$$
$$Hence\ depth\ of\ valley: 0.08\ km$$

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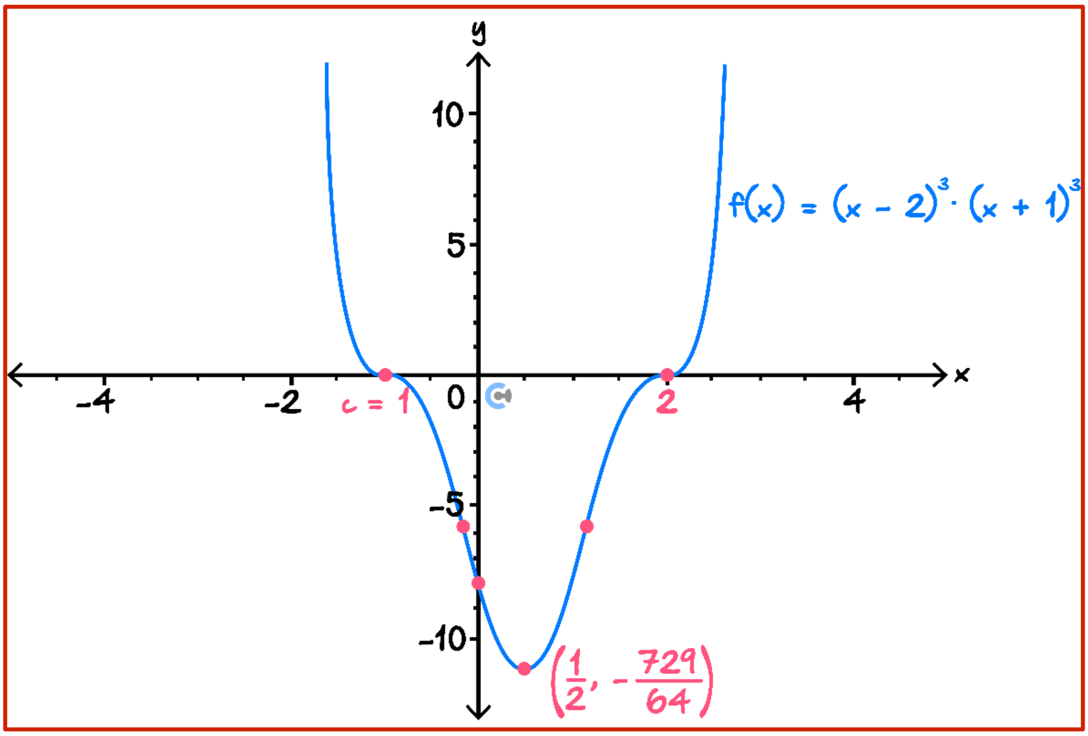
Question 16 (11 marks)

Consider the following function of the form $f(x) = a(x-b)^3(x-c)^3$ where $b > c$.



The turning point of the graph is given as $\left(\frac{1}{2}, -\frac{729}{64}\right)$.

a. Find the values of a, b, and c. (3 marks)

$b = 2$
$c = -1$

sub $\left(\frac{1}{2}, -\frac{729}{64}\right)$ solve $a = 1$

b.

i. Show that $f(m+2) = f(-m-1)$ is true for all values of m. (2 marks)

$$f(m+2) = (m+2-2)^3 \cdot (m+2+1)^3$$
$$= m^3 \cdot (m+3)^3$$

$$f(-m-1) = (-m-1-2)^3 \cdot (-m-1+1)^3$$
$$= (-m-3)^3 \cdot (-m)^3$$
$$= (m+3)^3 \cdot m^3$$

[0.4]

LHS = RHS

$\therefore$ Shown

ii. State the value of r such that $f(r+m) = f(r-m)$ for all values of m. (1 mark)

$$r = \frac{1}{2}$$

c. Consider $g(x) = x + k$.

i. Find the value(s) of k such that there are no negative x-intercepts for $f\big(g(x)\big)$. (2 marks)

$f(g(x)) = f(x+k)$ ← k left.

if $f(x)$ had $x = -1$ & 2 as x int

$f(x+k)$ " $x = -1-k$ & $2-k$ as x int

$0 \leq -1-k$

$k \leq -1$

ii. Find the value(s) of k such that there is only one negative x-intercept for $f\big(g(x)\big)$. (2 marks)

$-1-k < 0 \quad \cap \quad 2-k \geq 0$

$-1 < k \quad \cap \quad 2 \geq k$

$k \in (-1, 2]$

iii. Find the value of k such that $f\big(g(x)\big)$ is an even function. (1 mark)

$f\big(g(x)\big)$ is f translated k units to the left.

f will be even if it is translated $\frac{1}{2}$ units left.

Therefore, $k = \frac{1}{2}$.

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