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VCE Mathematical Methods ¾
Functions & Relations Exam Skills [0.2]
Workshop Solutions

Section A: Recap

Sub-Section: Maximal Domains

Starting with a domain!

Maximal Domain

- **Definition**: The largest possible set of input values (elements of the domain) for which the function is well-defined.
- **Three Important Rules:**

Functions	Maximal Domain
$\sqrt{z}$	$z \geq 0$
$\log(z)$	$z > 0$
$\frac{1}{z}$	$z \neq 0$

- **Steps:**

1. Find the restriction of the inside.
2. Sketch the graph if needed.
3. Solve for domain.

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Sub-Section: Domain of Sum, Difference and Product of Functions

What about a domain of the sum of two functions?

Sums, Differences and Products of Functions

- Rules:

$$(f+g)(x) = \boxed{f(x)+g(x)}$$

$$(f-g)(x) = \boxed{f(x)-g(x)}$$

$$(f\times g)(x) = \boxed{f(x)\times g(x)}$$

- Idea:

$$\text{Domain of sum or product of two functions} = \boxed{\text{Intersection}} \text{ of the two domains}$$

- Steps:

1. Find the domain of each function.

2. Find the intersection (draw a number line if needed).

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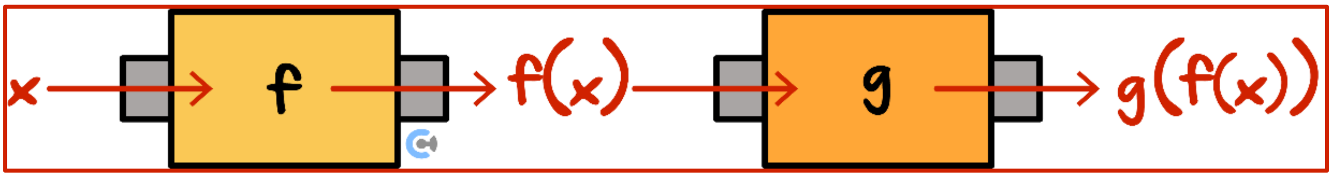
Sub-Section: Basics of Composition

What was the "composition" of functions?

Composite Functions



- **Definition**: A ___series___ of functions.
- **Representation of the above:**

$$y = g(f(x)) = g \circ f(x)$$

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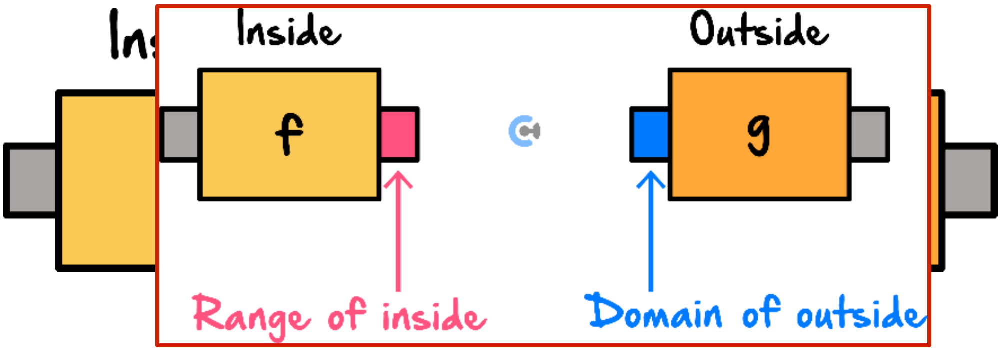
Sub-Section: Validity of Composite Functions

Did composite functions work all the time?

Validity of Composite Functions



- Output of $f(x)$: Range of Inside. *(Label Above)*
- Input of $g(x)$: Domain of outside. *(Label Above)*
- Composite Function is only valid if:

$$\textit{Range of Inside} \subseteq \textit{Domain of Outside}$$

- Acronym:

RIDO = Range of inside, domain of outside.

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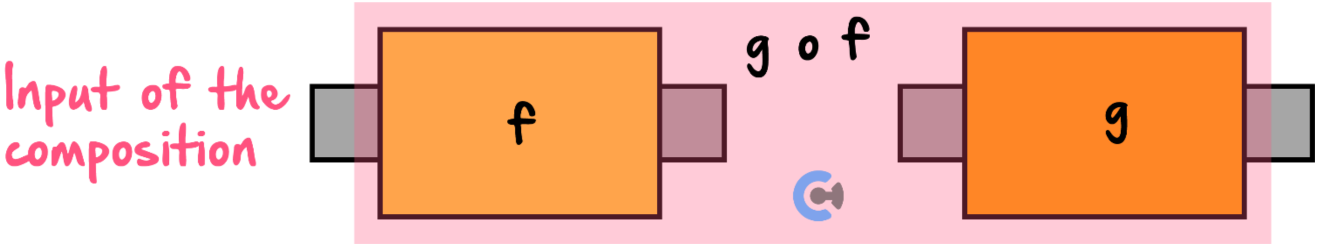
Sub-Section: Domain of Composite Functions

How did we find the domain of a composite function?

Domain of Composite Functions



$$\textbf{Domain of Composite} = \textbf{Domain of Inside}$$

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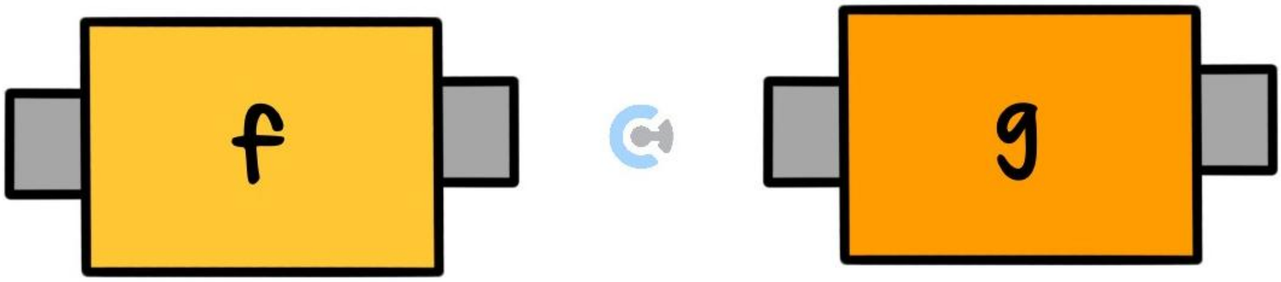
Sub-Section: Range of Composite Functions

Range of the Composite Functions



Range of Composite $\subseteq$ *Range of the Outside*

- **Finding the range of composition function**: Use the domain and the rule, just like another function.

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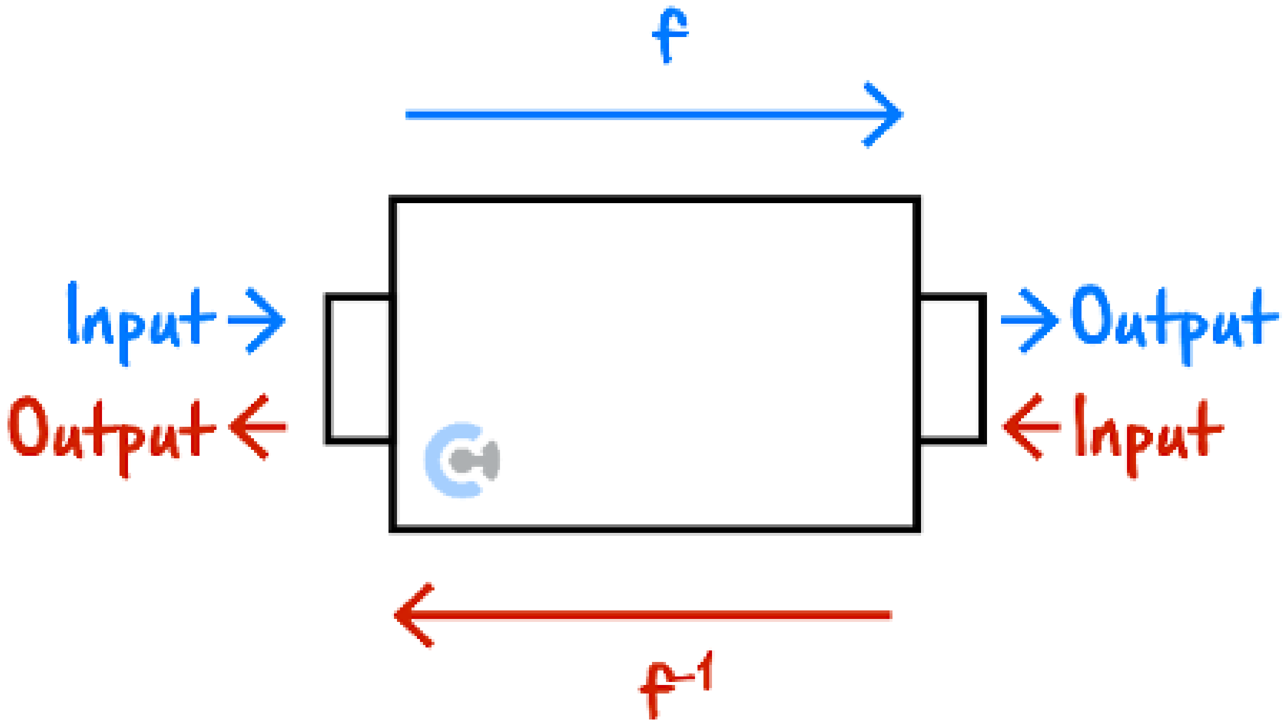
Sub-Section: Basics of Inverses

What does "Inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the opposite.



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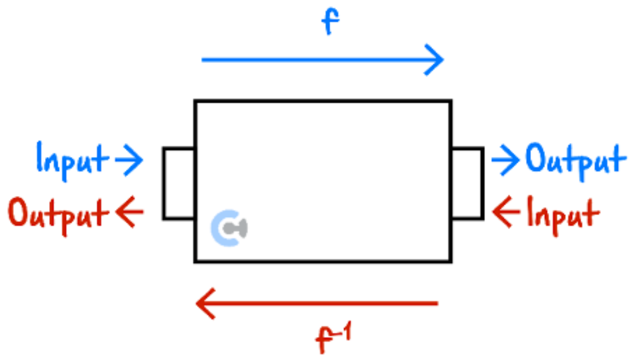
Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

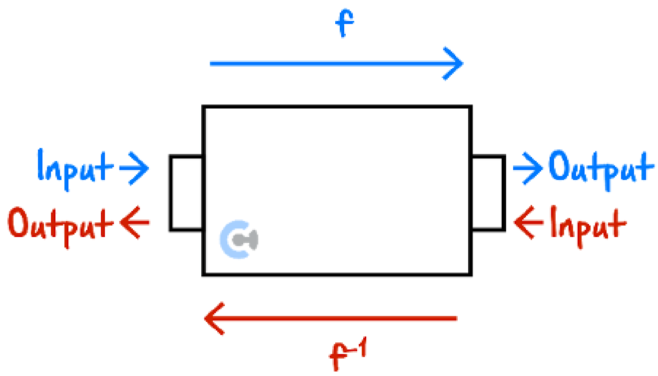
Solving for an Inverse Relation

- Swap x and y.



NOTE: $f(x) = y$.

Domain and Range of Inverse Functions



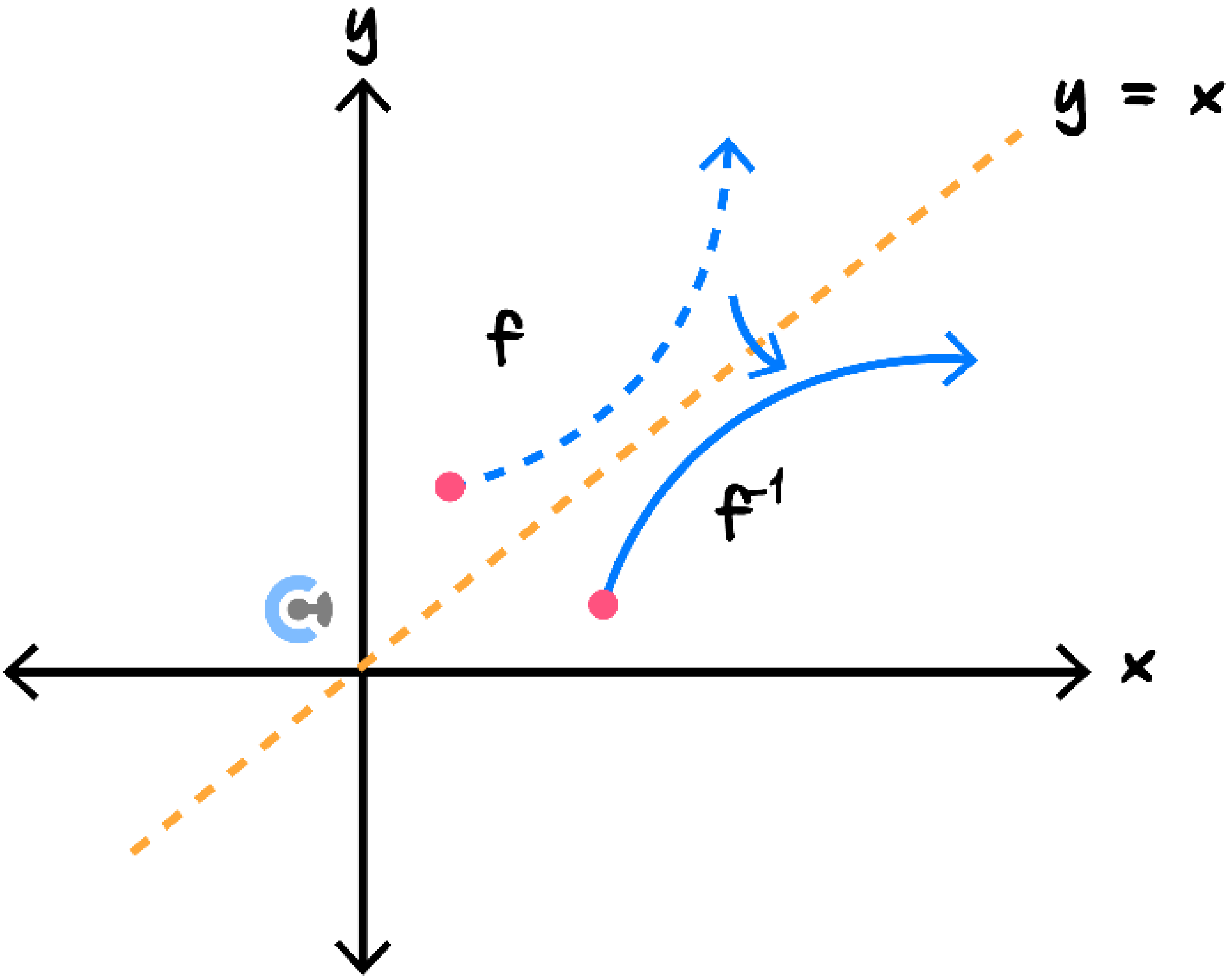
$Dom\ f^{-1} =$ $Ran\ f$

$Ran\ f^{-1} =$ $Dom\ f$

Sub-Section: Symmetry Around $y = x$

Why does this happen?

Symmetry of Inverse Functions



- Inverse functions are always symmetrical around $y = x$.

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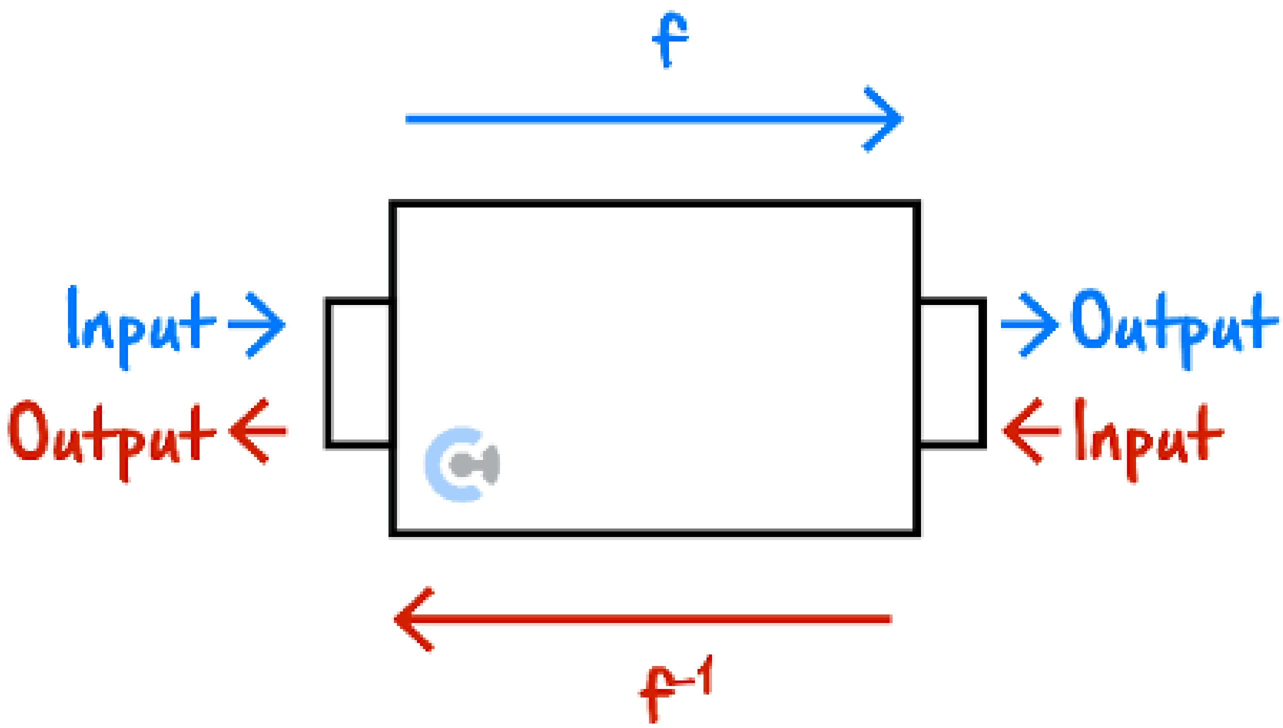
Sub-Section: Validity of Inverse Function

Does an inverse function always exist?

Validity of Inverse Functions



- Requirement for Inverse Function:

f needs to be **1:1**.

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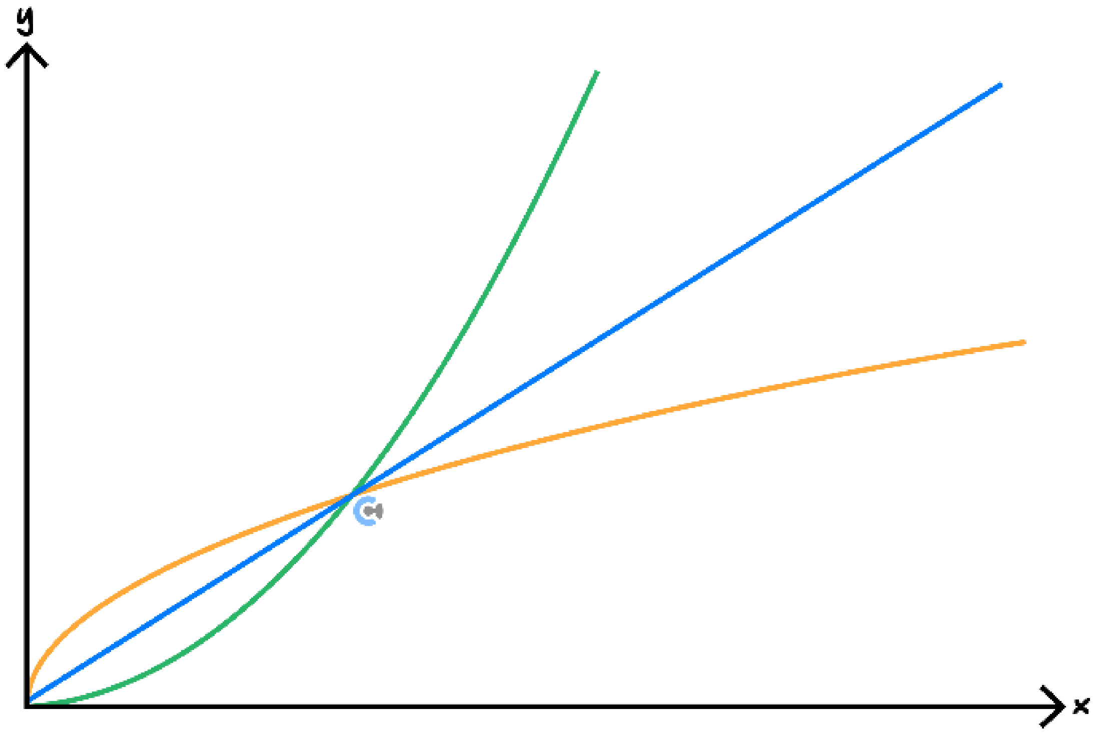
Sub-Section: Intersection Between Inverses

Where do inverses meet?

Intersection Between a Function and its Inverse



- Equate with $y = x$ instead.

$$f(x) = x \text{ OR } f^{-1}(x) = x$$

- We cannot do this when the function is decreasing function.

NOTE: This only works for an increasing function.

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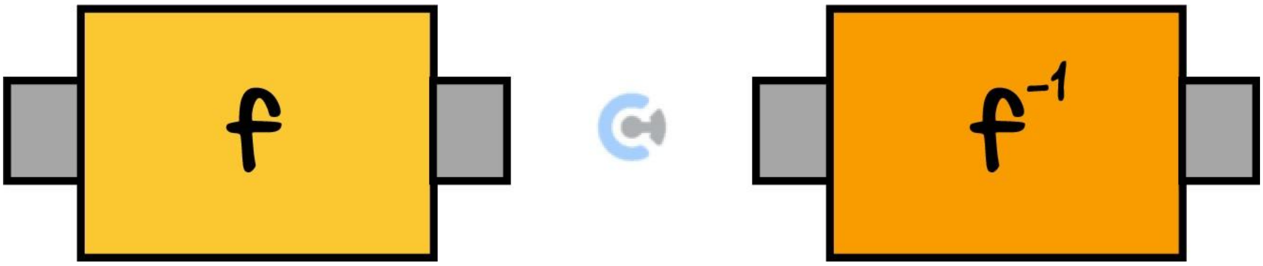
Sub-Section: Composition of Inverses

Composition of Inverse Functions



$$f \circ f^{-1}(x) = x, \quad \text{for all } x \in \text{dom } f^{-1}$$

$$f^{-1} \circ f(x) = x \quad \text{for all } x \in \text{dom } f$$

NOTE: Domain = Domain of Inside.

Space for Personal Notes

Sub-Section: Find a New Domain to Fix Composite Functions

Fixing Broken Composite Functions

- **Aim:** Restrict the domain of the inside function so that the range of the inside function fits inside the domain of the outside.

- **Steps:**

  1. **Write down the RIDO statement** with the domain of the outside (as it is fixed).

  2. **Sketch the inside function** to see what domain is needed.

Space for Personal Notes

Sub-Section: Find the Range of Complex Composite Functions

Finding Range of Complex Composite Functions

- **Aim:** Find the range of complicated functions.
- **Steps:**

1. Break the function into __composition__ of two simple functions.
2. Follow the __box diagram__ to find the range.

Space for Personal Notes

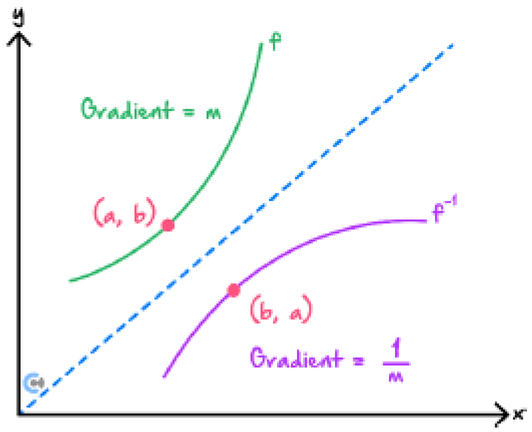
Sub-Section: Find the Gradient of Inverse Functions

This is a fun application of inverse with calculus!

REMINDER: Gradient of a Point

$$\text{Gradient at a point} = \frac{dy}{dx}$$

Gradient of an Inverse



$$\text{If Gradient of } f \text{ at } (a, f(a)) = m$$

$$\text{Gradient of } f^{-1} \text{ at } \boxed{(f(a), a) = \frac{1}{m}}$$

Space for Personal Notes

Section B: Warm Up

INSTRUCTION: 5 Minutes Writing.

Question 1

Consider $f(x) = \sqrt{2x}$ and $g(x) = 2x - 1$, both defined over their maximal domains.

a. Is $f(g(x))$ defined?

No. The range of g is R, but the maximal domain of f is only $x \geq 0$. Since range of inside is not a subset of domain of outside, $f(g(x))$ cannot exist.

b. Find the large restricted domain of g such that $f(g(x))$ is defined.

Step 1
Range of inside $\subseteq [0, \infty)$

Step 2
$2x - 1$ needs to be a subset of $[0, \infty)$

Step 3
Graph of $y = 2x - 1$ shows that $x \geq 1/2$ is the largest domain for which the range will be $[0, \infty)$

c. Find the range of $y = \log_3(x^2 + 9)$.

$[2, \infty)$

d. Consider the one-to-one function h with the following properties:

$$h(3) = 2 \text{ and } h'(3) = 5.$$

Find the gradient of h^{-1} at $x = 2$.

$$\frac{1}{5}$$

Space for Personal Notes

Section C: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 26 Minutes Writing.

Question 2 (7 marks)

Consider the two functions:

$$f : D \to \mathbb{R}, f(x) = \frac{1}{x+2} + 1$$

$$g: [0, \infty) \to \mathbb{R}, g(x) = \sqrt{x+4}$$

where D is a restricted domain of f.

a. Define g^{-1}, the inverse function of g. (2 marks)

$x = \sqrt{y+4} \implies y = x^2 - 4$. dom g^{-1} = ran $g = [0, \infty)$

$g^{-1} : [0, \infty) \to \mathbb{R}, g^{-1}(x) = x^2 - 4$.

b. The gradient of g^{-1} when $x = 3$ is 6. Find the gradient of g when $x = 5$. (2 marks)

$g(5) = 3 \implies g'(5) = \frac{1}{6}$

c. Find the restricted domain, D, of the function f such that the composite function $g \circ f$ is defined. (3 marks)

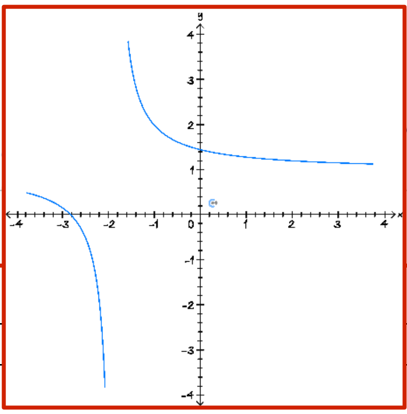
Solution: We require that ran $f \geq 0$, therefore $\frac{1}{x+2} + 1 \geq 0$. Solve

$$\frac{1}{x+2} + 1 = 0$$
$$x + 2 = -1$$
$$x = -3$$

Then by considering the shape/a rough sketch of the hyperbola we see that $x \leq -3$ or $x > -2$. Therefore,

$$D = (-\infty, -3] \cup (-2, \infty) = \mathbb{R} \setminus (-3, -2]$$

```
Plot[1/(x + 2) + 1, {x, -5, 5}]
```



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Question 3 (6 marks)

Consider the functions, $f(x) = \frac{1}{x-2}$ and $g(x) = 2x + 3$ defined on their maximal domains.

Let $h(x) = g(f(x))$.

a. Write down the rule and domain of $h(x)$. (2 marks)

Solution: $h : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $h(x) = \dfrac{2}{x-2} + 3$

b. Define the inverse function, h^{-1}, of h. (2 marks)

Solution:

$$x = \frac{2}{y-2} + 3$$
$$x - 3 = \frac{2}{y-2}$$
$$y = \frac{2}{x-3} + 2$$

dom h^{-1} = ran $h = \mathbb{R} \setminus \{3\}$. Therefore,

$$h^{-1} : \mathbb{R} \setminus \{3\} \to \mathbb{R},\ h^{-1}(x) = \frac{2}{x-3} + 2$$

c. Find the point(s) of intersection between h and h^{-1}. (2 marks)

Solution: Intersect on the line $y = x$.

$$\frac{2}{x-2} + 3 = x$$
$$2 = (x-2)(x-3)$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$
$$x = 1, 4$$

Therefore points of intersection are $(1, 1)$ and $(4, 4)$

Question 4 (7 marks)

Consider the function:

$$f: [a, \infty) \to \mathbb{R}, f(x) = \frac{1}{2}x^2 - 2x + 5$$

a. Find the largest value of a such that the inverse function f^{-1} exists. (1 mark)

Solution: $f(x) = \frac{1}{2}(x-2)^2 + 3$

Therefore, $a = 2$

b. Define f^{-1}. (2 marks)

Solution: dom f^{-1} = ran $f = [3, \infty)$ and ran f^{-1} = dom $f = [2, \infty)$

$$x = \frac{1}{2}(y-2)^2 + 3$$

$$2(x-3) = (y-2)^2$$

$$y = \pm\sqrt{2x-6} + 2$$

By considering ran f^{-1} conclude that

$$f^{-1}: [3, \infty) \to \mathbb{R}, f^{-1}(x) = 2 + \sqrt{2x-6}.$$

c. Given that f is an increasing function, show that f and f^{-1} do not have any points of intersection. (2 marks)

Solution: Would intersect on the line $y = x$. Consider

$$\frac{1}{2}x^2 - 2x + 5 = x$$

$$\frac{1}{2}x^2 - x + 5 = 0$$

Then $\Delta = 1 - 4 \times \frac{1}{2} \times 5 = -9 < 0$. So the equation has no real solutions. Therefore no intersection.

Let $g:\ \mathbb{R} \to \mathbb{R}, g(x) = \frac{1}{2}x^2 - 2x + 5$ and $h:\ \mathbb{R} \to \mathbb{R}, h(x) = 2^x$.

d. Find the range of $h(g(x))$. (1 mark)

$g(x)$ has a local minimum at $(2,3)$. Therefore ran $h(g(x)) = [8, \infty)$

e. Find the minimum value of c such that $h\big(g(x+c)\big)$ is one-to-one over the interval $[1, \infty)$. (1 mark)

$c = 1$

Space for Personal Notes

Section D: Technology Warmup

INSTRUCTION: 5 Minutes Writing.

Calculator Commands: Finding the domain and range

TI

- domain $(f(x), x)$, $f\ Min$ and $Fmax$.

Define $f(x)=\sqrt{9-x^2}$	*Done*
domain$(f(x),x)$	$-3\le x\le 3$
fMin$(f(x),x)$	$x=-3$ or $x=3$
fMax$(f(x),x)$	$x=0$
$f(3)$	0
$f(0)$	3

TI-UDF

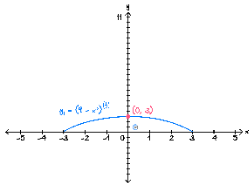
Analyse a Function: Find intercepts, critical points and their nature, maximal domain, asymptote.	`analyse(<function>, <variable>)` `analysed(<function>, <variable>, <lower bound>, <upper bound>)`

$$analysed\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

- Start Point: $\left[-5 \quad \frac{262}{77}\right]$
- End Point: $\left[5 \quad \frac{-316}{529}\right]$
- Maximal Domain:
  $x\neq-1.68469$ and
  $x\neq-0.629579$ and
  $x\neq0.314273$ and
  $-5\le x\le 5$

Casio Classpad

- Graph the function and use G-Solve to find min and max values for the range.



Mathematica

```
In[127]:= f[x_] := √9 - x^2
In[128]:= FunctionDomain[f[x], x]
Out[128]= -3 ≤ x ≤ 3

In[129]:= FunctionRange[f[x], x, y]
Out[129]= 0 ≤ y ≤ 3
```

Mathematica UDF :

- FInfo [f [x], $\{x, x\ min, x\ max\}, y$]

  Returns useful information about a function, including derivative, domain, range, period, horizontal intercepts, vertical intercepts, stationary points, inflexion points, left and sided asymptotes, oblique asymptotes and vertical asymptotes.

$$\text{FInfo}\left[\frac{x^2-1}{x\left(x^2-3\right)}, \{x, -\text{Infinity}, \text{Infinity}\}, y\right]$$

The function is $\frac{x^2-1}{x\left(x^2-3\right)}$

The derivative is $-\frac{x^4+3}{x^2\left(x^2-3\right)^2}$

Domain: $x<-\sqrt{3} \vee -\sqrt{3}<x<0 \vee 0<x<\sqrt{3} \vee x>\sqrt{3}$

Range: $y\in\mathbb{R}$

Period: 0

Horizontal Intercepts: $\{-1, 1\}$

Vertical Intercepts: None

Stationary points: $\{\}$

Inflexion points: {{-0.871..., -0.123...}, {0.871..., 0.123...}}

Left sided asymtote: $y=0$

Right sided asymtote: $y=0$

Oblique asymtote: $\{0\}$

Vertical asymtote: $\{x=0, x=-\sqrt{3}, x=\sqrt{3}\}$

Calculator Commands: Finding the composite function

TI

Define $f(x)=\ln(x)$	*Done*
Define $g(x)=x^2+3$	*Done*
$f(g(x))$	$\ln(x^2+3)$

CASIO

```
define f(x) = ln(x)
                                done
define g(x) = x^2+3
                                done
f(g(x))
                             ln(x^2+3)
```

Mathematica

```
In[141]:= f[x_] := Log[x]
In[142]:= g[x_] := x^2 + 3
In[143]:= f[g[x]]
Out[143]= Log[3 + x^2]
```

Calculator Commands: Finding the inverse function

TI

Define $f(x)=x^2+4\cdot x+9$	*Done*
solve$(f(y)=x,y)$	$y=-(\sqrt{x-5}+2)$ or $y=\sqrt{x-5}-2$

CASIO

```
define f(x) = x^2+4x+9
                                done
solve(f(y)=x, y)
               {y=-√(x-5)-2, y=√(x-5)-2}
```

Mathematica

```
In[154]:= f[x_] := x^2 + 4 x + 9
In[155]:= Solve[f[y] == x, y]
Out[155]= {{y → -2 - √(-5 + x)}, {y → -2 + √(-5 + x)}}
```

NOTE: It doesn't tell us which branch is correct.

Space for Personal Notes

Question 5 Tech-Active.

Find the domain and range of $f(x) = \sqrt{\frac{x^2-1}{x^2}}$.

```
In[20]:= f[x_] := Sqrt[(x^2 - 1)/(x^2)]

In[21]:= FunctionDomain[f[x], x]
Out[21]= x ≤ -1 || x ≥ 1

In[22]:= FunctionRange[f[x], x, y]
Out[22]= 0 ≤ y < 1
```

Space for Personal Notes

Section E: Exam 2 (23 Marks)

INSTRUCTION: 23 Marks. 30 Minutes Writing.

Question 6 (1 mark)

The function f defined by $f\ A \rightarrow \mathbb{R}, f(x) = (x-2)^2 + 3$ will have an inverse function if its domain A is:

A. $\mathbb{R}$

B. $(-\infty, 3]$

C. $[3, 6]$

D. $[0, \infty)$

Question 7 (1 mark)

The linear function $f : D \rightarrow \mathbb{R}, f(x) = 2 - x$ has a range of $[-5,1)$. The domain of f is:

A. $(-5,1]$

B. $(-1,5]$

C. $(1,7]$

D. $[7,1)$

Space for Personal Notes

Question 8 (1 mark)

Let f be a one-to-one differentiable function, and the following values are known:

$$f(2) = 3, f(3) = 4, f'(2) = 5, \text{ and } f'(3) = 6$$

Let $g(x) = f^{-1}(x)$. The value of $g'(3)$ is:

A. $\frac{1}{5}$

B. $\frac{1}{7}$

C. $\frac{1}{4}$

D. $\frac{1}{6}$

Question 9 (1 mark)

Let $f : [-1,3] \to \mathbb{R}, f(x) = 2x - 1$ and $g: D \to \mathbb{R}, g(x) = x^2 - 1$.

The largest interval D such that $f \circ g$ exists is:

A. $(0,2)$

B. $[0,2]$

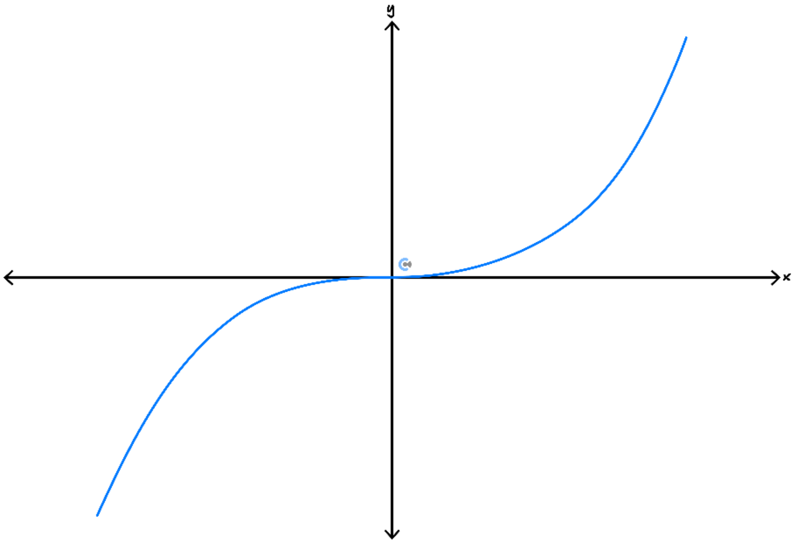
C. $[-2,2]$

D. $[0,3]$

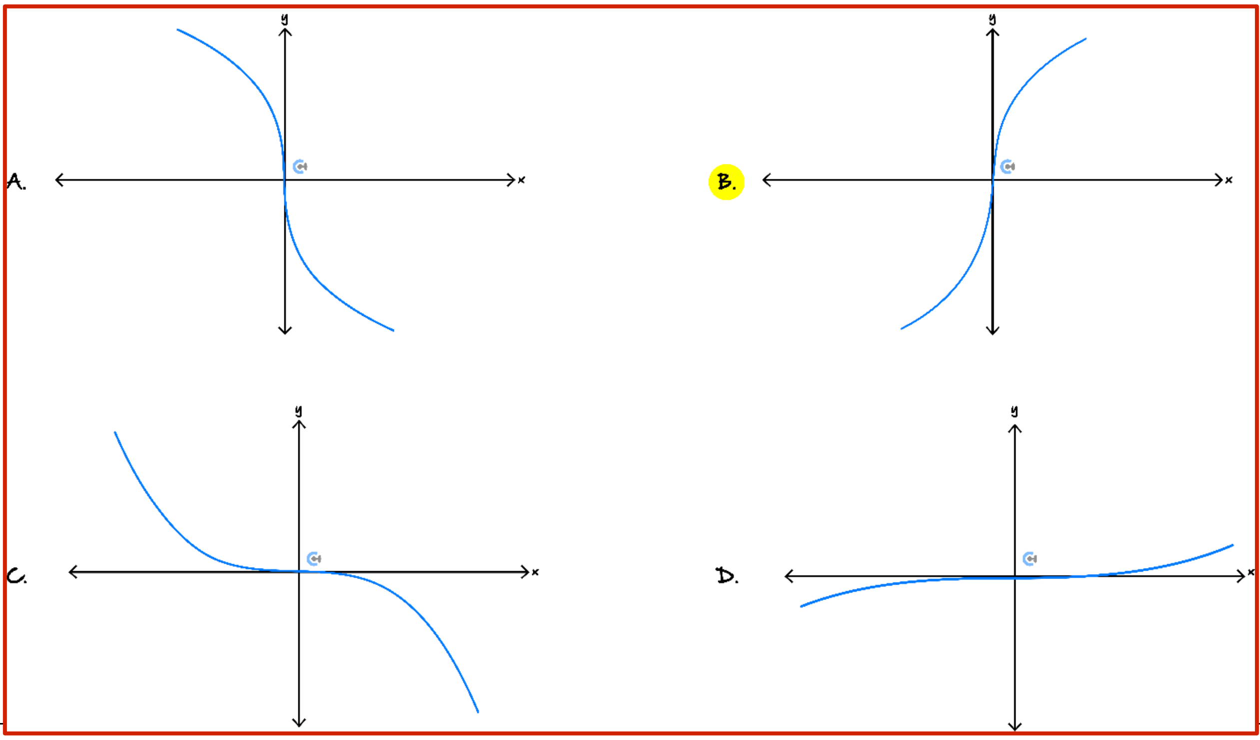
Space for Personal Notes

Question 10 (1 mark)

Part of the graph of $y = f(x)$ is shown below.



The inverse function f^{-1} is best represented by:



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Question 11 (8 marks)

Health insurance provides MediBear (C_m) and Bopa (C_b) offer cost-based health insurance plans represented by the functions:

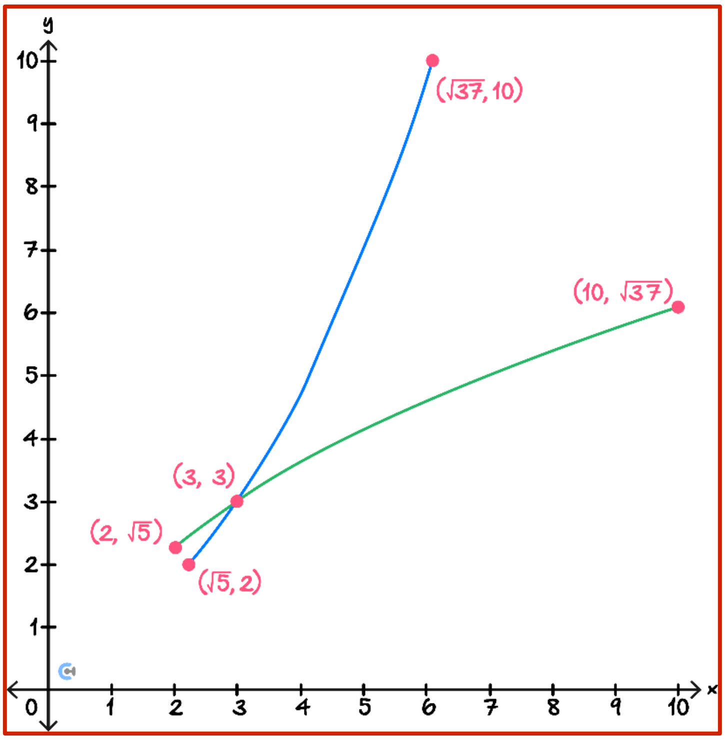
$$C_m = \sqrt{4x - 3}, 2 \leq x \leq 10 \text{ and } C_b = C_m^{-1}(x)$$

where C represents the amount paid for the plan by the customers in tens of dollars and x represents the plan's benefits rating.

a. Define the function C_b. (2 marks)

$C_b : [\sqrt{5}, \sqrt{37}] \to \mathbb{R}, C_b(x) = \frac{1}{4}(x^2 + 3)$

b. Graph the two functions on the axes below. Label all endpoints with coordinates. (2 marks)



c. What is the maximum plan benefit rating for a plan offered by Bopa? Give your answer correct to one decimal place. (1 mark)

$\sqrt{37} \approx 6.1$

d. What plan benefit rating results in both plans having the same cost? (1 mark)

$3.$

e. Find the values of x for which $C_m < C_b$. (2 marks)

$[3, 10]$

Space for Personal Notes

Question 12 (10 marks)

Consider the function $f : [a, \infty) \to \mathbb{R}, f(x) = x^2 - 2x - 2$.

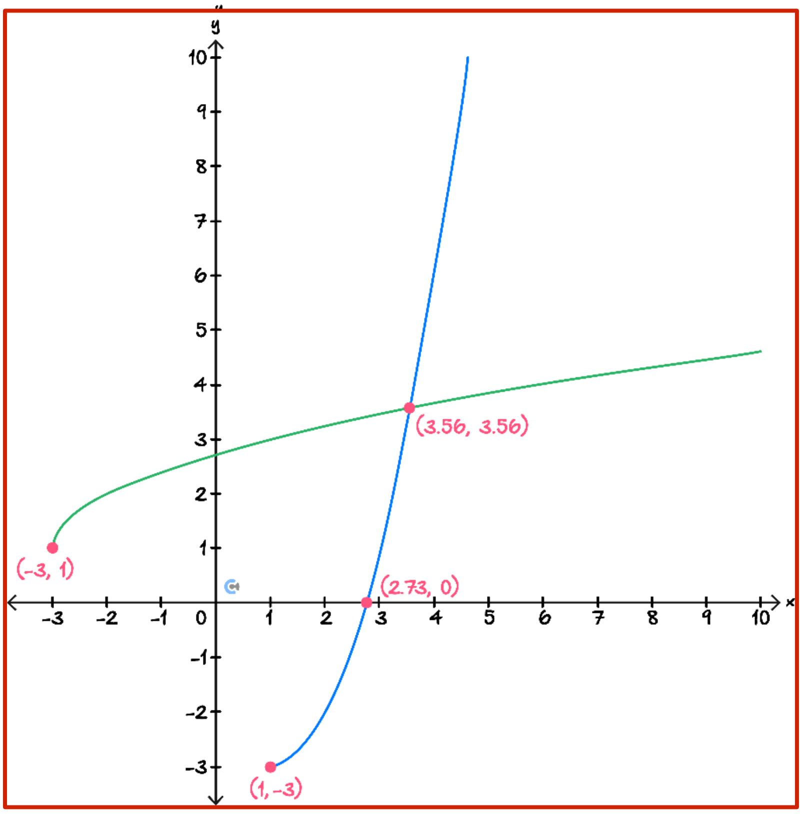
a. State the smallest value of a for which f has an inverse function. (1 mark)

$a = 1$

b. Define the inverse function, f^{-1}. (2 marks)

$f^{-1} : [-3, \infty) \to \mathbb{R},\ f^{-1}(x) = 1 + \sqrt{x+3}$

c. Sketch the graphs of f and its inverse on the axes below. Label all axes intercepts, endpoints and points of intersection with coordinates correct to two decimal places. (3 marks)



d. Let f and f^{-1} intersect at the point P. It is known that f has a gradient of $1+\sqrt{17}$ at P. Find the acute angle made by the tangents to f and f^{-1} at P. Give your answer in degrees correct to two decimal places. (2 marks)

$$\theta = \left|\tan^{-1}(1+\sqrt{17}) - \tan^{-1}\left(\frac{1}{1+\sqrt{17}}\right)\right| \approx 67.91^\circ$$

Consider the function $g : [2, \infty) \to \mathbb{R}, g(x) = \log_2(x+1)$.

e. Find all possible values for a such that $g(f(x))$ does not exist. (1 mark)

To not exist we need ran $f < 2 \implies a \in (1-\sqrt{5}, 1+\sqrt{5})$

f. If $a = 4$, find the range of $g(f(x))$ when it exists. (1 mark)

$[\log_2(7), \infty)$

Let's take a BREAK (Extension Stream)!

Space for Personal Notes

Section F: Extension Exam 1 (13 Marks)

INSTRUCTION: 13 Marks. 20 Minutes Writing.

Question 13 (5 marks)

Consider the two functions $f(x) = \frac{1}{x-2}$ and $g(x) = 1 + \cos(x)$ defined on their maximal domains.

a. Determine whether or not the functions $f \circ g(x)$ or $g \circ f(x)$ exist, and justify your answer. If the composite function exists, state its rule and domain. (2 marks)

Solution: $f \circ g(x)$ does not exist because ran $g = [0, 2] \not\subseteq \mathbb{R} \setminus \{2\} = \text{dom } f$.
$g \circ f(x)$ does exist since ran $f = \mathbb{R} \setminus \{0\} \subseteq \text{dom } g = \mathbb{R}$.
$g \circ f(x) = 1 + \cos\left(\frac{1}{x-2}\right)$, with domain $\mathbb{R} \setminus \{2\}$.

b. Find the values of x for which $f^{-1}(x) > f(x)$. (3 marks)

Solution: $x = \frac{1}{y-2} \implies y = \frac{1}{x} + 2$. Solve,

$$\frac{1}{x-2} = x$$
$$x^2 - 2x - 1 = 0$$
$$(x-1)^2 = 2$$
$$x = 1 \pm \sqrt{2}$$

Consider the shapes of the two graphs to conclude that $f^{-1}(x) > f(x)$ for

$$x \in (-\infty, 1-\sqrt{2}) \cup (0, 2) \cup (1+\sqrt{2}, \infty)$$

Question 14 (4 marks)

Let $f: (-\infty, k) \to \mathbb{R},\ f(x) = \frac{1}{(x-k)^2}$, where k is a real constant.

a. Find the rule for f^{-1} in terms of k. (1 mark)

$$f^{-1}(x) = k - \frac{1}{\sqrt{x}}$$

b. Find the exact value of k so that $f = f^{-1}$ has one unique solution.
Express your answer in the form $\frac{a}{b^c}$, for positive integers a, b and rational number c.

HINT: There will be one unique solution if $y = x$ is tangent to f and f^{-1} when they intersect. (3 marks)

Solution: Equation 1 is: $\frac{1}{(x-k)^2} = x$

Equation 2 is: $f'(x) = 1 \implies -\frac{2}{(x-k)^3} = 1$

E2 $\implies (x-k)^3 = -2$. Then multiply E1 by $\frac{1}{x-k}$

$$-\frac{1}{2} = \frac{x}{x-k}$$
$$\implies k = 3x$$

Sub $k = 3x$ into E2 $\implies (-2x)^3 = -2 \implies x = \frac{1}{2^{2/3}}$. Then,

$$k = \frac{3}{2^{2/3}}$$

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Question 15 (4 marks)

Let $f:\left[-\frac{a}{4}, a\right] \to \mathbb{R}, f(x) = \sqrt{4x+a} - 2$, where a is a positive real number.

a. The graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$ may have up to two points of intersection.
Find the x-coordinates of any possible points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ in terms of a. (2 marks)

We solve,

$$\sqrt{4x+a} - 2 = x$$
$$(x+2)^2 = 4x + a$$
$$x^2 + 4x + 4 = 4x + a$$
$$x^2 = a - 4$$
$$x = \pm\sqrt{a-4}$$

b. Determine the set of values of a for which the graphs of f and f^{-1} have two points of intersection. (2 marks)

Solution:
From **part a.** the x-value of any point of intersection must satisfy $x^2 = a - 4$. This will have no real solutions if $a < 4$, and one real solution if $a = 4$. Therefore we must have $a > 4$ for two possible points of intersection.

Now we must consider whether the points of intersection are within the domain of both f and f^{-1} by considering endpoints.
dom f^{-1} = ran $f = [-2, \sqrt{5a} - 2]$ and dom $f = \left[-\frac{a}{4}, a\right]$
The point $(\sqrt{a-4}, \sqrt{a-4})$ is always in the domain of both f and f^{-1}.

$$-2 = -\frac{a}{4} \implies a = 8.$$

When $a > 8$ the point $(-\sqrt{a-4}, -\sqrt{a-4})$ is not in the doman of f^{-1}.
Two points of intersection for $4 < a \leq 8$.

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Section G: Extension Exam 2 (16 Marks)

INSTRUCTION: 16 Marks. 20 Minutes Writing.

Question 16 (1 mark)

Let $f(x) = \sqrt{ax+b}$ and let g be the inverse function of f.
Given that $f(0) = 1$ and $g'(x) > 0$, then all possible values of a are:

A. $a \in \mathbb{R}^-$

B. $a \in \mathbb{R}^+$

C. $a \in \mathbb{R}\backslash\{0\}$

D. $a \in [0,1)$

Question 17 (1 mark)

The range of the function given by $f : (0,3] \to \mathbb{R}, f(x) = x^2 - 2x + b$ is:

A. $(b-1, b+3)$

B. $[b-1, b+3]$

C. $(b, 3]$

D. $(b-1, b+3]$

Question 18 (1 mark)

The functions $f(x) = \log_2(a-x)$ and $g(x) = -\sqrt{x+a}$ are defined on their maximal domains and $a \in \mathbb{R}^+$. The domain of $\frac{f}{g}$ is:

A. $[-a, a)$

B. $[-a, a]$

C. $(-a, a)$

D. $\mathbb{R}\backslash\{a\}$

Question 19 (1 mark)

Let f be a one-to-one differentiable function, and the following values are known:

$$f(a) = b, f(b) = c, f'(c) = d, \text{ and } f'(b) = k.$$

Let $g(x) = f^{-1}(x)$. The value of $g'(c)$ is:

A. $\frac{1}{a}$

B. $\frac{1}{k}$

C. $\frac{1}{d}$

D. $\frac{1}{b}$

Question 20 (1 mark)

Consider the functions $f(x) = x^2 + b$, where $b \in \mathbb{R}$ and $g(x) = \sqrt{(x-2)(x-3)}$ defined on their maximal domains. The composite functions $g(f(x))$ will have domain equal to $\mathbb{R}$ if:

A. $b < -2$

B. $b < -1$

C. $b > 3$

D. $b > 2$

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Question 21 (11 marks)

Consider the function $f : [\sqrt{2}, \infty) \to \mathbb{R}, f(x) = \sqrt{2x^2 - 4}$.

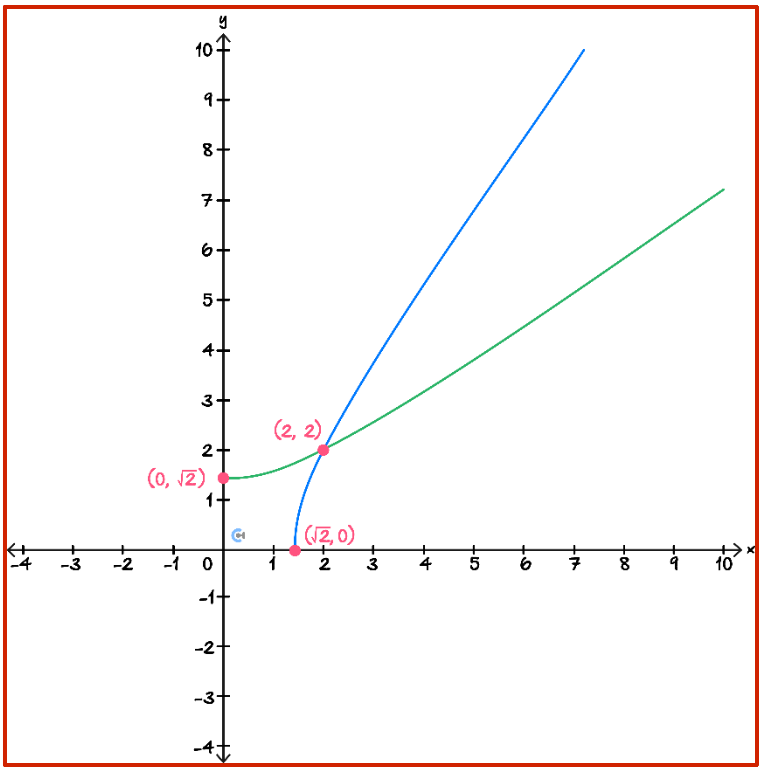
a. Define f^{-1}, the inverse function of f. (2 marks)

$$f^{-1} : [0, \infty) \to \mathbb{R},\ f^{-1}(x) = \sqrt{\frac{x^2+4}{2}}$$

b. Find the coordinates of the point P, which is the intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$. (1 mark)

$(2, 2)$

c. Sketch the graphs of $y = f(x), y = f^{-1}(x)$, on the axes below. Label all axes intercepts. (2 marks)



d. Find the acute angle between the tangents to the curves $y = f(x)$ and $y = f^{-1}(x)$ at the point P. Give your answer in degrees correct to two decimal places. (2 marks)

$f'(2) = 2$. Therfore $\theta = \left|\tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)\right| = 36.87°$

Now, consider the one-to-one increasing function $g : D \to \mathbb{R}$, where $g(x) = \sqrt{kx^2 - 4}$ and $k \in \mathbb{R}^+$.

e.

i. Find the domain D in terms of k. (1 mark)

Solution: $kx^2 \geq 4 \implies x \leq -\frac{2}{\sqrt{k}}$ or $x \geq \frac{2}{\sqrt{k}}$. But it is an increasing one-to-one function, therefore

$$D = \left[\frac{2}{\sqrt{k}}, \infty\right).$$

ii. Find the value(s) of k such that g and g^{-1} do not intersect. (1 mark)

Solution: $g^{-1}(x) = \sqrt{\frac{x^2+4}{k}}$

Solving $g(x) = g^{-1}(x) \implies x = \frac{2}{\sqrt{k-1}}$, which is defined for $k >$ 1

Therfore no intersection for $k \in (0, 1]$

f. The two curves $y = g(x)$ and $y = g^{-1}(x)$ intersect at $x = c$, where $c > 1$. The angle between the tangents at $x = c$ is θ.

It is known that $\tan(\theta) = \frac{9}{40}$. Determine the value of c and k. (2 marks)

Solution: We solve the simultaneous equations

$$\tan(\theta) = \left| \frac{g'(c) - \frac{1}{g'(c)}}{1+1} \right| = \frac{9}{40} \quad \text{and} \quad g(c) = g^{-1}(c)$$

$c = 4$ and $k = \frac{5}{4}$.

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