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VCE Mathematical Methods ¾
Polynomials [0.7]
Workshop

Error Logbook:

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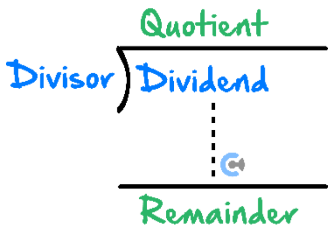
Section A: Recap

Roots of Polynomial Functions

$$\textit{Roots} = x\textit{-intercepts}$$

Polynomial Long Division



$$\frac{\textit{Dividend}}{\textit{Divisor}} = \textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

TIPS:

- Always focus on the highest degree term first.
- Always remember to fill in any missing powers of x in the numerator or denominator with "placeholders" that have a coefficient of 0.

Remainder Theorem

- **Definition**:
  - Find the remainder of the long division without needing long division.

$$\textit{When } P(x) \textit{ is divided by } (x-\alpha), \textit{ the remainder is } P(\alpha).$$

divisor

- **Steps**:
  1. Find x-values which make the divisor equal to 0.
  2. Substitute it into the dividend function.

Definition

Factor Theorem

- For every x-intercept, there is a factor.

$$If\ P(\alpha) = 0, then\ (x - \alpha)\ is\ a\ factor\ of\ P(x).$$

remainder $= 0$

Definition

Factorising Cubic Polynomials

- **Steps:**
  1. Find a single root by trial and error. $\pm 1, \pm 2, \pm 3$ (start small)
     - (Factor theorem: Substitute into the function and see if we get zero.)
  2. Use long division to find the quadratic factor.
  3. Factorise the remaining factor.

Definition

Rational Root Theorem

e.g. $x^3 + 4x^2 - 2x + 6$

- Rational root theorem **narrows down** the possible roots.

Potential roots $= \pm \frac{1, 2, 3, 6}{1}$

$$Potential\ root = \pm \frac{Factors\ of\ constant\ term\ a_0}{Factors\ of\ leading\ coefficient\ a_n}$$

- If the roots are rational numbers, the roots can only be $\pm \frac{\text{factors of constant term } a_0}{\text{factors of leading coefficient } a_n}$.

Definition

Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Space for Personal Notes

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

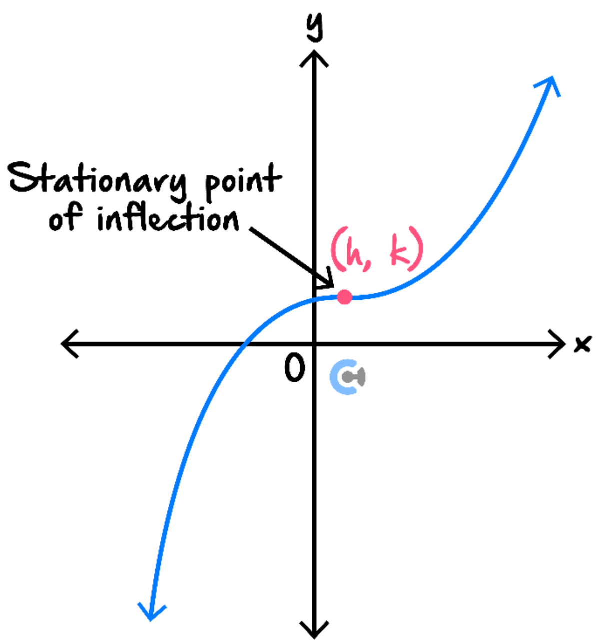
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

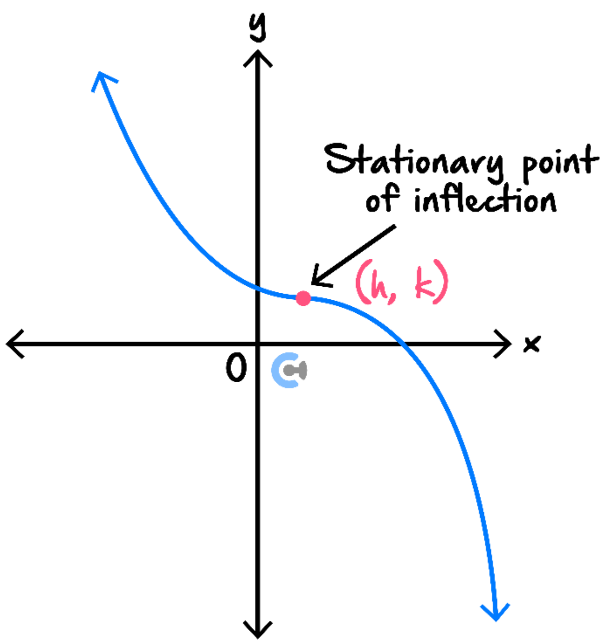
Definition

3, 5, 7, 9 ...

Graphs of $a(x-h)^n + k$, where n is Odd and Positive

- All graphs look like a "cubic".





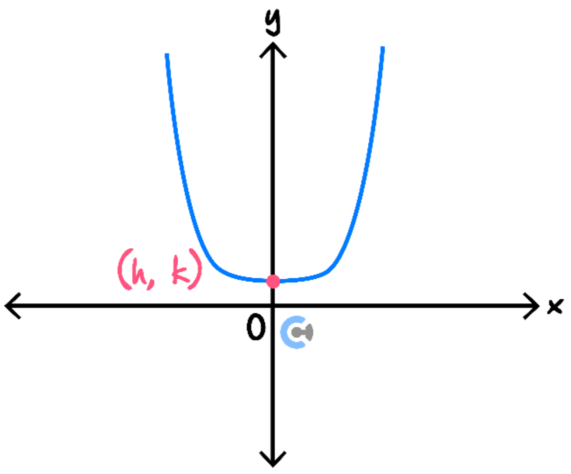
- The point (h, k) gives us the stationary point of inflection.
- n cannot be 1 for this shape to occur!

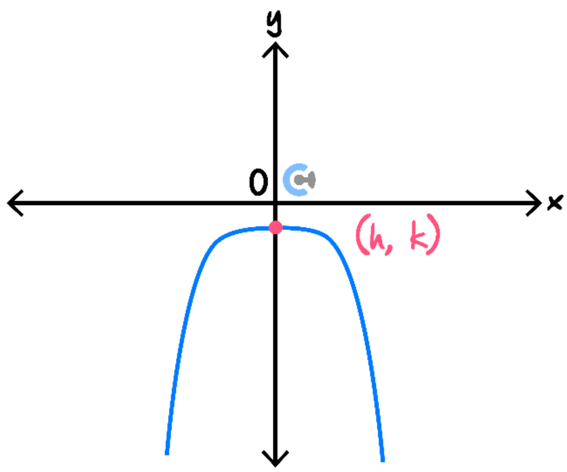
Definition

2, 4, 6, 8, 10 ...

Graphs of $a(x-h)^n + k$, where n is Even and Positive

- All graphs look like a "quadratic".



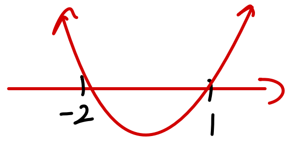


- The point (h, k) gives us the turning point.

Graphs of Factorised Polynomials

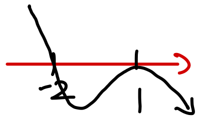
- **Steps:**

1. Plot x-intercepts.
2. Determine whether the polynomial is positive or negative.
3. Use the repeated factors to deduce the shape:



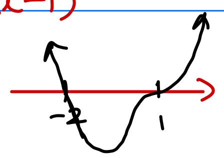
"cut"

- Non-repeated: Only x-intercept. $y = (x+2)(x-1)$
- Even repeated: x-intercept and a turning point. $y = (x+2)(x-1)^4$

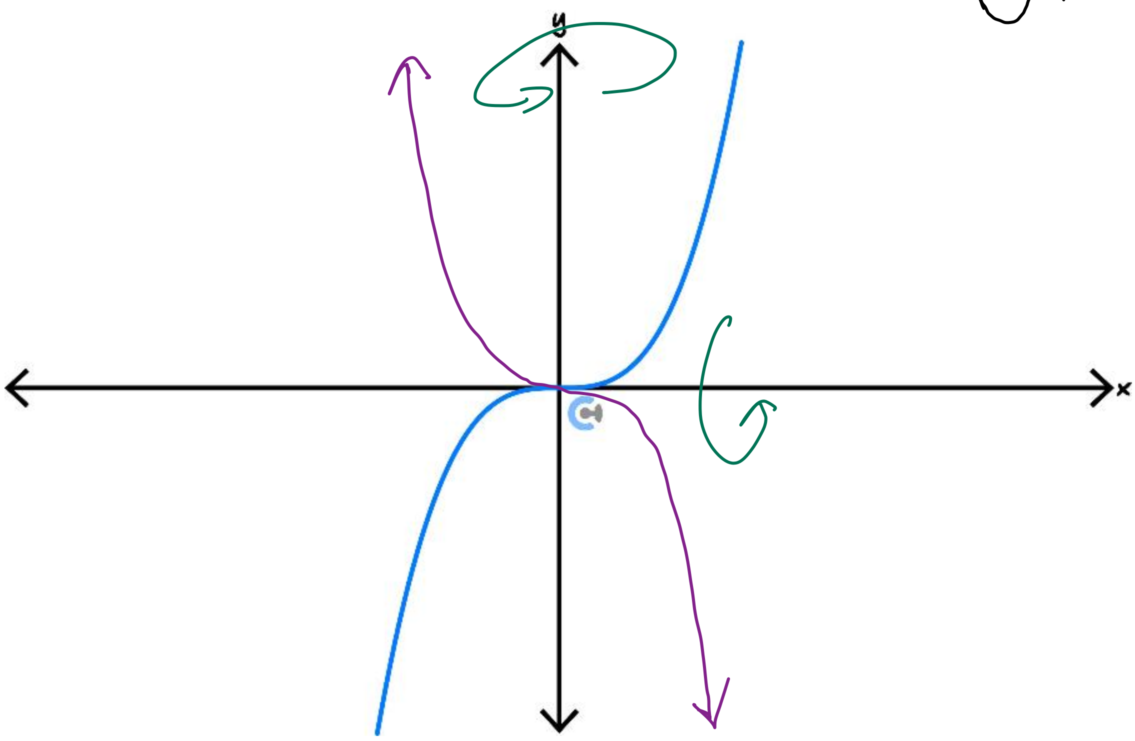


- Odd repeated: x-intercept and a stationary point of inflection.

$y = (x+2)(x-1)^3$



Odd Functions

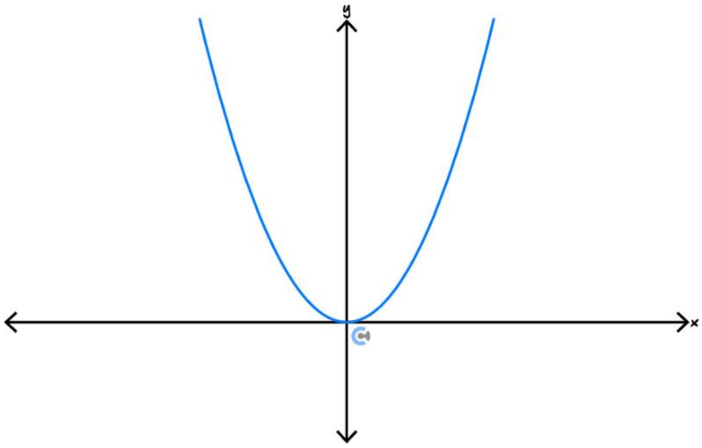


- E.g., $x^3, x^5, x^7 - x^3$. They are all odd powers.

$$f(-x) = -f(x)$$

- Property: Reflecting around the y-axis is the same as reflecting around the x-axis.

Even Functions



- E.g., $x^2, x^4, -x^{10}, x^4 - 4$. They are all even powers.

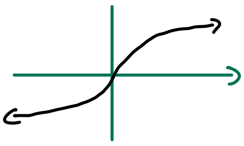
$$\boxed{f(-x) = f(x)}$$

- Property: It is symmetrical around the y-axis.

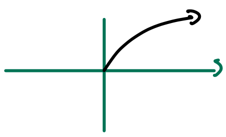
Power Functions

$$y = x^{\frac{n}{m}}$$

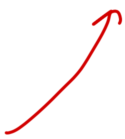
- m: Dictates the number of **tails**.
  - **Odd m = Two tails.** e.g. $y = x^{1/3}$



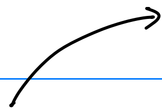
  - **Even m = One tail.** e.g. $y = x^{1/2}$



- n: Dictates the **range**.
  - Odd n: The range could be all real. above or below x-axis
  - Even n: The range must be non-negative. on/above x-axis
- $\frac{n}{m}$ *(Power)*:
  - Power > 1: Looks like a polynomial function. e.g. $\frac{6}{5}, \frac{11}{8}$



  - Power < 1: Looks like a root function. e.g. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$



Section B: Warm Up

Question 1

a. Find the remainder of the division $\frac{f(x)}{g(x)}$ where $f(x) = x^3 + 3x^2 + 2$ and $g(x) = x - 1$.

1

$f(1) = 1^3 + 3(1)^2 + 2$

$= 1 + 3 + 2$

$= 6$ ← remainder

b. Use polynomial long division to write $f(x) = \frac{x^3+2x^2+3x+2}{x+2}$ in the form $f(x) = Q(x) + \frac{a}{x+2}$ for quadratic function Q and integer a.

$$\begin{array}{r|l} & x^2 + 3 \\ x+2 & x^3+2x^2+3x+2 \\ & -(x^3+2x^2) \\ \hline & 0 + 3x + 2 \\ & -(3x+6) \\ \hline & -4 \end{array}$$

$f(x) = x^2 + 3 + \frac{-4}{x+2}$

c.

i. Find all the roots of $f(x) = x^3 - 2x^2 - x + 2$.

$x^3 - 2x^2 - x + 2 = 0$

$x^2(x-2) - (x-2) = 0$ factorise by grouping

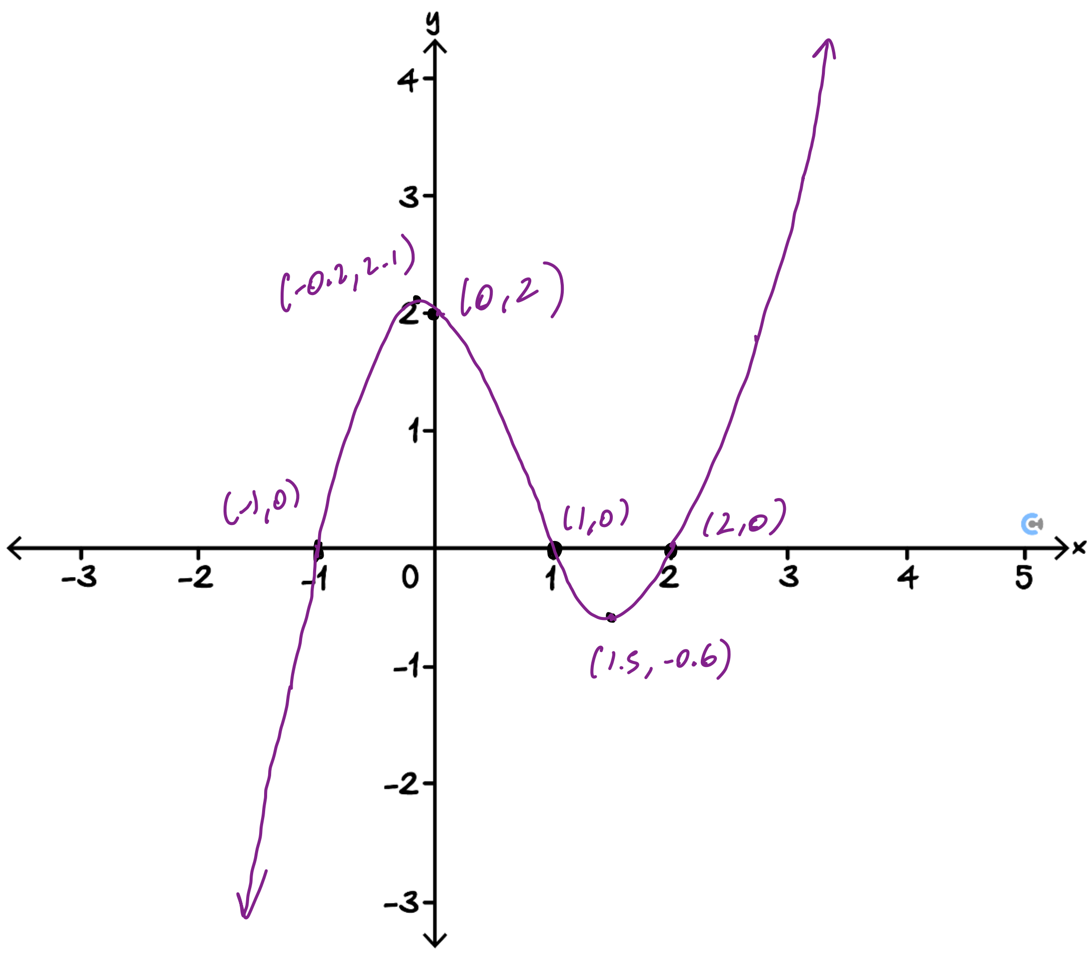
$(x^2-1)(x-2) = 0$

$x^2 - 1 = 0$ $\quad x - 2 = 0$

$\boxed{x = \pm 1 \quad x = 2}$ //

Try: $x = \pm 1, \pm 2, \pm 3$

ii. Sketch the graph of $y = f(x)$ on the axes below. Turning points occur at approximately $(-0.2, 2.1)$ and $(1.5, -0.6)$.



d. Factorise the expression $x^3 + 27$.

$= x^3 + 3^3$

$= (x+3)(x^2 - 3x + 9)$

e. Expand the expression $(x-2)^3$.

$= x^3 - 3x^2(2) + 3x(2)^2 - 2^3$

$= x^3 - 6x^2 + 12x - 8$

f. Show that the function $f(x) = x^3 - 3x$ is odd.

want: $f(-x) = -f(x)$

LHS: $f(-x) = (-x)^3 - 3(-x)$

$= -x^3 + 3x$

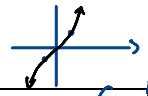
RHS: $-f(x)$

$= -(x^3 - 3x)$

$= -x^3 + 3x$

$\therefore f(-x) = -f(x)$

f is odd function!



$\left((-1)^{1/3}\right)^5 = -1$

all positive

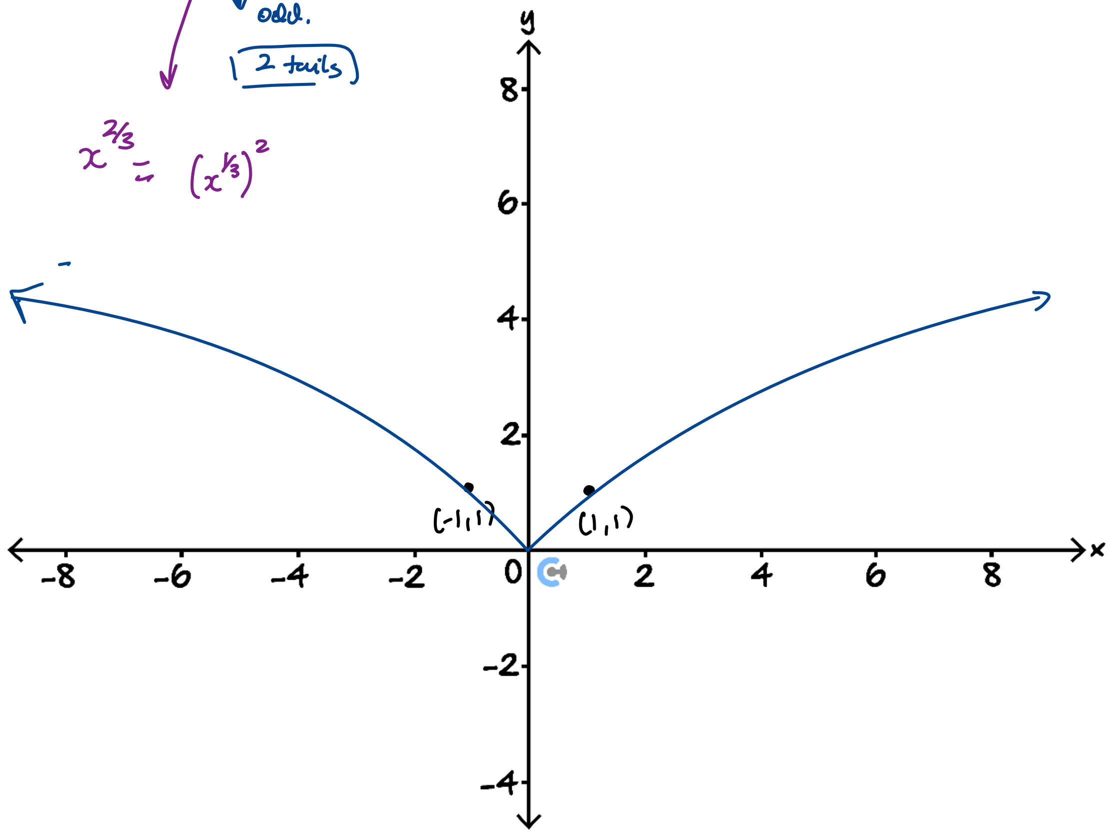
< 1 "root function"

g. Sketch the graph of $y = x^{\frac{2}{3}}$ on the axes below.

odd.

2 tails

$x^{2/3} = \left(x^{1/3}\right)^2$



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Section C: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 25 Minutes Writing.

Question 2 (3 marks)

$f(2) = -4$

Let $f(x) = x^3 - ax^2 - 5x + b$, where a and b are constants. When $f(x)$ is divided by $x - 2$, the remainder is -4 and when $f(x)$ is divided by $x + 1$, the remainder is 8. Find the value of a and b.

$f(-1) = 8$

$2^3 - a(2)^2 - 5(2) + b = -4$

$(-1)^3 - a(-1)^2 - 5(-1) + b = 8$

$8 - 4a - 10 + b = -4$ $\quad$ $b - 4a = -2$ --①

$-1 - a + 5 + b = 8$ $\quad$ $b - a = 4$ --②

①−②: $-3a = -6$, $\boxed{a = 2}$

$\boxed{b = 6}$

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Question 3 (5 marks)

Consider the function given by $f(x) = 2x^3 - 3x^2 - 11x + 6$.

a. Find all x-intercepts of $f(x)$. (3 marks)

$f(3) = 2(3)^3 - 3(3)^2 - 11(3) + 6$

$= 54 - 27 - 33 + 6$

$= 0$ $\therefore$ $(x-3)$ is a factor

Try: $\pm 1, \pm 2, \pm 3$

$$\begin{array}{r|l} & \boxed{2x^2 + 3x - 2} \\ x-3 & 2x^3 - 3x^2 - 11x + 6 \\ & -(2x^3 - 6x^2) \\ \hline & 3x^2 - 11x + 6 \\ & 3x^2 - 9x \\ \hline & -2x + 6 \\ & -2x + 6 \\ \hline & 0 \end{array}$$

$f(x) = (x-3)(2x^2+3x-2)$

$= (x-3)(2x-1)(x+2)$

$f(x) = 0$

$\boxed{x = 3, \frac{1}{2}, -2}$

b. Hence, find all x-intercepts for $f(g(x))$ where $g(x) = x^2 + 1$. (2 marks)

$f(g(x)) = 0$

$g(x) = 3, \frac{1}{2}, -2$

$f(3) = 0$

$f(\frac{1}{2}) = 0$

$f(-2) = 0$

$x^2 + 1 = 3$ $x^2 + 1 = \frac{1}{2}$ $x^2 + 1 = -2$

$x^2 = 2$ $x^2 = -\frac{1}{2}$ $x^2 = -3$

$\boxed{x = \pm\sqrt{2}}$ no sol^ns no sol^ns

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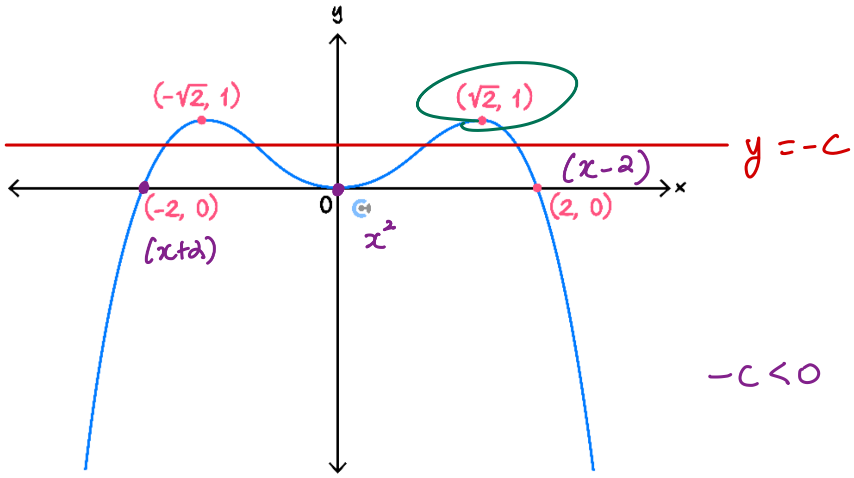
Question 4 (3 marks)

The function $f: R \to R, f(x)$ is a polynomial function of degree 4. Part of the graph of f is shown below.

The graph of f touches the x-axis at the origin.



$-c < 0$ or $0 \leq -c \leq 1$

a. Find the rule of f. (2 marks)

$f(x) = ax^2(x+2)(x-2)$

passes $(\sqrt{2}, 1)$:

$1 = a(\sqrt{2})^2(\sqrt{2}+2)(\sqrt{2}-2)$

$1 = 2a(\sqrt{2}^2 - 2^2)$

$1 = 2a(2-4)$

$1 = 2a \cdot -2$, $-4a = 1$, $a = \frac{-1}{4}$

$f(x) = \frac{-1}{4}x^2(x+2)(x-2)$

b. Find the values of c for which $f(x) + c = 0$, where $c \in \mathbb{R}$, has an even number of real solutions. (1 mark)

$f(x) = -c$

$-c < 0$ or $0 < -c \leq 1$

$c > 0$ or $0 > c \geq -1$

$c \in [-1, \infty) \setminus \{0\}$

Question 5 (3 marks)

Consider the function given by $f(x) = x^4 + x^2 + 2$. (2 marks)

a. Show that $f(x)$ is an even function. symmetry about y-axis

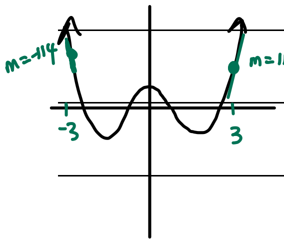
$f(-x) = f(x)$

$f(-x) = (-x)^4 + (-x)^2 + 2$

$= x^4 + x^2 + 2$ $\therefore f(x)$ is even.

$= f(x)$

b. The gradient of f when $x = 3$ is 114. State the gradient of f when $x = -3$. (1 mark)



$m = -114$

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Question 6 (6 marks)

Consider the function $h: [-3,1] \to \mathbb{R}, h(x) = \frac{1}{2}x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{9}{2}$.

a. Given that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, write $h(x)$ in the form of $a(x+b)^3 + c$. (3 marks)

$h(x) = \frac{1}{2}(x^3 + 3x^2 + 3x + 9)$

$= \frac{1}{2}(\underbrace{x^3 + 3x^2 + 3x + 1^3}_{} + 8)$

$= \frac{1}{2}((x+1)^3 + 8)$

$= \frac{1}{2}(x+1)^3 + 4$

$(-1, 4)$ P.O.I

b. Sketch $y = h(x)$ on the axes below. Label any endpoints, axes, intercepts, and stationary points. (2 marks)

$x=1: \quad y = \frac{1}{2}(2)^3 + 4$

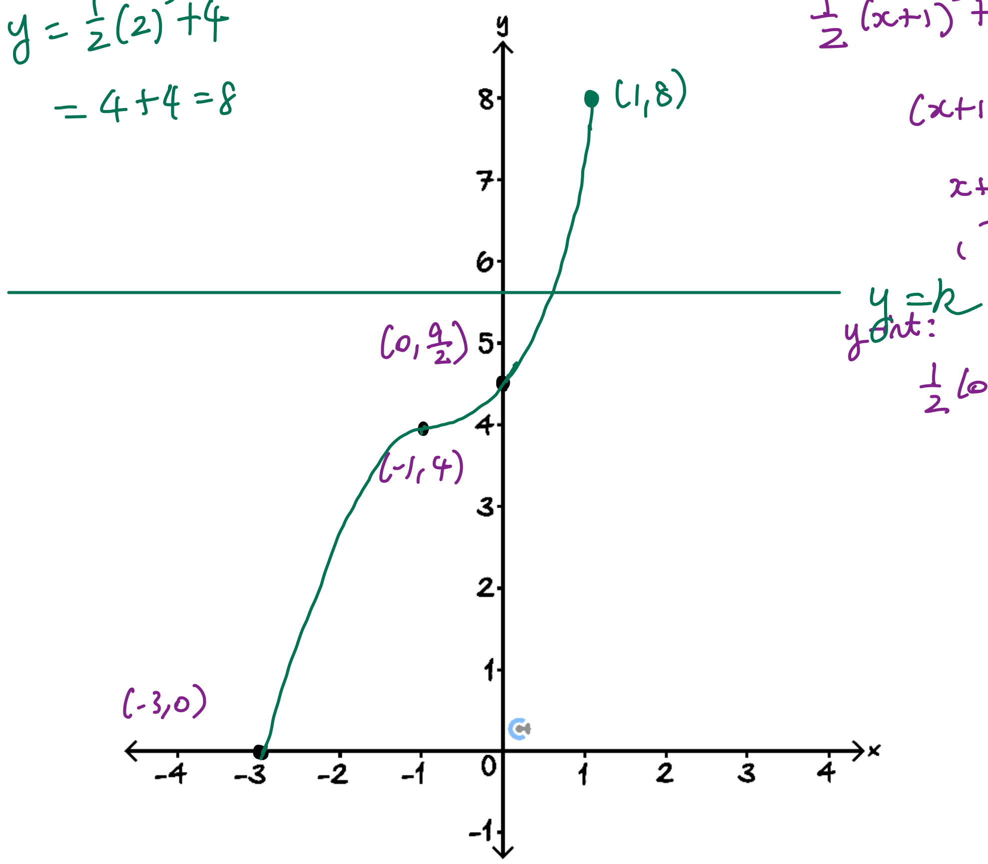
$= 4 + 4 = 8$

$\frac{1}{2}(x+1)^3 + 4 = 0$

$(x+1)^3 = -8$

$x + 1 = -2$

$\boxed{x = -3}$



y int:

$\frac{1}{2}(0+1)^3 + 4 = \frac{9}{2}$

c. How many solution(s) will $h(x) = k$ always have for $k \in [0,8]$? (1 mark)

1

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Section D: Tech Active Exam Skills

Calculator Commands: Factor and Expand

- All the technologies have a "factor" and "expand" function.
- The general syntax is $\boldsymbol{factor}(\boldsymbol{expr})$, or $\boldsymbol{expand}(\boldsymbol{expr})$. menu → 3 → 2, 3
- **Example:** Factorise the expression $2x^3 - 23x^2 + 33x + 108$ and expand the expression $(x+1)^5(x-2)^3$.
- **TI:**

$\text{factor}(2\cdot x^3-23\cdot x^2+33\cdot x+108)$	$(x-9)\cdot(x-4)\cdot(2\cdot x+3)$
$\text{expand}((x+1)^5\cdot(x-2)^3)$	$x^8-x^7-8\cdot x^6+2\cdot x^5+25\cdot x^4+11\cdot x^3-26\cdot x^2-28\cdot x-8$

- **Casio:**

$\text{factor}(2x^3-23x^2+33x+108)$

$(x-4)\cdot(x-9)\cdot(2\cdot x+3)$

$\text{expand}((x+1)^5(x-2)^3)$

$x^8-x^7-8\cdot x^6+2\cdot x^5+25\cdot x^4+11\cdot x^3-26\cdot x^2-28\cdot x-8$

- **Mathematica:**

In[7]:= Factor[108 + 33 x - 23 x^2 + 2 x^3]

Out[7]= $(-9+x)(-4+x)(3+2x)$

In[8]:= Expand[(x + 1)^5 (x - 2)^3]

Out[8]= $-8-28x-26x^2+11x^3+25x^4+2x^5-8x^6-x^7+x^8$

Space for Personal Notes

Calculator Commands: Turning Point

- ALWAYS sketch the graph first to get an idea of the nature of the turning point.

- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.

- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.

- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	Done
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	
	done
$\text{solve}(\frac{d}{dx}(f(x))=0, x)$	
	$\{x=1\}$
$f(1)$	
	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]

In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]

Out[5]= {{x → 1, y → e}}
```

TI UDF: We can use the analyse function.

Analyse a Function

$$\mathit{analysed}\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

- Start Point: $\begin{bmatrix}-5 & \frac{262}{77}\end{bmatrix}$
- End Point: $\begin{bmatrix}5 & \frac{-316}{529}\end{bmatrix}$
- Maximal Domain:
  $x\neq-1.68469$ and
  $x\neq-0.629579$ and
  $x\neq0.314273$ and
  $-5\leq x\leq5$
- Asymptotes: (4)
  $x=-1.68469$ (Vertical)
  $x=-0.629579$ (Vertical)
  $x=0.314273$ (Vertical)
  $y=\frac{4}{3}-\frac{x}{3}$ (Oblique)
- x –Intercepts: (4)
  $\begin{bmatrix}-1.3772 & 0\end{bmatrix},\begin{bmatrix}-0.273891 & 0\end{bmatrix},$
  $\begin{bmatrix}1 & 0\end{bmatrix},\begin{bmatrix}2.65109 & 0\end{bmatrix}$
- Vertical Intercept: $\begin{bmatrix}0 & 1\end{bmatrix}$
- Derivative:
  $$\frac{-\left(3\cdot x^6+12\cdot x^5-26\cdot x^3-24\cdot x^2-6\cdot x-4\right)}{\left(3\cdot x^3+6\cdot x^2+x-1\right)^2}$$
- Inflection Points: (2)
  $\begin{bmatrix}-1.11377 & 1.48672\end{bmatrix}$ (Increasing)
  $\begin{bmatrix}-0.11198 & 0.604642\end{bmatrix}$ (Increasing)
- Stationary Points: (2)
  $\begin{bmatrix}-3.45719 & 3.17894\end{bmatrix}$ (Local min.)
  $\begin{bmatrix}1.6173 & 0.124612\end{bmatrix}$ (Local max.)

Done

Overview:

This program will find for a given function:

- Coordinates of endpoints.
- The maximal domain.
- The equations of straight-line asymptotes.
- The rule of the derivative.
- Inflection points and their nature.
- Stationary points and their nature.

There are two analysis programs:

- Analyse which analyses a function over the domain R or the maximal domain.
- Analysed which analyses over a domain with specified start and end points.
- Both are found in the methods_func library. You can switch between the two on the calculator page by adding/removing the 'd' to reference the appropriate program.

Input:

$$analyse(< function >, < variable >)$$

$$analysed(< function >, < variable >, < lower\ bound >, < upper\ bound >)$$

- **Other notes:**
  - It is recommended to use the analysed program when working with trigonometric functions.
  - Be careful when using functions with parameters since some parts of the programs may not be able to give a solution. :/
  - If at least one of the bounds is "?", the asymptote finder will be disabled and the program will analyse over the maximal domain.

Calculator Commands: Using Sliders/Manipulate on CAS

- **Mathematica**

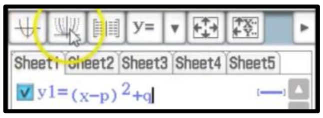
```
Manipulate[Plot[function, {x, xmin, xmax}],

    {unknown, lowerbound, upperbound}]
```

  - **NOTE:** The function **must** be typed out instead of using its saved name.

- **TI-Nspire**

- **Casio Classpad**



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Calculator Commands: Finding the equation of a polynomial that passes through points.

- Given n points, we can find a degree $n-1$ polynomial that passes through all these points.
- **Example:** Find the equation of the quadratic function that passes through the points $(0,6)$, $(2,2)$, and $(3,3)$.
- **TI:**

menu → 3 → 7 (solve system of equations)

multiple eqns

solving for multi. variables

Define $f(x)=a\cdot x^2+b\cdot x+c$	*Done*
solve$(f(0)=6$ and $f(2)=2$ and $f(3)=3,a,b,c)$	$a=1$ and $b=-4$ and $c=6$
$f(x)\|a=1$ and $b=-4$ and $c=6$	$x^2-4\cdot x+6$

- **Casio:**

```
define f(x) = a*x^2 + b*x + c                         done

{f(0)=6
 f(2)=2
 f(3)=3 |a, b, c
                                          {a=1, b=-4, c=6}
f(x)|{a=1, b=-4, c=6}
                                                x^2-4·x+6
```

- **Mathematica:**

```
In[9]:=  f[x_] := a x^2 + b x + c

In[10]:= Solve[f[0] == 6 && f[2] == 2 && f[3] == 3]
Out[10]= {{a → 1, b → -4, c → 6}}

In[11]:= f[x] /. {a → 1, b → -4, c → 6}
Out[11]= 6 - 4 x + x^2
```

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Section E: Exam 2 (27 Marks)

INSTRUCTION: 27 Marks. 34 Minutes Writing.

Question 7 (1 mark)

$f(4)=20$ $\quad f(\frac{1}{3})=0$

Let $f(x) = x^3 + ax^2 + bx + 2$. It is known that $\frac{f(x)}{4-x}$ has a remainder of 20, and f has a factor of $3x - 1$. Find the values of a and b.

A. $a = -\frac{97}{66}$ and $b = -\frac{371}{66}$.

B. $a = -\frac{17}{6}$ and $b = -\frac{1}{6}$.

C. $a = \frac{7}{2}$ and $b = -\frac{13}{2}$.

D. $a = \frac{259}{78}$ and $b = -\frac{563}{78}$.

solve$\left(\begin{cases} f(4)=20 \\ f(\frac{1}{3})=0 \end{cases}, \{a,b\}\right)$

Question 8 (1 mark)

1 sol^n $\quad$ 2 sol^n $\quad (-1)^2+2k(-1)+10 \neq 0$

$1-2k+10 \neq 0$

Consider the following quadratic $y = (x+1)^2(x^2 + 2kx + 10)$. It is known that the quadratic has three distinct x-intercepts. What are the possible value(s) of k?

$2k \neq -11$

A. $k < -2\sqrt{10} \cup k > 2\sqrt{10}$.

$\Delta > 0$ $\quad k \neq \frac{-11}{2}$

B. $k = \pm 2\sqrt{10}$.

$4k^2 - 4(1)(10) > 0$

C. $k = \pm\sqrt{10}$.

D. $k < -\sqrt{10} \cup k > \sqrt{10}$. $\setminus \{-\frac{11}{2}\}$

Question 9 (1 mark)

The function $f(x) = x^5 + (k-1)x^4 + 3x^3 + x$ is an odd function when:

A. $k \in R$.

solve$(f(-x) = -f(x), k)$

B. $k = -1$.

C. $k = 1$.

D. $k \leq 0$.

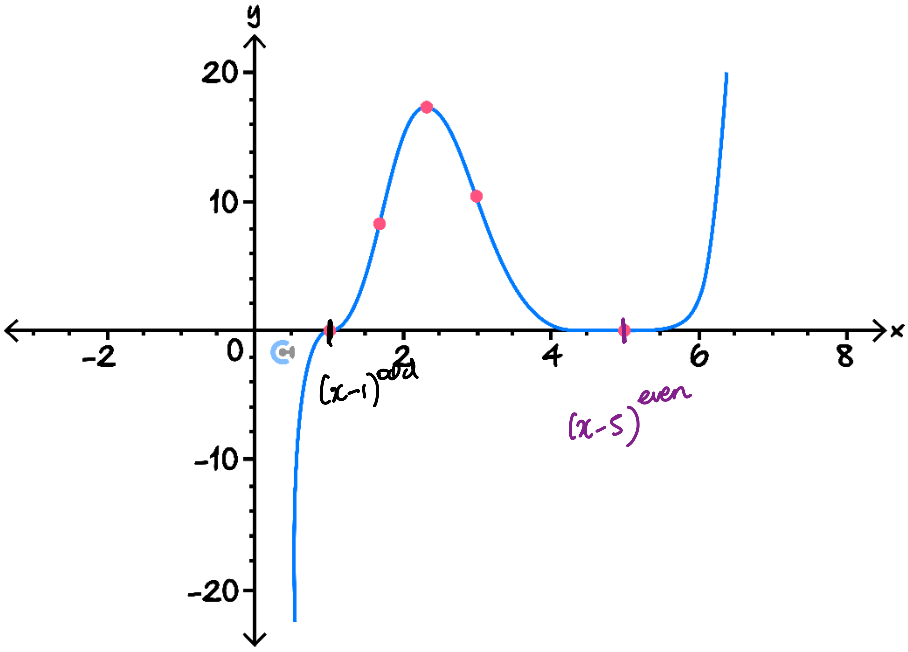
$x=-1$

$$\underbrace{(x+1)^2}_{1\ \text{sol}^{n}}\underbrace{(x^2+2kx+10)^2}_{2\ \text{sol}^{n}} \Big\}\ 3\ \text{sol}^{n}$$

$\Delta > 0$

Question 10 (1 mark)

Given that $a < 0$, which one of the following equations can correspond to the given graph?



A. $y = -a(x+1)^3(x-5)^2$.

B. $y = a(5-x)(x-1)$. X

C. $y = a(5-x)^4(x-1)$. X

D. $y = -a(5-x)^6(x-1)^3$.

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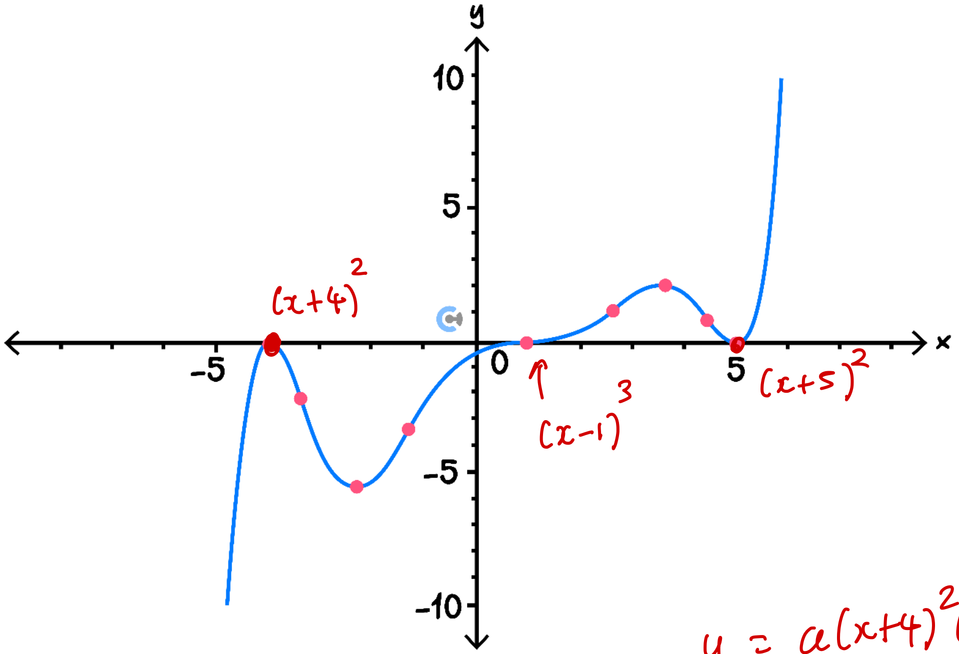
Question 11 (1 mark)

What is the minimum degree of the following polynomial?



$y = a(x+4)^2(x-1)^3(x+5)^2$

A. 5.

B. 6.

C. 7.

D. 8.

Question 12 (1 mark) $w^3 - 9x^2 + 15x = -w$

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when:

A. $-7 < w < 25$.

B. $w \leq -7$.

C. $w \geq 25$.

D. $w < -7$ or $w > 25$.

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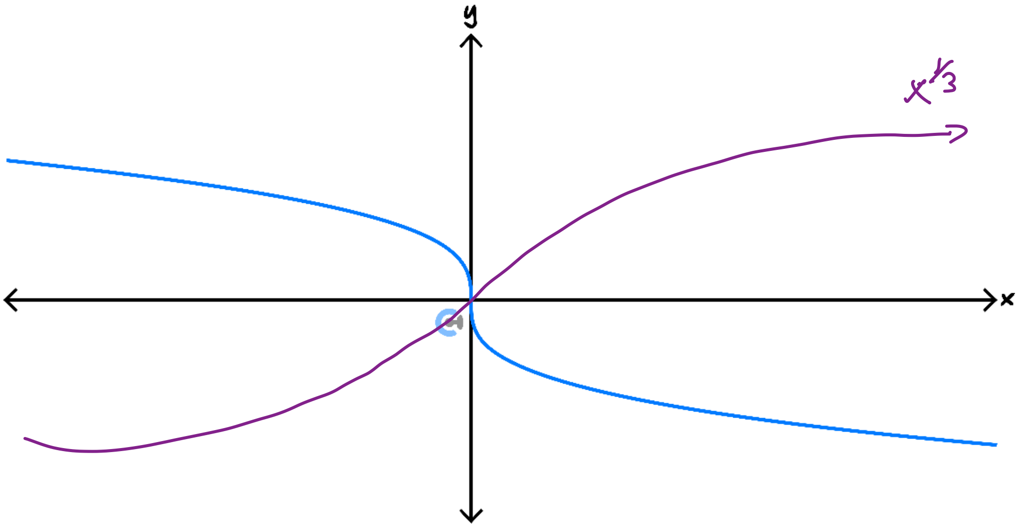
Question 13 (1 mark)

The following graph could have a rule:



A. $y = x^{1/3}$.

B. $y = x^{2/3}$.

C. $y = -x^{2/3}$.

D. $y = -x^{1/3}$.

Question 14 (1 mark)

$\frac{p}{q} < 1$ $\frac{m}{n} > 1$

Consider the function $f: R^+ \to R, f(x) = x^{\frac{p}{q}}$ and $g: R^+ \to R, g(x) = x^{\frac{m}{n}}$, where $p, q, m,$ and n are positive integers, and $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in simplest form.

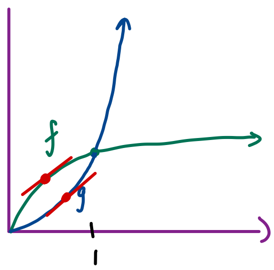
If $\{x: f(x) > g(x)\} = (0, 1)$ and $\{x: g(x) > f(x)\} = (1, \infty)$, which of the following must be **false**?

A. $m > p$ and $q = n$.

B. $pn < qm$.

C. $f'(c) = g'(c)$ for some $c \in (0,1)$.

D. $f'(d) = g'(d)$ for some $d \in (1, \infty)$.



Derivative = gradient

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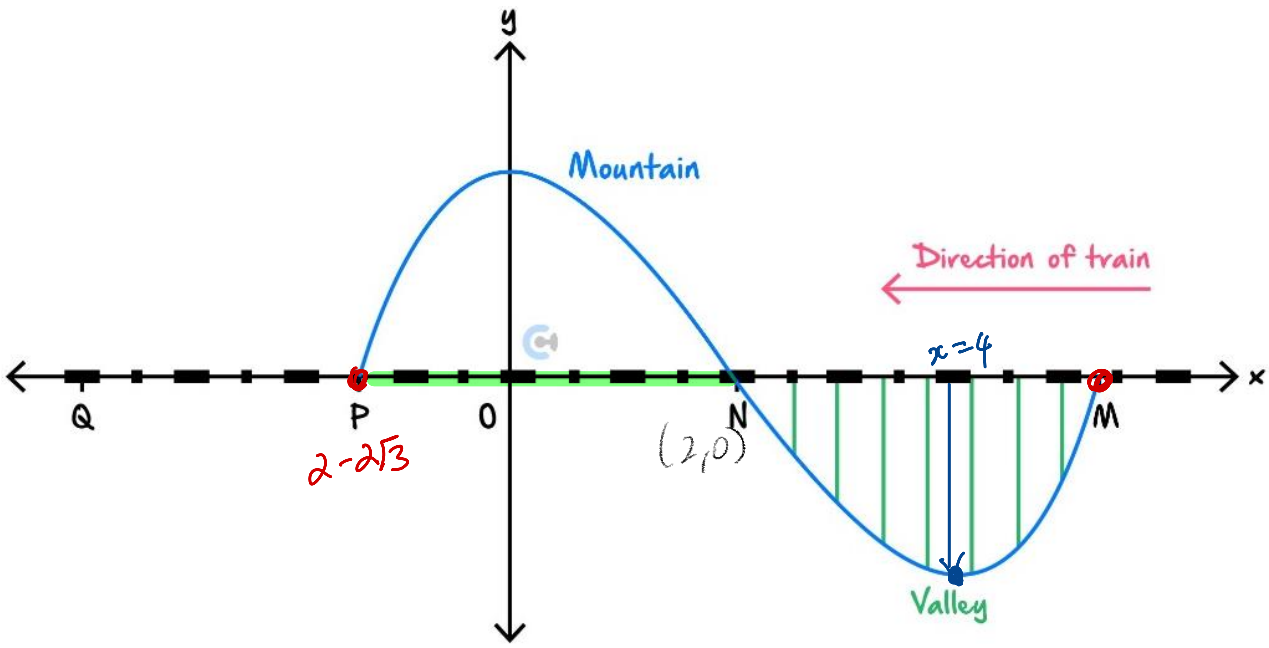
Question 15 (8 marks)



A train is travelling along a straight-level track from M towards Q.

The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.

Section NP passes through a tunnel in a mountain.

Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph:

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b, \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

$f(x) = \frac{1}{200}\left(a \cdot x^3 + b \cdot x^2 + c\right)$

a. The curve defined from M to P passes through $N(2,0)$. The gradient of the curve at N is -0.06 and the the curve has a turning point at $x = 4$.

Gradient $=0$

From this information, write down three simultaneous equations in a, b, and c, and hence, show that $a = 1, b = -6$, and $c = 16$. (4 marks)

$f(2)=0 \longrightarrow 0 = \frac{1}{200}(8a+4b+c)$

$f'(2) = -0.06 \quad -0.06 = \frac{1}{200}(12a+4b)$

$f'(4) = 0 \quad 0 = \frac{1}{200}(48a+8b)$

$8a+4b+c=0$

$-12 = 12a+4b$

$48a+8b=0, \quad 6a+b=0, \quad b=-6a$

$-12 = 12a+4(-6a)$

$-12 = -12a, \quad \boxed{a=1}$

$b = -6(1) = -6$

$8(1)+4(-6)+c=0$

$8-24+c=0$

$\boxed{c=16}$

b. Find giving exact values:

i. The coordinates of M and P. (2 marks)

$f(x)=0$

$x = 2 \pm 2\sqrt{3}$

$M(2+2\sqrt{3}, 0)$

$P(2-2\sqrt{3}, 0)$

ii. The length of the tunnel. (1 mark)

Length $= 2-(2-2\sqrt{3})$

$= 2\sqrt{3}$ km

iii. The maximum depth of the valley below the train track. (1 mark)

$f(4) = -0.08$

max depth $= 0.08$ km.

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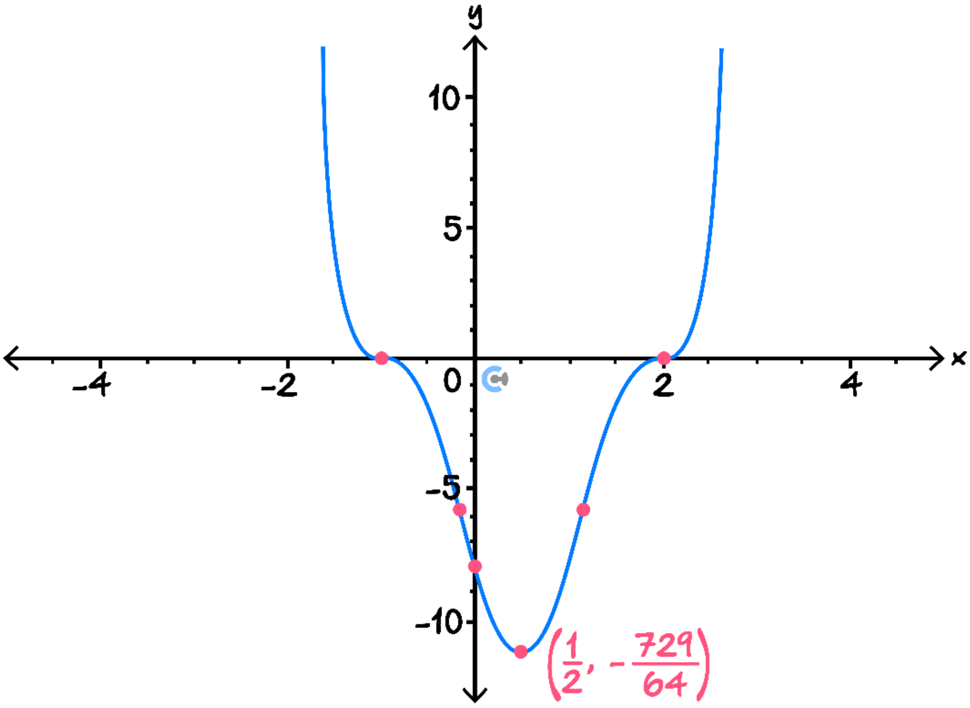
Question 16 (11 marks)

Consider the following function of the form $f(x) = a(x-b)^3(x-c)^3$ where $b > c$.



The turning point of the graph is given as $\left(\frac{1}{2}, -\frac{729}{64}\right)$.

a. Find the values of a, b, and c. (3 marks)

$f(x) = a(x-2)^3(x-(-1))^3$

$f\left(\frac{1}{2}\right) = -\frac{729}{64}$, solve for a:

$a = 1$

$\boxed{a=1,\ b=2,\ c=-1}$

b.

i. Show that $f(m+2) = f(-m-1)$ is true for all values of m. (2 marks)

$f(m+2) = (m+2-2)^3(m+2+1)^3$

$= m^3(m+3)^3$

$f(-m-1) = (-m-1-2)^3(-m-1+1)^3$

$= (-m-3)^3(-m)^3$

$= -m^3(-(m+3))^3 = -m^3 \cdot -(m+3)^3$

$= m^3(m+3)^3$

$\therefore f(m+2) = f(-m-1)$ as req.

ii. State the value of r such that $f(r+m) = f(r-m)$ for all values of m. (1 mark)

$r = \frac{1}{2}$

c. Consider $g(x) = x + k$.

i. Find the value(s) of k such that there are no negative x-intercepts for $f(g(x))$. (2 marks)

$f(g(x)) = 0$

$f(x+k) = 0$

↑ k left

$k \leq -1$

$f(x) = (x-2)^3(x+1)^3$

$-k-1 \geq 0$

$\boxed{k \leq -1}$

ii. Find the value(s) of k such that there is only one negative x-intercept for $f(g(x))$. (2 marks)

$f(g(x)) = (x+k-2)^3(x+k+1)^3$

$= (x-(2-k))^3(x-(-k-1))^3$

$2-k \geq 0 \quad \cap \quad -k-1 < 0$

$k \leq 2 \quad \cap \quad k > -1$, $\boxed{k \in (-1, 2]}$

iii. Find the value of k such that $f(g(x))$ is an even function. (1 mark)

symmetrical about y-axis.

$f(g(x)) = f(x+k)$ k left.

$k = \frac{1}{2}$! move ½ left.

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