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VCE Mathematical Methods ¾
Functions & Relations Exam Skills [0.2]
Workshop

Section A: Recap

Sub-Section: Maximal Domains

Starting with a domain!

Maximal Domain

- **Definition**: The largest possible set of input values (elements of the domain) for which the function is well-defined.
- **Three Important Rules:**

Functions	Maximal Domain
$\sqrt{z}$	
$\log(z)$	
$\frac{1}{z}$	

- **Steps:**
  1. Find the restriction of the inside.
  2. Sketch the graph if needed.
  3. Solve for domain.

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Sub-Section: Domain of Sum, Difference and Product of Functions

What about a domain of the sum of two functions?

Sums, Differences and Products of Functions

- Rules:

$$(f+g)(x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$(f-g)(x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$(f \times g)(x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

- Idea:

$$\textit{Domain of sum or product of two functions} = \_\_\_\_\_\_\_\_\_\_\_\_ \textit{ of the two domains}$$

- Steps:

1. Find the domain of each function.
2. Find the intersection (draw a number line if needed).

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Sub-Section: Basics of Composition

What was the "composition" of functions?

Composite Functions

- **Definition**: A _______________ of functions.
- **Representation of the above:**

$$y = \underline{\hspace{6cm}}$$

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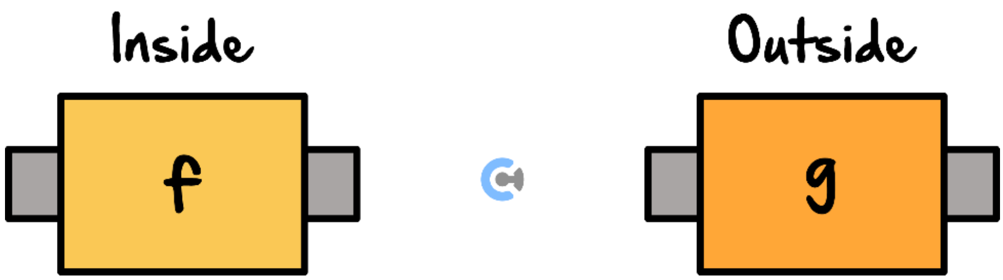
Sub-Section: Validity of Composite Functions

Did composite functions work all the time?

Validity of Composite Functions



- Output of $f(x)$: ______________________________. *(Label Above)*
- Input of $g(x)$: ______________________________. *(Label Above)*
- Composite Function is only valid if:

  ______________________________ $\subseteq$ ______________________________

- Acronym:

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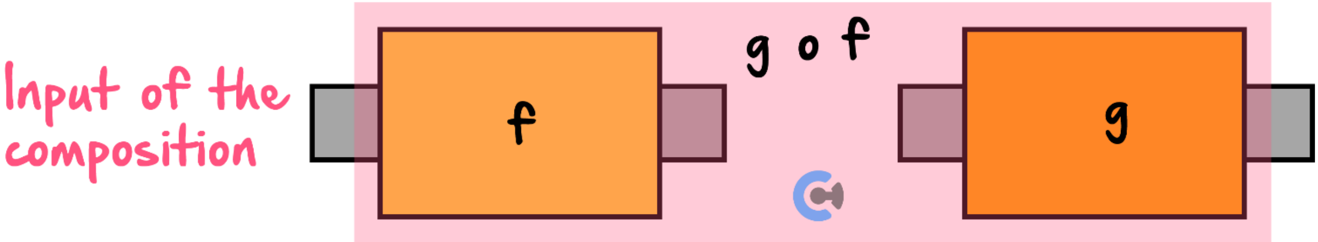
Sub-Section: Domain of Composite Functions

How did we find the domain of a composite function?

Domain of Composite Functions



$$\textbf{Domain of Composite} = \textbf{Domain of Inside}$$

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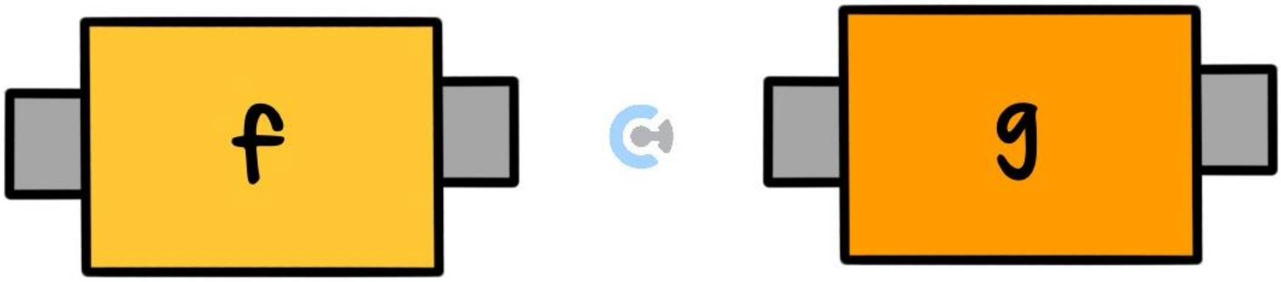
Sub-Section: Range of Composite Functions

Range of the Composite Functions



Range of Composite* $\subseteq$ *Range of the Outside

- **Finding the range of composition function**: Use the domain and the rule, just like another function.

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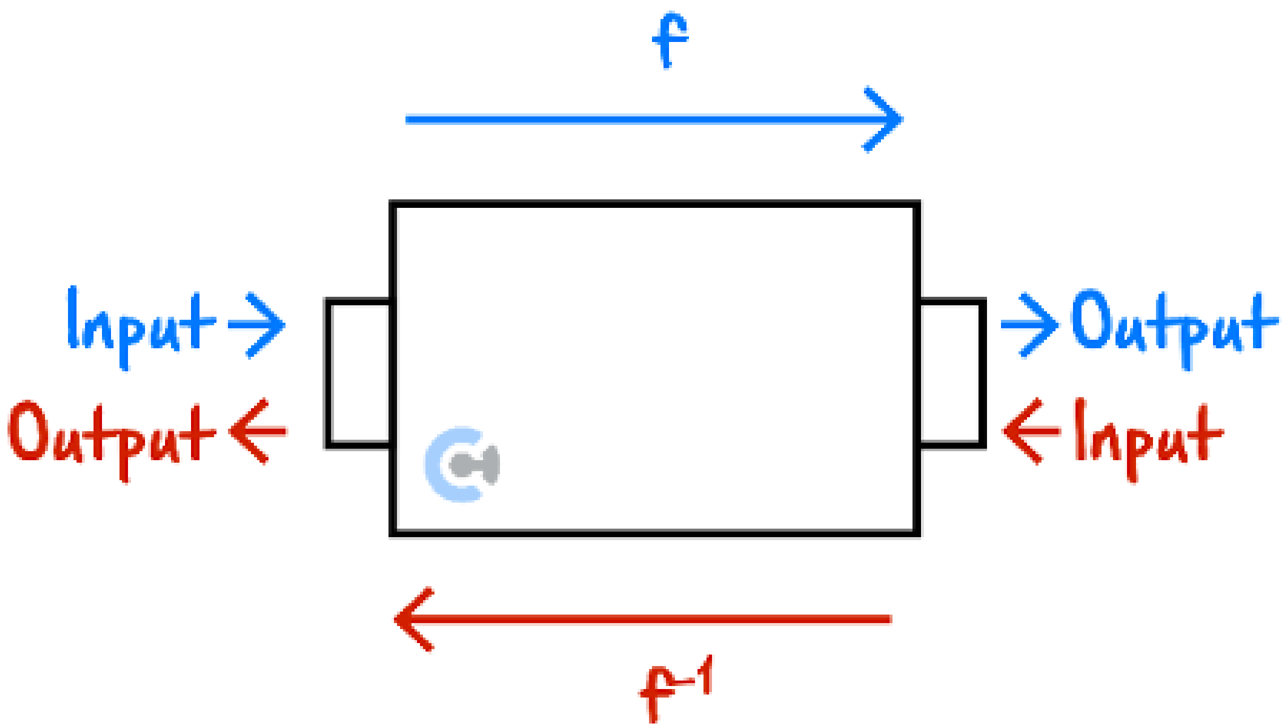
Sub-Section: Basics of Inverses

What does "Inverse" mean?

Inverse Relation

- **Definition**: Inverse is a relation which does the ________________.



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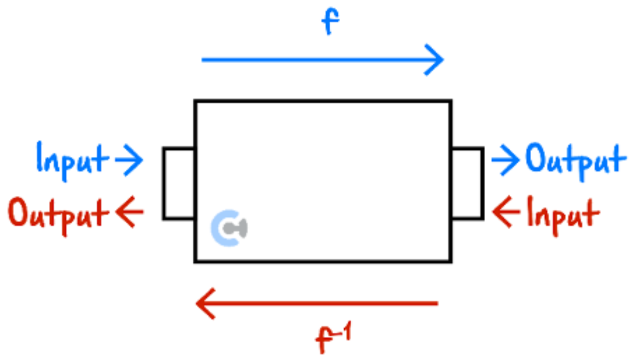
Sub-Section: Swapping x and y

Is there a better way of solving for an inverse relation?

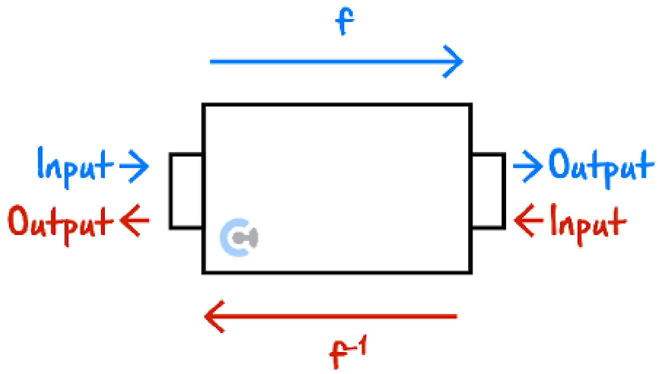
Solving for an Inverse Relation

- Swap x and y.



NOTE: $f(x) = y$.

Domain and Range of Inverse Functions



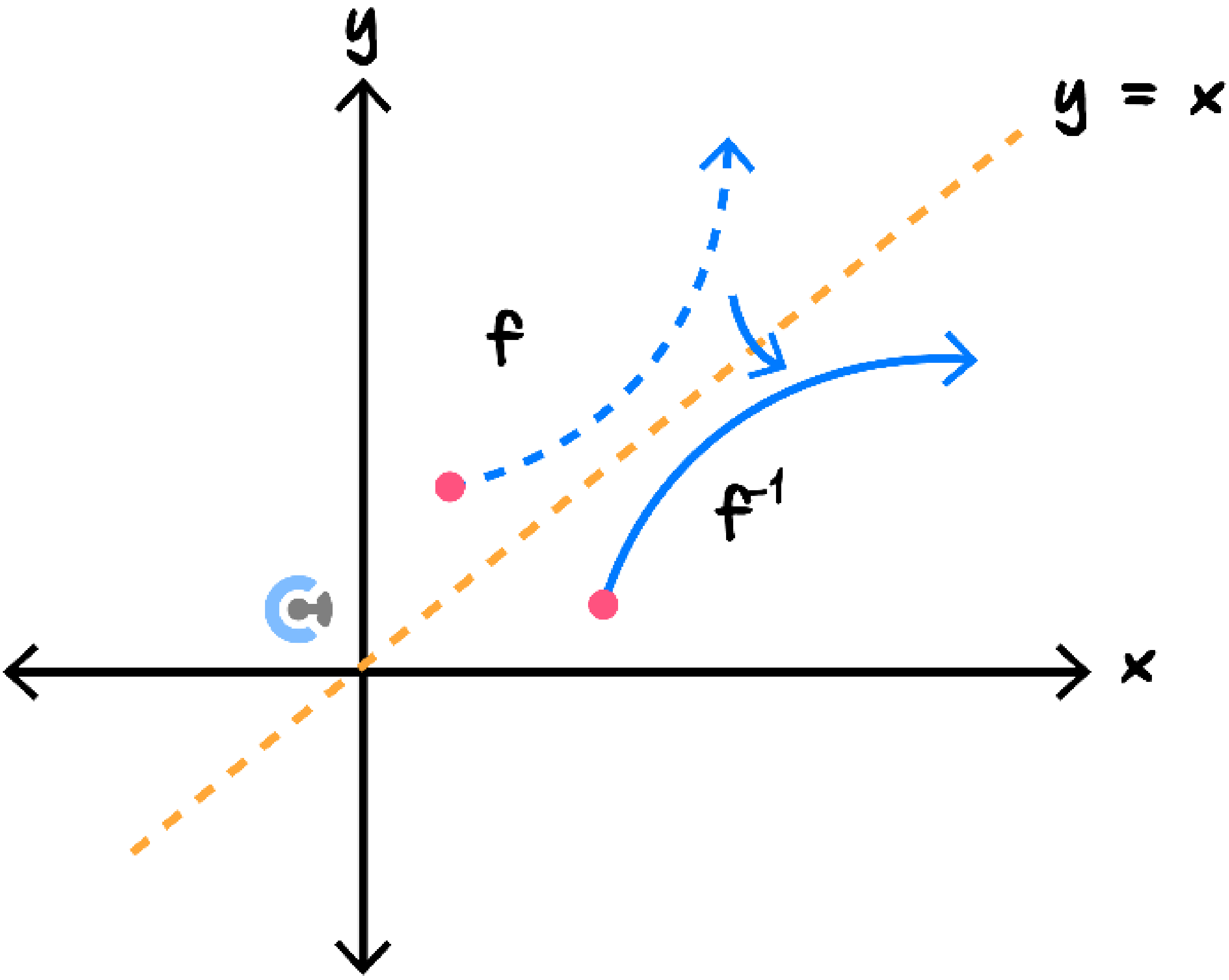
$$Dom\ f^{-1} = \underline{\hspace{4cm}}$$

$$Ran\ f^{-1} = \underline{\hspace{4cm}}$$

Sub-Section: Symmetry Around $y = x$

Why does this happen?

Symmetry of Inverse Functions



- Inverse functions are always symmetrical around $y = x$.

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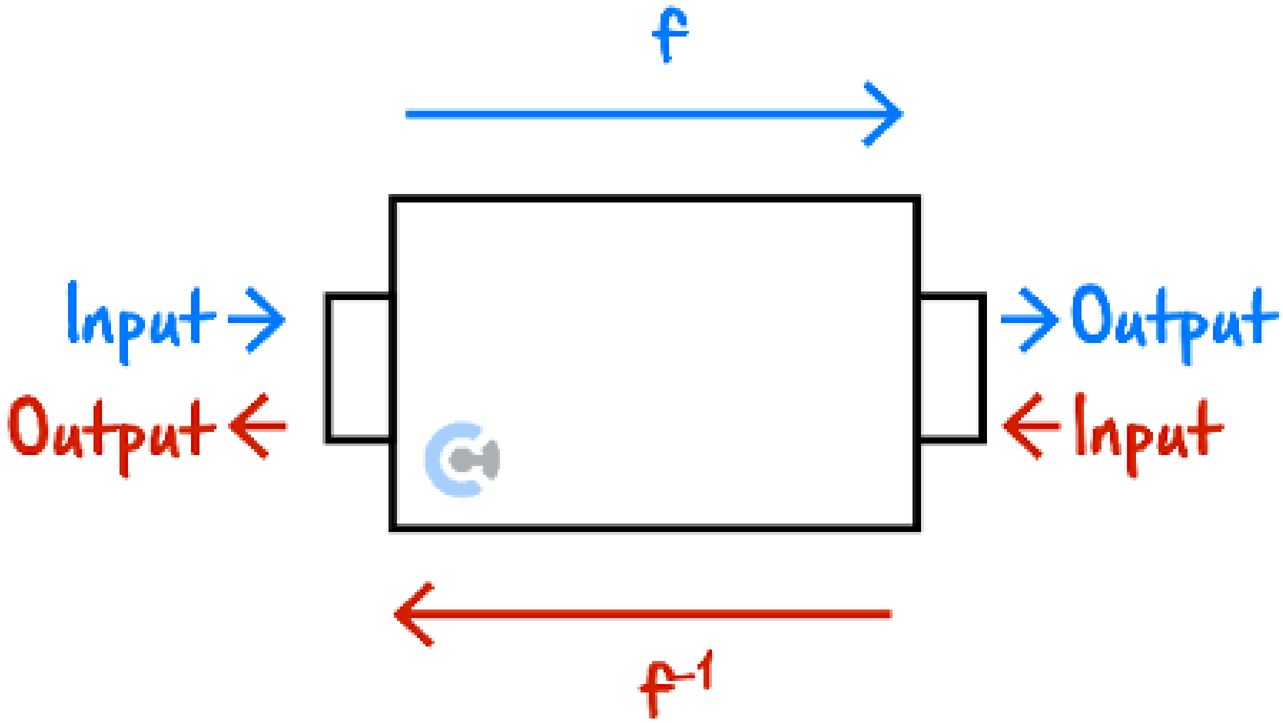
Sub-Section: Validity of Inverse Function

Does an inverse function always exist?

Validity of Inverse Functions



- Requirement for Inverse Function:

$$f \text{ needs to be } \_\_\_\_\_\_\_\_.$$

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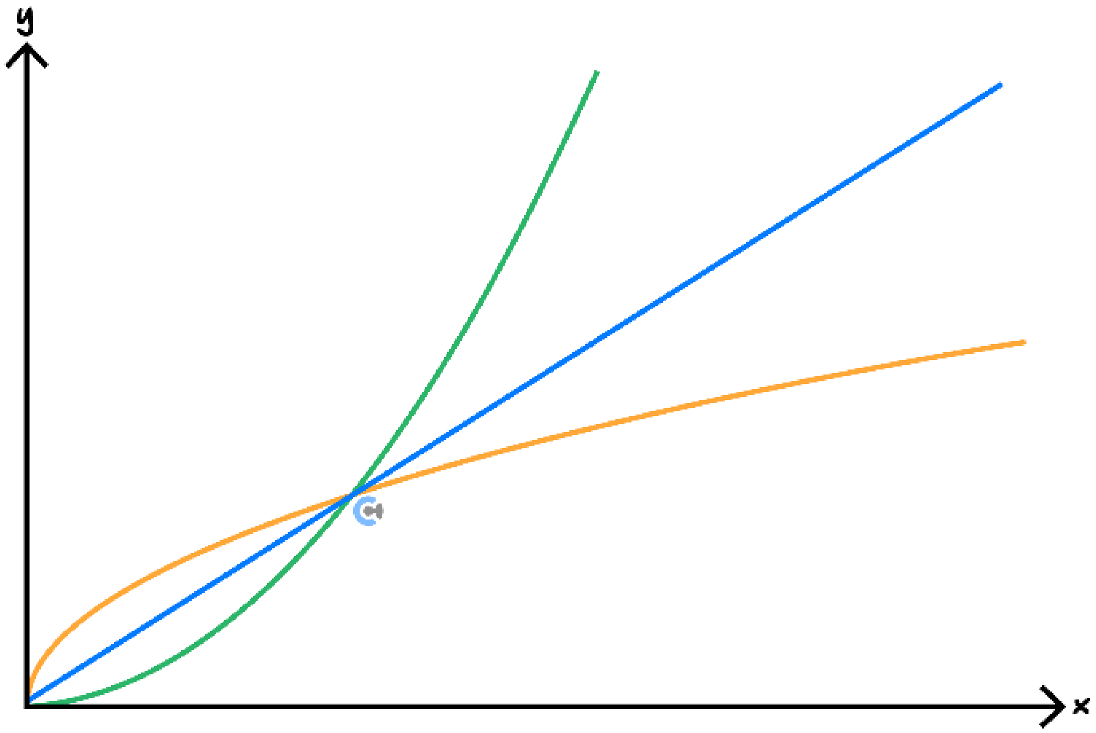
Sub-Section: Intersection Between Inverses

Where do inverses meet?

Intersection Between a Function and its Inverse



- Equate with ________ instead.

$$f(x) = x \text{ OR } f^{-1}(x) = x$$

- We cannot do this when the function is ____________________ function.

NOTE: This only works for an increasing function.

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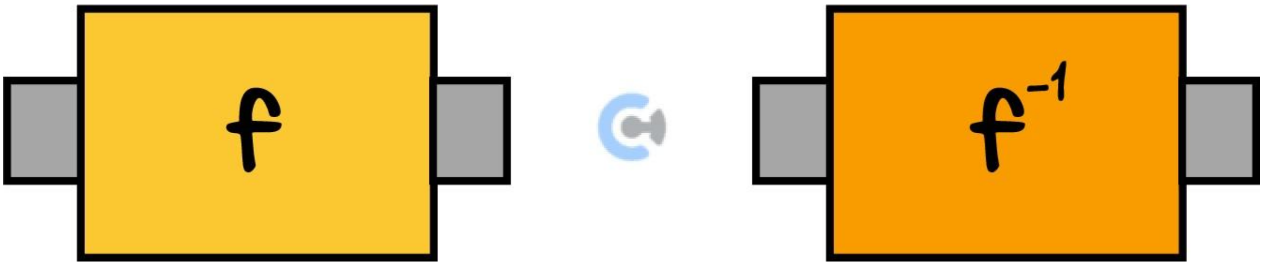
Sub-Section: Composition of Inverses

Composition of Inverse Functions



$$f \circ f^{-1}(x) = \_\_, \quad for\ all\ x \in \_\_\_\_\_\_\_$$

$$f^{-1} \circ f(x) = \_\_, \quad for\ all\ x \in \_\_\_\_\_\_\_$$

NOTE: Domain = Domain of Inside.

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Sub-Section: Find a New Domain to Fix Composite Functions

Fixing Broken Composite Functions

- **Aim:** Restrict the domain of the inside function so that the range of the inside function fits inside the domain of the outside.
- **Steps:**
  1. **Write down the RIDO statement** with the domain of the outside (as it is fixed).
  2. **Sketch the inside function** to see what domain is needed.

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<u>Sub-Section</u>: Find the Range of Complex Composite Functions

<u>Finding Range of Complex Composite Functions</u>

- **Aim:** Find the range of complicated functions.
- **Steps:**

1. Break the function into ________________ of two simple functions.
2. Follow the ___________________ to find the range.

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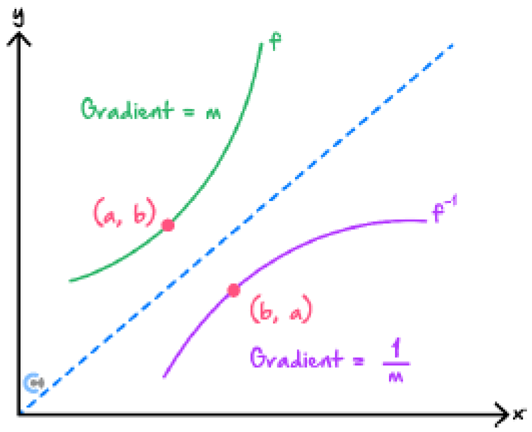
Sub-Section: Find the Gradient of Inverse Functions

This is a fun application of inverse with calculus!

REMINDER: Gradient of a Point

$$\text{Gradient at a point } = \frac{dy}{dx}$$

Gradient of an Inverse



$$\text{If Gradient of } f \text{ at } (a, f(a)) = m$$

$$\text{Gradient of } f^{-1} \text{ at } \rule{6cm}{0.4pt}$$

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<u>Section B:</u> Warm Up

INSTRUCTION: 5 Minutes Writing.

Question 1

Consider $f(x) = \sqrt{2x}$ and $g(x) = 2x - 1$, both defined over their maximal domains.

a. Is $f(g(x))$ defined?

b. Find the large restricted domain of g such that $f(g(x))$ is defined.

c. Find the range of $y = \log_3(x^2 + 9)$.

d. Consider the one-to-one function h with the following properties:

$$h(3) = 2 \text{ and } h'(3) = 5.$$

Find the gradient of h^{-1} at $x = 2$.

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Section C: Exam 1 (20 Marks)

INSTRUCTION: 20 Marks. 26 Minutes Writing.

Question 2 (7 marks)

Consider the two functions:

$$f : D \to \mathbb{R}, f(x) = \frac{1}{x+2} + 1$$

$$g : [0, \infty) \to \mathbb{R}, g(x) = \sqrt{x+4}$$

where D is a restricted domain of f.

a. Define g^{-1}, the inverse function of g. (2 marks)

b. The gradient of g^{-1} when $x = 3$ is 6. Find the gradient of g when $x = 5$. (2 marks)

c. Find the restricted domain, D, of the function f such that the composite function $g \circ f$ is defined. (3 marks)

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Question 3 (6 marks)

Consider the functions, $f(x) = \frac{1}{x-2}$ and $g(x) = 2x + 3$ defined on their maximal domains.

Let $h(x) = g(f(x))$.

a. Write down the rule and domain of $h(x)$. (2 marks)

b. Define the inverse function, h^{-1}, of h. (2 marks)

c. Find the point(s) of intersection between h and h^{-1}. (2 marks)

Question 4 (7 marks)

Consider the function:

$$f:[a,\infty)\to\mathbb{R}, f(x)=\frac{1}{2}x^2-2x+5$$

a. Find the largest value of a such that the inverse function f^{-1} exists. (1 mark)

b. Define f^{-1}. (2 marks)

c. Given that f is an increasing function, show that f and f^{-1} do not have any points of intersection. (2 marks)

Let $g: \mathbb{R} \to \mathbb{R}, g(x) = \frac{1}{2}x^2 - 2x + 5$ and $h: \mathbb{R} \to \mathbb{R}, h(x) = 2^x$.

d. Find the range of $h(g(x))$. (1 mark)

e. Find the minimum value of c such that $h\big(g(x+c)\big)$ is one-to-one over the interval $[1, \infty)$. (1 mark)

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Section D: Technology Warmup

INSTRUCTION: 5 Minutes Writing.

Calculator Commands: Finding the domain and range

TI

- domain $(f(x), x)$, $f\ Min$ and $Fmax$.

Define $f(x)=\sqrt{9-x^2}$	*Done*
domain$(f(x),x)$	$-3\le x\le 3$
fMin$(f(x),x)$	$x=-3$ or $x=3$
fMax$(f(x),x)$	$x=0$
$f(3)$	0
$f(0)$	3

TI-UDF

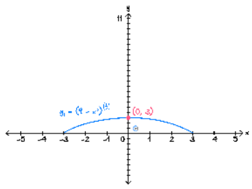
Analyse a Function: Find intercepts, critical points and their nature, maximal domain, asymptote.	analyse(<function>, <variable>) analysed(<function>, <variable>, <lower bound>, <upper bound>)

$$analysed\left(\frac{x^4-2\cdot x^3-3\cdot x^2+3\cdot x+1}{-3\cdot x^3-6\cdot x^2-x+1},x,-5,5\right)$$

- Start Point: $\begin{bmatrix}-5 & \frac{262}{77}\end{bmatrix}$
- End Point: $\begin{bmatrix}5 & \frac{-316}{529}\end{bmatrix}$
- Maximal Domain:
  $x\neq-1.68469$ and
  $x\neq-0.629579$ and
  $x\neq0.314273$ and
  $-5\le x\le 5$

Casio Classpad

- Graph the function and use G-Solve to find min and max values for the range.



Mathematica

```
In[127]:= f[x_] := √(9 - x^2)
In[128]:= FunctionDomain[f[x], x]
Out[128]= -3 ≤ x ≤ 3
In[129]:= FunctionRange[f[x], x, y]
Out[129]= 0 ≤ y ≤ 3
```

Mathematica UDF :

- FInfo [f [x], $\{x, x\ min, x\ max\}, y$]

Returns useful information about a function, including derivative, domain, range, period, horizontal intercepts, vertical intercepts, stationary points, inflexion points, left and sided asymptotes, oblique asymptotes and vertical asymptotes.

$$\text{FInfo}\left[\frac{x^2-1}{x\left(x^2-3\right)}, \{x, -\text{Infinity}, \text{Infinity}\}, y\right]$$

The function is $\frac{x^2-1}{x\left(x^2-3\right)}$

The derivative is $-\frac{x^4+3}{x^2\left(x^2-3\right)^2}$

Domain: $x<-\sqrt{3} \lor -\sqrt{3}<x<0 \lor 0<x<\sqrt{3} \lor x>\sqrt{3}$

Range: $y\in\mathbb{R}$

Period: 0

Horizontal Intercepts: $\{-1, 1\}$

Vertical Intercepts: None

Stationary points: $\{\}$

Inflexion points: $\{\{-0.871\ldots, -0.123\ldots\}, \{0.871\ldots, 0.123\ldots\}\}$

Left sided asymtote: $y=0$

Right sided asymtote: $y=0$

Oblique asymtote: $\{0\}$

Vertical asymtote: $\{x=0, x=-\sqrt{3}, x=\sqrt{3}\}$

Calculator Commands: Finding the composite function

TI	CASIO	Mathematica
Define $f(x)=\ln(x)$ Done Define $g(x)=x^2+3$ Done $f(g(x))$ $\ln(x^2+3)$	define f(x) = ln(x) done define g(x) = x^2+3 done f(g(x)) $\ln(x^2+3)$	In[141]:= f[x_] := Log[x] In[142]:= g[x_] := x^2 + 3 In[143]:= f[g[x]] Out[143]= Log[3 + x^2]

Calculator Commands: Finding the inverse function

TI	CASIO	Mathematica
Define $f(x)=x^2+4\cdot x+9$ Done solve$(f(y)=x,y)$ $y=-(\sqrt{x-5}+2)$ or $y=\sqrt{x-5}-2$	define f(x) = x^2+4x+9 done solve(f(y)=x, y) $\{y=-\sqrt{x-5}-2,\ y=\sqrt{x-5}-2\}$	In[154]:= f[x_] := x^2 + 4 x + 9 In[155]:= Solve[f[y] == x, y] Out[155]= $\{\{y \to -2 - \sqrt{-5+x}\}, \{y \to -2 + \sqrt{-5+x}\}\}$

NOTE: It doesn't tell us which branch is correct.

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Question 5 Tech-Active.

Find the domain and range of $f(x) = \sqrt{\frac{x^2-1}{x^2}}$.

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Section E: Exam 2 (23 Marks)

INSTRUCTION: 23 Marks. 30 Minutes Writing.

Question 6 (1 mark)

The function f defined by $f\ A \to \mathbb{R}, f(x) = (x-2)^2 + 3$ will have an inverse function if its domain A is:

A. $\mathbb{R}$

B. $(-\infty, 3]$

C. $[3, 6]$

D. $[0, \infty)$

Question 7 (1 mark)

The linear function $f : D \to \mathbb{R}, f(x) = 2 - x$ has a range of $[-5,1)$. The domain of f is:

A. $(-5,1]$

B. $(-1,5]$

C. $(1,7]$

D. $[7,1)$

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Question 8 (1 mark)

Let f be a one-to-one differentiable function, and the following values are known:

$$f(2) = 3, f(3) = 4, f'(2) = 5, \text{ and } f'(3) = 6$$

Let $g(x) = f^{-1}(x)$. The value of $g'(3)$ is:

A. $\frac{1}{5}$

B. $\frac{1}{7}$

C. $\frac{1}{4}$

D. $\frac{1}{6}$

Question 9 (1 mark)

Let $f: [-1,3] \to \mathbb{R}, f(x) = 2x - 1$ and $g: D \to \mathbb{R}, g(x) = x^2 - 1$.

The largest interval D such that $f \circ g$ exists is:

A. $(0,2)$

B. $[0,2]$

C. $[-2,2]$

D. $[0,3]$

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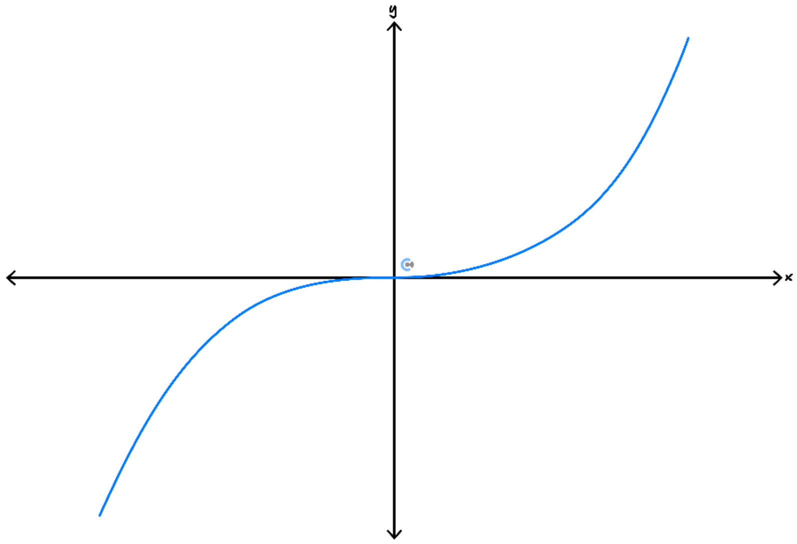
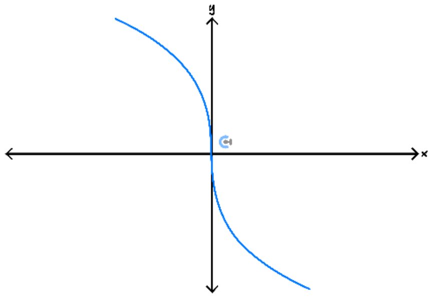
Question 10 (1 mark)

Part of the graph of $y = f(x)$ is shown below.

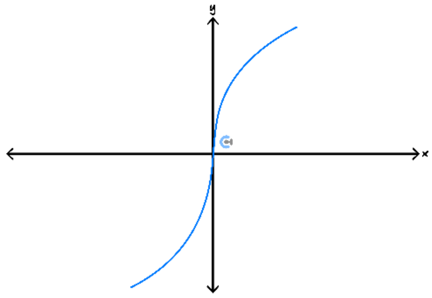


The inverse function f^{-1} is best represented by:

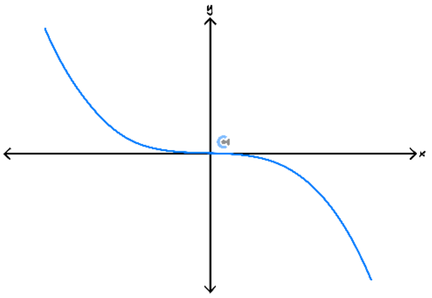
A.



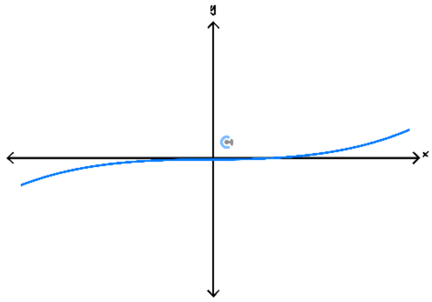
B.



C.



D.



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Question 11 (8 marks)

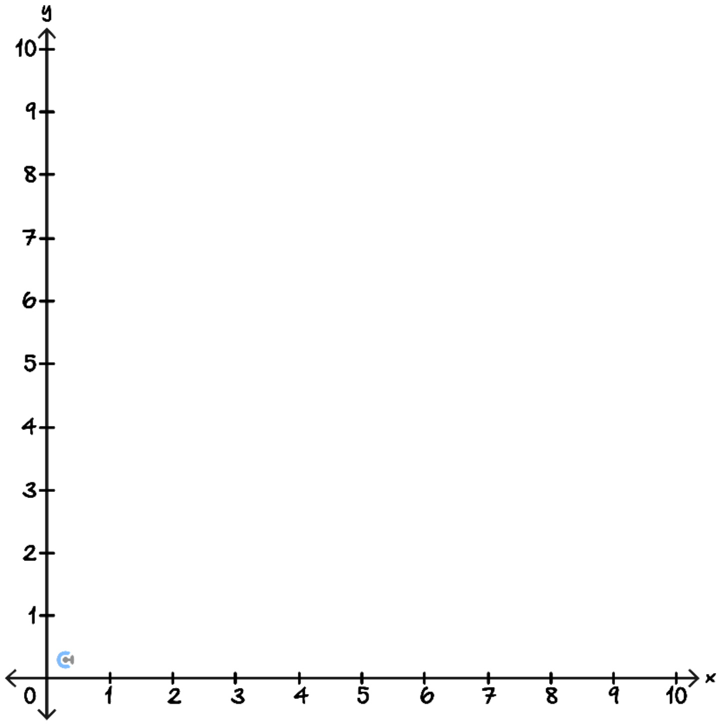
Health insurance provides MediBear (C_m) and Bopa (C_b) offer cost-based health insurance plans represented by the functions:

$$C_m = \sqrt{4x-3}, 2 \leq x \leq 10 \text{ and } C_b = C_m^{-1}(x)$$

where C represents the amount paid for the plan by the customers in tens of dollars and x represents the plan's benefits rating.

a. Define the function C_b. (2 marks)

b. Graph the two functions on the axes below. Label all endpoints with coordinates. (2 marks)



c. What is the maximum plan benefit rating for a plan offered by Bopa? Give your answer correct to one decimal place. (1 mark)

d. What plan benefit rating results in both plans having the same cost? (1 mark)

e. Find the values of x for which $C_m < C_b$. (2 marks)

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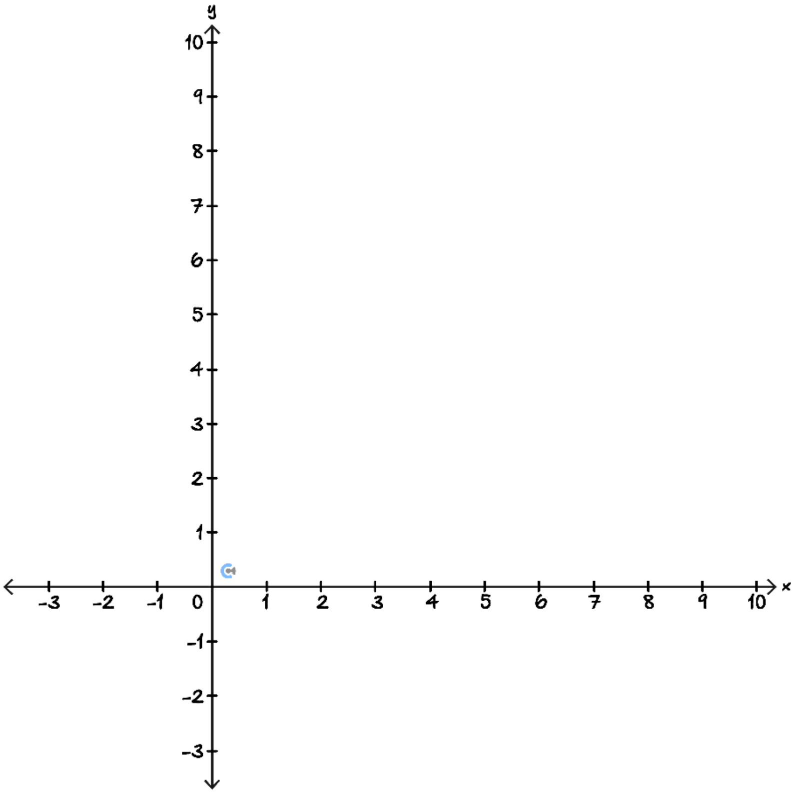
Question 12 (10 marks)

Consider the function $f:[a,\infty)\to\mathbb{R}, f(x)=x^2-2x-2$.

a. State the smallest value of a for which f has an inverse function. (1 mark)

b. Define the inverse function, f^{-1}. (2 marks)

c. Sketch the graphs of f and its inverse on the axes below. Label all axes intercepts, endpoints and points of intersection with coordinates correct to two decimal places. (3 marks)



d. Let f and f^{-1} intersect at the point P. It is known that f has a gradient of $1+\sqrt{17}$ at P. Find the acute angle made by the tangents to f and f^{-1} at P. Give your answer in degrees correct to two decimal places. (2 marks)

Consider the function $g : [2, \infty) \to \mathbb{R}, g(x) = \log_2(x+1)$.

e. Find all possible values for a such that $g(f(x))$ does not exist. (1 mark)

f. If $a = 4$, find the range of $g\big(f(x)\big)$ when it exists. (1 mark)

Let's take a BREAK (Extension Stream)!

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Section F: Extension Exam 1 (13 Marks)

INSTRUCTION: 13 Marks. 20 Minutes Writing.

Question 13 (5 marks)

Consider the two functions $f(x) = \frac{1}{x-2}$ and $g(x) = 1 + \cos(x)$ defined on their maximal domains.

a. Determine whether or not the functions $f \circ g(x)$ or $g \circ f(x)$ exist, and justify your answer. If the composite function exists, state its rule and domain. (2 marks)

b. Find the values of x for which $f^{-1}(x) > f(x)$. (3 marks)

Question 14 (4 marks)

Let $f: (-\infty, k) \to \mathbb{R},\ f(x) = \frac{1}{(x-k)^2}$, where k is a real constant.

a. Find the rule for f^{-1} in terms of k. (1 mark)

b. Find the exact value of k so that $f = f^{-1}$ has one unique solution.
Express your answer in the form $\frac{a}{b^c}$, for positive integers a, b and rational number c.

HINT: There will be one unique solution if $y = x$ is tangent to f and f^{-1} when they intersect. (3 marks)

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Question 15 (4 marks)

Let $f: \left[-\frac{a}{4}, a\right] \to \mathbb{R}, f(x) = \sqrt{4x + a} - 2$, where a is a positive real number.

a. The graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$ may have up to two points of intersection.
Find the x-coordinates of any possible points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ in terms of a. (2 marks)

b. Determine the set of values of a for which the graphs of f and f^{-1} have two points of intersection. (2 marks)

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Section G: Extension Exam 2 (16 Marks)

INSTRUCTION: 16 Marks. 20 Minutes Writing.

Question 16 (1 mark)

Let $f(x) = \sqrt{ax+b}$ and let g be the inverse function of f.
Given that $f(0) = 1$ and $g'(x) > 0$, then all possible values of a are:

A. $a \in \mathbb{R}^-$

B. $a \in \mathbb{R}^+$

C. $a \in \mathbb{R}\backslash\{0\}$

D. $a \in [0,1)$

Question 17 (1 mark)

The range of the function given by $f: (0,3] \to \mathbb{R}, f(x) = x^2 - 2x + b$ is:

A. $(b-1, b+3)$

B. $[b-1, b+3]$

C. $(b, 3]$

D. $(b-1, b+3]$

Question 18 (1 mark)

The functions $f(x) = \log_2(a-x)$ and $g(x) = -\sqrt{x+a}$ are defined on their maximal domains and $a \in \mathbb{R}^+$. The domain of $\frac{f}{g}$ is:

A. $[-a, a)$

B. $[-a, a]$

C. $(-a, a)$

D. $\mathbb{R}\backslash\{a\}$

Question 19 (1 mark)

Let f be a one-to-one differentiable function, and the following values are known:

$$f(a) = b, f(b) = c, f'(c) = d, \text{ and } f'(b) = k.$$

Let $g(x) = f^{-1}(x)$. The value of $g'(c)$ is:

A. $\frac{1}{a}$

B. $\frac{1}{k}$

C. $\frac{1}{d}$

D. $\frac{1}{b}$

Question 20 (1 mark)

Consider the functions $f(x) = x^2 + b$, where $b \in \mathbb{R}$ and $g(x) = \sqrt{(x-2)(x-3)}$ defined on their maximal domains. The composite functions $g(f(x))$ will have domain equal to $\mathbb{R}$ if:

A. $b < -2$

B. $b < -1$

C. $b > 3$

D. $b > 2$

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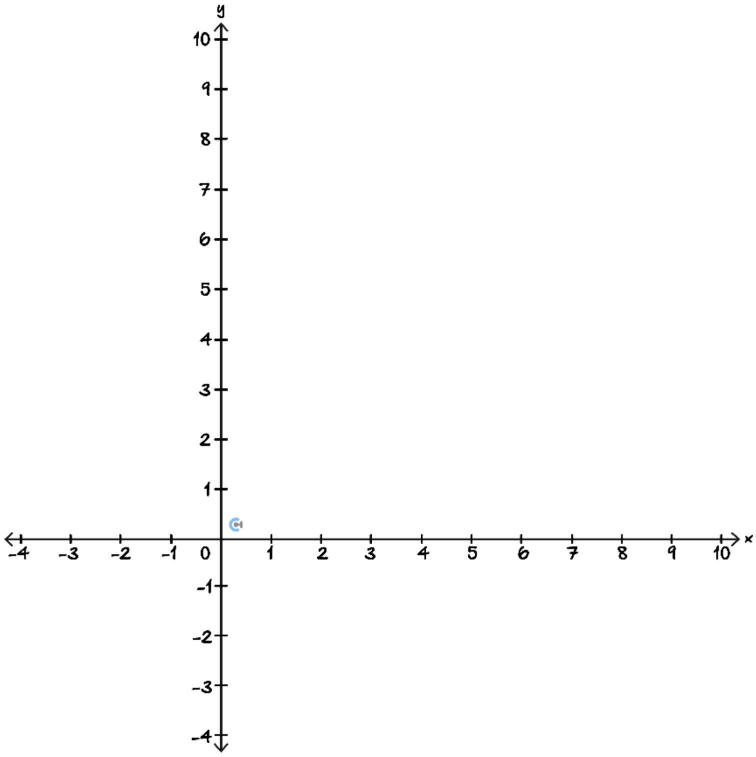
Question 21 (11 marks)

Consider the function $f : \left[\sqrt{2}, \infty\right) \to \mathbb{R}, f(x) = \sqrt{2x^2 - 4}$.

a. Define f^{-1}, the inverse function of f. (2 marks)

b. Find the coordinates of the point P, which is the intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$. (1 mark)

c. Sketch the graphs of $y = f(x), y = f^{-1}(x)$, on the axes below. Label all axes intercepts. (2 marks)



d. Find the acute angle between the tangents to the curves $y = f(x)$ and $y = f^{-1}(x)$ at the point P. Give your answer in degrees correct to two decimal places. (2 marks)

Now, consider the one-to-one increasing function $g : D \to \mathbb{R}$, where $g(x) = \sqrt{kx^2 - 4}$ and $k \in \mathbb{R}^+$.

e.

i. Find the domain D in terms of k. (1 mark)

ii. Find the value(s) of k such that g and g^{-1} do not intersect. (1 mark)

f. The two curves $y = g(x)$ and $y = g^{-1}(x)$ intersect at $x = c$, where $c > 1$. The angle between the tangents at $x = c$ is θ.

It is known that $\tan(\theta) = \frac{9}{40}$. Determine the value of c and k. (2 marks)

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VCE Mathematical Methods ¾

Free 1-on-1 Support

Be Sure to Make The Most of These (Free) Services!

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➤ Book via bit.ly/contour-methods-consult-2025 (or QR code below). ➤ One active booking at a time (must attend before booking the next).	➤ Message +61 440 138 726 with questions. ➤ Save the contact as "Contour Methods".

Booking Link for Consults
bit.ly/contour-methods-consult-2025

Number for Text-Based Support
+61 440 138 726