CONTOUREDUCATION

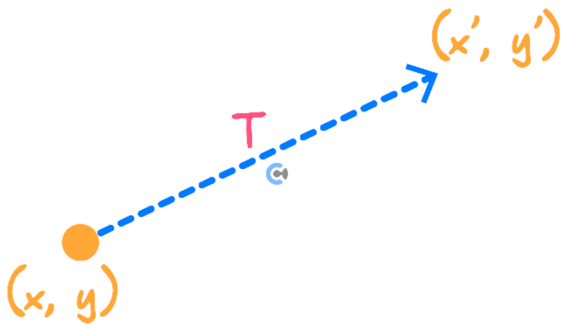
Website: contoureducation.com.au | **Phone:** 1800 888 300
Email: hello@contoureducation.com.au

VCE Mathematical Methods ¾
Transformations [0.3]
Workshop

Section A: Recap

Sub-Section: Image and Pre-Image

Image and Pre-Image



- The original coordinate is called the ________________.
- The transformed coordinate is called the ______________.

Pre-Image: (x, y)

Image: (x', y')

Space for Personal Notes

Sub-Section: Dilation

Dilation

Dilation by a factor a from the x-axis: $y' = ay$

Dilation by a factor b from the y-axis: $x' = bx$

NOTE: We are applying the transformations on (x, y) not (x', y').

Space for Personal Notes

Sub-Section: Reflection

Reflection

Reflection in the x-axis: $y' = -y$

Reflection in the y-axis: $x' = -x$

Space for Personal Notes

Sub-Section: Translation

Translation

Translation by c units in the positive direction of the x-axis: $x' = x + c$

Translation by d units in the positive direction of the y-axis: $y' = y + d$

Space for Personal Notes

Sub-Section: The Order of Transformations

The Order of Transformation

Order = BODMAS Order

Space for Personal Notes

Sub-Section: Interpreting the Transformation of Points

Interpretation of Transformations

- When the ________________ x' and y' are the subjects, we can read the transformation ________.

$$x' = x + 5 \rightarrow 5 \text{ right}$$

- When the ____________________ x and y are the subjects instead, we must read the transformation in the ________________ way.
- This includes the order of transformation!

$$x = x' - 5 \rightarrow 5 \text{ right}$$

NOTE: This includes the order of transformation!

TIP: It is best to make x' and y' the subject before you interpret the transformations.

Space for Personal Notes

Sub-Section: Applying Transformations to Functions

Transformation of Functions

- The aim is to get rid of the old variables, x and y, and have the new variables, x' and y', instead.

$$y = f(x) \rightarrow y' = f(x')$$

- Steps:

1. Transform the points.
2. Make x and y the subjects.
3. Substitute them into the function.

Space for Personal Notes

Sub-Section: Finding the Applied Transformations

Now, let's go backwards!

Reverse Engineering

- Steps:

1. Add the dashes (') back to the transformed function.
2. Make $f(\ )$ the subject.
3. Equate the LHS of the original and transformed functions to the RHS of the original and transformed functions.
4. Make x' and y' the subjects and interpret the transformations.

Space for Personal Notes

Section B: Warmup

Question 1

Consider the transformation:

$$T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (2x + 1, 3y - 2)$$

a. Find the image of the point $P(1,2)$ under T.

b. Describe the transformation, T, in DRT order.

c. Find the image of the curve, $y = \frac{1}{3}x^2$ under the transformation T.

Section C: Exam 1 (21 Marks)

Question 2 (4 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 4$.

a. Find the coordinates of all the axes intercepts of f. (1 mark)

b. Let g be the image of the graph of f under the following sequence of transformations:

- Dilation by a factor of $\frac{1}{2}$ from the y-axis.
- Dilation by a factor of 3 from the x-axis.
- Translation 1 unit to the left.

Find the rule for $g(x)$. (2 marks)

c. State the coordinates for the axes intercepts of g. (1 mark)

Question 3 (3 marks)

Consider the function: $f(x) = \frac{1}{2}(x+1)^2 - \frac{3}{2}$

Apply the following sequence of transformations to $f(x)$:

Dilation by a factor 3 from the x-axis.
Translated 4 units in the negative direction of the x-axis.
Reflection in the y-axis.
Translated 2 units in the positive direction of the y-axis.
Dilation by a factor of $\frac{1}{3}$ from the y-axis.

Space for Personal Notes

Question 4 (4 marks)

Let $f(x) = \frac{1}{3x+3}$.

a. The transformation T_1 given by:

$$T_1: \mathbb{R}^2 \to \mathbb{R}^2, T_1(x, y) = (x + a, by),$$

maps the graph of $y = f(x)$ onto the graph of $y = \frac{1}{x}$.

Find the values of a and b. (2 marks)

b. The transformation T_2 given by:

$$T_2: \mathbb{R}^2 \to \mathbb{R}^2, T_2(x, y) = (c(x + d), y),$$

maps the graph of $y = \frac{1}{x}$ onto the graph of $y = f(x)$.

Find the values of c and d. (2 marks)

Space for Personal Notes

Question 5 (7 marks)

Consider the cubic function:

$$f(x) = x^3 - 2x^2 - x + 2$$

a. Find the x-intercepts of the graph $y = f(x)$. (3 marks)

Let $g(x) = 2f(2x - k)$.

b. Find the sequence of transformations required for $f(x)$ to transform to $g(x)$. Give your answer in DRT order. (2 marks)

c. Find the value(s) of k such that, there is only one negative x-intercept for $g(x)$. (2 marks)

Question 6 (3 marks)

The image of the curve $y = \sqrt{16 - x^2}$ under a transformation T, has the equation $y = \sqrt{55 - 6x - x^2}$.

Find the sequence of transformations that make up T, with dilations before translations.

Section D: Tech Active Exam Skills

Calculator Tip: Finding Transformed Functions

- Save the function as $f(x)$.
- Substitute the x and y in terms of x' and y'.
- Solve for y!
- Can also apply the transformations directly to $f(x)$. Must make sure you interpret the transformations correctly or you can easily make a mistake doing this.

Mathematica UDF:

- ApplyTransformList[]

ApplyTransformList[*f[x] , {x, y}, list of transforms*]
Applies the list of transforms to f[x] in the chronological order.

```
ApplyTransformList[x^2, {x, y}, {x - 1, 2 x, y + 3}]
```

$4 + x + \frac{x^2}{4}$

```
ApplyTransformInvList[f[x], {x, y}, {x - 1, 2 x, y + 3}]
```

```
-3 + f[2 (-1 + x)]
```

```
ApplyTransformInvList[Sin[x], {x, y}, {x - π / 2, 2 y, y - 1 }]
```

$\text{Sin}\left[\frac{x}{2}\right]^2$

- ApplyTransformInvList[]

ApplyTransformInvList[*f[x] , {x, y}, list of transforms*]
Applies the list of transforms to f[x] in reverse order and as the inverse to the transforms of *ApplyTransformList.*

```
In[∘]:= ApplyTransformInvList[x^2, {x, y}, {x - 1, 2 * x, y + 3}]
Out[∘]= 1 - 8 x + 4 x^2

In[∘]:= ApplyTransformInvList[f[x], {x, y}, {x - 1, 2 * x, y + 3}]
Out[∘]= -3 + f[2 (-1 + x)]

In[∘]:= ApplyTransformInvList[2 * Cos[x] - 1, {x, y}, {x - Pi / 2, 2 * y, y - 1 }]
Out[∘]= Sin[x]
```

TI UDF:

- transform()

Transform a Function

$transform\left(\sin(x), x, \left\{x-\frac{\pi}{2}, 2\cdot y, y-1\right\}\right)$

▸ Translation $\frac{\pi}{2}$ units along the neg. x−dir.

$\cos(x)$

▸ Dilation by factor of 2 from the x−axis

$2\cdot\cos(x)$

▸ Translation −1 unit along the neg. y−dir.

$2\cdot\cos(x)-1$

Overview:

Apply any sequence of transformations to a function. The program will display the transformed function after each step.

Input:

```
transform(<function>, <variable>,
      <list of transformations>)
```

Other notes:

- The list of transformations can either be presented in a (horizontal or vertical) matrix of expressions or a list of expressions

- transform_inv()

Invert a Transformation

$transform\_inv\left(x^2, x, \{x-1, 2\cdot x, y+3\}\right)$

- Inverted Transformations:

$\left\{y-3, \frac{x}{2}, x+1\right\}$

- Translation -3 units along the neg. y-dir.

x^2-3

- Dilation by factor of $\frac{1}{2}$ from the y-axis

$4\cdot x^2-3$

- Translation 1 unit along the pos. x-dir.

$4\cdot x^2-8\cdot x+1$

Overview:

Find the preimage of a function under a list of transformations. The program will display the list of inverted transformations and the transformed function after each step.

Input:

```
transform_inv(<function>, <variable>,
      <list of transformations>)
```

Other notes:

- The list of transformations can either be presented in a row or column matrix, or a list of expressions

Space for Personal Notes

Section E: Exam 2 (21 Marks)

Question 7 (1 mark)

Find the possible transformation(s) for the function $f(x) = x^2$ to transform into $g(x) = 4x^2 + 4$.

A. Dilation by a factor of 4 from the y-axis, translation of 4 units in the positive direction of the y-axis.

B. Dilation by a factor of 4 from the y-axis, translation of 4 units in the negative direction of the y-axis.

C. Dilation by a factor of $\frac{1}{4}$ from the y-axis, translation of 4 units in the positive direction of the y-axis.

D. Dilation by a factor of $\frac{1}{2}$ from the y-axis, translation of 4 units in the positive direction of the y-axis.

Question 8 (1 mark)

Given that $f(x)$ is a function with a local minimum point at $(-2,3)$. The graph of $y = -2f(3x + 2) - 2$ must have which of the following?

A. Local minimum at $(-4, -8)$.

B. Local minimum at $\left(-\frac{4}{3}, -8\right)$.

C. Local maximum at $(-4, -8)$.

D. Local maximum at $\left(-\frac{4}{3}, -8\right)$.

Question 9 (1 mark)

There exists a function where dilating by a factor of 2 from the x-axis gives the same image as dilating it by a factor of $\frac{1}{4}$ from the y-axis. Which of the following could be the function?

A. $f(x) = x^2$

B. $f(x) = 2\sqrt{x}$

C. $f(x) = \sqrt{x} - 4$

D. $f(x) = \frac{1}{x}$

Question 10 (1 mark)

Which one of the following sequences of transformations is different from the rest?

A. Dilation by a factor of 2 from the x-axis, dilation by a factor of $\frac{1}{3}$ from the y-axis, reflection in the y-axis, translation 2 right, translation 4 up.

B. Dilation by a factor of 2 from the x-axis, dilation by a factor of $\frac{1}{3}$ from the y-axis, translation 2 left, translation 4 up, reflection in the y-axis.

C. Reflection in the y-axis, translation 6 left, translation 2 up, dilation by a factor of 2 from the x-axis, dilation by a factor of $\frac{1}{3}$ from the y-axis.

D. Translation 6 left, translation 2 up, reflection in the y-axis, dilation by a factor of $\frac{1}{3}$ from the y-axis, dilation by a factor of 2 from the x-axis.

Question 11 (1 mark)

The graph of the function f is obtained from the graph of the function g with rule $g(x) = 3\cos\left(x - \frac{\pi}{6}\right)$ by a dilation of a factor of $\frac{1}{2}$ from the x-axis, a reflection in the y-axis, a translation of $\frac{\pi}{6}$ units in the negative x-direction and a translation of 4 units in the negative y-direction, in that order.

The rule of f is:

A. $f(x) = \frac{3}{2}\cos\left(-x - \frac{\pi}{3}\right) - 4$

B. $f(x) = \frac{3}{2}\cos(-x) - 4$

C. $f(x) = \frac{3}{2}\cos(x) - 4$

D. $f(x) = -3\cos\left(\frac{x}{2} - \frac{\pi}{3}\right) - 4$

E. $f(x) = \frac{3}{2}\cos\left(-x + \frac{\pi}{3}\right) - 4$

Space for Personal Notes

Question 12 (1 mark)

The curve with the equation $y = e^x$ is transformed by a dilation from the y-axis by a scale factor of 2, a translation by one unit to the left in the x-direction and a translation of two units downwards in the y-direction. The equation of the transformed curve is:

A. $y = 0.5e^{x-1} - 2$

B. $y = 2e^{x-1} - 2$

C. $y = e^{0.5(x+1)} - 2$

D. $y = e^{2(x+1)} - 2$

Question 13 (7 marks)

Consider the function, $f: \mathbb{R} \to \mathbb{R}, f(x) = (x-1)(x+1)(2x-1)(x+2)$.

a. State the values of x for which, $f(x) = 0$. (1 mark)

b. The graph of $y = f(x)$ is translated a units to the right, where $a \in \mathbb{R}$, to become the graph $y = g(x)$. Find the values of a for which, the graph $y = g(x)$ has:

i. Three positive x-intercepts. (2 marks)

ii. Four negative x-intercepts. (1 mark)

Let h be the function, $h : \mathbb{R} \to \mathbb{R}, h(x) = (x-1)^2(x+2)^2$, which has a local maximum at $\left(-\frac{1}{2}, \frac{81}{16}\right)$.

Let k be the function, $k : \mathbb{R} \to \mathbb{R}, k(x) = 2x^2(2x+3)^2$, which has a local maximum at $\left(-\frac{3}{4}, \frac{81}{32}\right)$.

c. Using translations only, describe a sequence of transformations on k, for which its image would have a local maximum at the same coordinates as that of h. (1 mark)

d. Find a sequence of transformations in the order DRT that maps the graph of $y = h(x)$ to the graph of $y = k(x)$. (2 marks)

Space for Personal Notes

Question 14 (8 marks)

Consider the function, $f: [-4,4] \to R, f(x) = x^2 + k$, where k is a real number.

a. Consider the transformation, $T(x, y) = \left(2x + 1, \frac{1}{3}y - 1\right)$. Find the transformed function of $y = f(x)$ under the transformation T, and also state its domain. (3 marks)

b. Find the inverse transformation of T and call it T^{-1}. (2 marks)
NOTE: The inverse transformation is a transformation which works in opposite to the original transformation.

c. Using T^{-1}, find the equation of the pre-image of $y = f(x)$ under the transformation T.
State the domain also. (3 marks)
NOTE: Pre-image is the function you would have had before the transformation.

Section F: Extension Exam 1 (14 Marks)

Question 15 (5 marks)

$$f(x) = 2x^2 + 4$$

$$g(x) = (4x - 3)^2 - 1$$

a. Identify a sequence of transformations that take $f(x)$ to $g(x)$ **without** the use of dilation from the y-axis. (2 marks)

b. Identify a sequence of transformations that take $f(x)$ to $g(x)$ **without** the use of dilation from the x-axis. (2 marks)

c. Assume that the domain of f and the domain of g are appropriately restricted such that, f^{-1} and g^{-1} both exist. Identify the transformations that take $f^{-1}(x)$ to $g^{-1}(x)$ **without** the use of dilation from the x-axis. (1 mark)

Question 16 (5 marks)

Consider the function, $f(x) = 2\sqrt{x-k}$, where $k \in R$.

a. Find a sequence of transformations that map $y = x^2$ where $x \geq 0$,to $y = f^{-1}(x)$. (2 marks)

b. Find the value(s) of k such that, $f(x)$ and $f^{-1}(x)$ do not intersect each other. (3 marks)

Question 17 (4 marks)

The image of the curve, $y = 3\sqrt{x^2 + 4x + 7} - 1$ under a transformation T, has the equation $y = \sqrt{4x^2 - 16x + 19}$. Find the sequence of transformations that make up T, with dilations before translations.

Section G: Extension Exam 2 (17 Marks)

Question 18 (1 mark)

The function tangent to $g(x)$ at $x = 1$ has an equation $y = 2x - 4$. What is the equation of the tangent of $2g(2x) + 1$ at $x = \frac{1}{2}$?

A. $y = 4x - 4$

B. $y = 4x - 5$

C. $y = 8x - 7$

D. $y = 8x - 11$

Question 19 (1 mark)

The transformation which maps $f(x) = \log_2(x)$ to $g(x) = 2\log_2(2x)$ is:

A. Dilated by factor 2 from the x-axis, translated 2 units in the positive direction of the y-axis.

B. Dilated by factor 2 from the x-axis, dilated by factor 2 from the y-axis.

C. Dilated by factor 2 from the x-axis, translated 1 unit in the positive direction of the y-axis.

D. Dilated by factor $\frac{1}{2}$ from the x-axis, dilated by factor 2 from the y-axis.

Question 20 (1 mark)

The transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps the graph of $y = 3 - \sqrt{\frac{x+1}{2}}$, onto the graph of $y = \sqrt{x}$ has the rule:

A. $T(x, y) = (2x + 1, -y - 3)$

B. $T(x, y) = \left(\frac{x+1}{2}, 3 - y\right)$

C. $T(x, y) = (2x - 1, 3 - y)$

D. $T(x, y) = \left(\frac{x+1}{2}, -y - 3\right)$

Question 21 (1 mark)

The image of the curve, $y = \sqrt{x^2 + 4}$ under the transformation T, has the equation $y = \sqrt{x^2 + 4x + 40}$.
The transformation T could be described as:

A. A dilation by factor 3 from the y-axis followed by dilation by factor 2 from the x-axis and a translation 3 units to the right.

B. A dilation by factor $\frac{1}{3}$ from the y-axis followed by a dilation by factor 3 from the x-axis and a translation 2 units to the right.

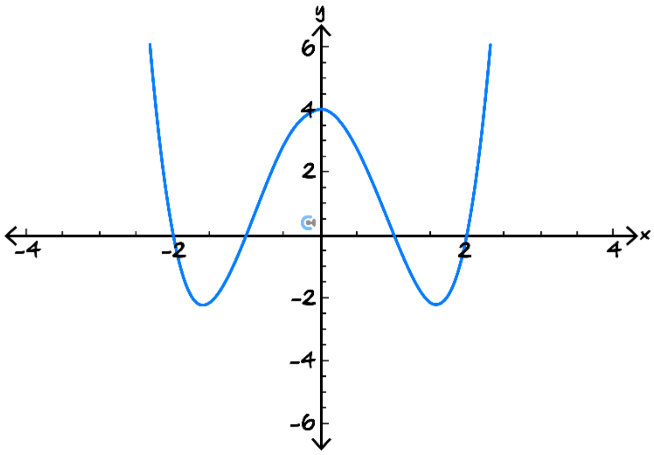
C. A dilation by factor 2 from the y-axis followed by a dilation by factor 3 from the x-axis and a translation 2 units to the left.

D. A dilation by factor 3 from the y-axis followed by a dilation by factor 3 from the x-axis and a translation 2 units to the left.

Space for Personal Notes

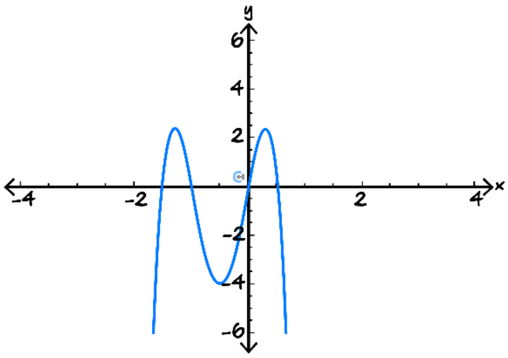
Question 22 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

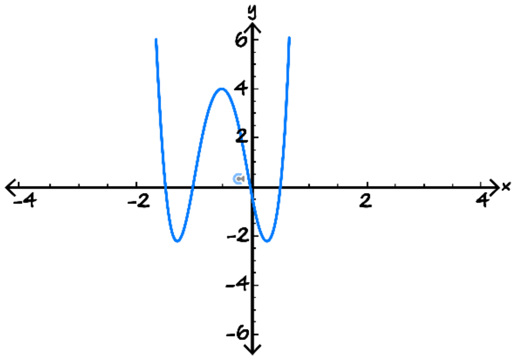


The corresponding part of the graph of $y = -f(2x - 1)$ is best represented by:

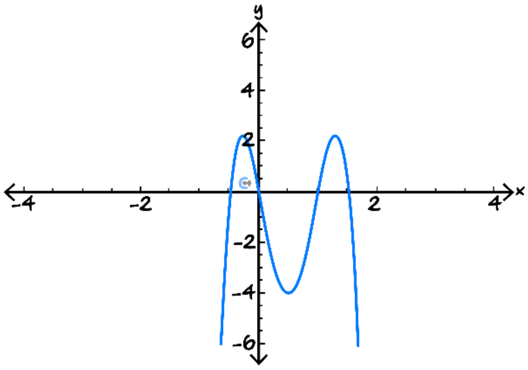
A.



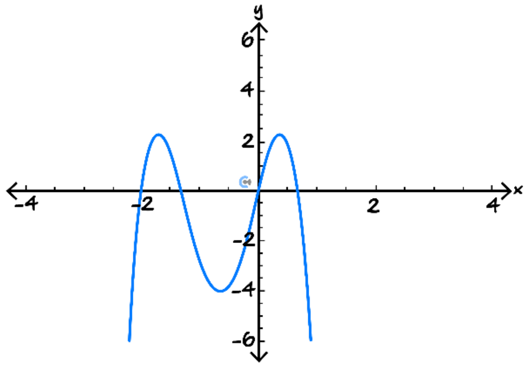
B.



C.



D.



Space for Personal Notes

Question 23 (12 marks)

Let $f(x) = \log_e(x+3) + \log_e(x)$.

a. State the domain of $f(x)$. (1 mark)

Consider the function f_1 where, $f_1(x) = \log_e(x+3+k) + \log_e(x+k)$ and k is a negative real constant.

b. State the transformation required to get the graph of f to the graph of f_1. Give your answer in terms of k. (1 mark)

c. When $k = -2$, the line $y = \frac{5x}{4} - \frac{15}{4} + \log_e(4)$ is tangent to the graph of $y = f_1(x)$ when $x = 3$. When $k = -3$, find the equation of the line that is tangent to the graph of $y = 2f_1(x) + 1$ when $x = 4$. (2 marks)

d. Find the value of x for which, $f_1'(x) = 1$. Express your answer in terms of k. (2 marks)
NOTE: $f'(x)$ is the derivative of f.

e. Hence or otherwise, find the value of k so that, the graphs of f_1 and ${f_1}^{-1}$ have only one point of intersection. Give your answer correct to three decimal places. (2 marks)
NOTE: f and its inverse will be tangential if their point of intersection have the same gradient.

Now consider the function f_2 where, $f_2(x) = \log_e\left(\frac{x}{a} + 3\right) + \log_e\left(\frac{x}{a}\right)$ and a is a positive real constant.

f. State the transformation required to get the graph of f to the graph of f_2. Give your answer in terms of a. (1 mark)

g. Find the value of a so that, the graphs of f_2 and ${f_2}^{-1}$ have only one point of intersection. Give the coordinates of this point of intersection. Give your answers correct to three decimal places. (3 marks)

Space for Personal Notes