CONTOUR EDUCATION

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VCE Mathematical Methods ¾
SAC 1 Revision IV [0.19]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________________ Notes:	Pg / Q #: ____________________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
Pg / Q #: ____________________ Notes:	Pg / Q #: ____________________ Notes:

Section A: SAC 1 Success

Welcome to the third SAC 1 workshop!

Context: SAC 1 Workshops

- SAC 1 - 50% of SACs, 20% of the study score.
- Will be running all the way till mid-June.
- After that, the workshop will turn into SAC 2 Workshop (Integration).
- Make sure to complete the SAC $1 \sim 8$.

Successful SAC

Study Score = How much you know × How much you show

- Answer everything you know.
- Answer without mistakes.
- Time Management is **key!**

Tutor's Comment: Explain how even if you know everything, if you cannot show in the SAC, you will get 0.

Analogy: Skipping Questions

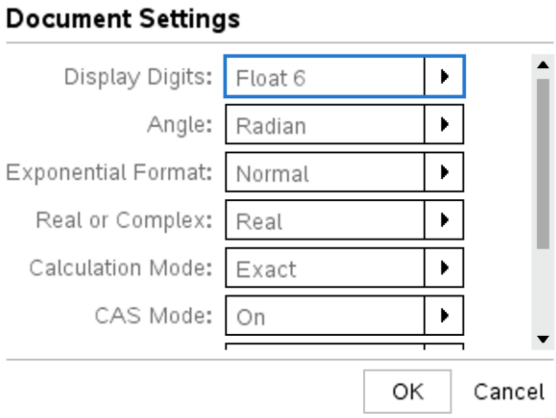
- Let's say if you were to fight them and win, you get assigned marks.

- Who would you fight first?
- Skip the hard question with little marks if it doesn't make sense during the reading time.

<u>SAC Proficiency List</u>

Before the SAC

- Prepare your stationery, including a ruler, eraser, and your mechanical pencil lead.
- Skim through the bound reference (if applicable).
- Do not speak to other people and lock in.
- TI & Mathematica Only: Check your Contour UDFS.
- TI Only: Check technology settings.



Reading Time

- **Detailed strategy** on how to exactly solve the question on your technology – Don't just **read**, think about how to solve it and use what technology commands.
- Identify questions to **skip.**

  For difficult SACs, it's not necessarily about getting 100%. It's about getting better than everyone else. While others are stuck on a hard 1-marker question, you can do an easy 3-marker first.

- Identify questions to **start first** – You don't have to start from Q1!
- Look for potential pitfalls – **Units, domain restriction of the unknown, variable and function meaning.**

Writing Time

- **Circle what the question is asking** for in the question.
- Spend the first **50%** of the time on all the **easy questions** you identified.
- Spend the next **25%** of doing the **difficult questions** you left blank.

- Spend the last 25% of the time on **checking your answers**.
- Check your answer by reading the question again and see if you answered the question.
- Check in the order of:

  Domino effect (check a, b, c first) $>$ ***Questions with high marks*** $(3+)$ $>$ *Hard Questions.*

- $\mathbf{TI}$ ONLY: Use new document - **doc** $\mathbf{4, 1.}$

After the SAC

- Think about how each mark loss can be prevented using this proficiency list.
- Think about the big picture and improve the marks -

  Instead of spending 10 minutes on 10c) (1 mark), I should have checked 5a) (3 marks).

Space for Personal Notes

Section B: SAC Questions - Tech Active (52 Marks)

INSTRUCTION:

- 52 Marks. 15 Minutes Reading, 75 Minutes Writing.

Question 1 (16 marks)

A function $y = \frac{x^3}{(x+2)^2}$ is to undergo the following sequence of transformations:

1. Dilation factor $\frac{1}{2}$ from the x-axis.

2. Reflection in the x-axis.

3. Reflection in the line $y = x$.

4. Dilation factor 8 from the y-axis.

5. Reflection in the y-axis.

a. Show that this sequence of transformations can be represented by the transformation: (2 marks)

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (4y, x)$$

We apply the transformations to the point (x, y) in order:

$(x, y) \to \left(x, \frac{1}{2}y\right) \to \left(x, -\frac{1}{2}y\right) \to \left(-\frac{1}{2}y, x\right) \to (-4y, x) \to (4y, x)$

[1M for swapping x and y when reflecting in y=x, 1M for clear logical sequence]

b. Describe a sequence of 2 transformations that would result in the same transformation as T. (2 marks)

A reflection in the line $y = x$ **[1M]**
followed by a dilation by factor 4 from the y-axis **[1M]**

c.

i. Determine the equation of image of y after applying the transformations represented by T. Leave your answer in the form $x = f(y)$. (2 marks)

We have that $x' = 4y \implies y = \frac{1}{4}x'$ and $y' = x \implies x = y'$. **[1M]**
Substitute into the equation and remove the dashes to get:

$$\frac{1}{4}x = \frac{y^3}{(y+2)^2}$$

$$x = \frac{4y^3}{(y+2)^2} \quad \textbf{[1A]}$$

ii. By carefully considering the graph of the preimage and the linear transformations used to transform it, explain why you would not expect the image to be a function. (2 marks)

The pre-image is **not** a 1:1 function. **[1M]**
The transformation involves reflecting in the line $y = x$ which produces the inverse of the pre-image, but the pre-image is not 1:1 so its image will not be a function. **[1M]** .

d.

i. Find a sequence of transformations which take the graph of $y = 5e^{\frac{x}{2}-3} + 4$ to the graph of $y = e^x$. (2 marks)

- A translation 4 units down (or a dilation by factor $\frac{1}{5}$ from the x-axis)
- A dilation by factor $\frac{1}{5}$ from the x-axis (a translation $\frac{4}{5}$ units down)

[1M]

Then we have $x' = \frac{x}{2} - 3 = \frac{1}{2}(x-6)$, so

- A translation 6 units to the left (or a dilation by factor $\frac{1}{2}$ from the y-axis)
- A dilation by factor $\frac{1}{2}$ from the y-axis (a translation 3 units left)

[1M]

There could be many different combinations.

ii. The transformation $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x, y) = (ax + b, cy + d)$ takes the graph of $y = e^x$ to the graph of $y = -3e^{-4(x+6)} + 2$.

Find the values of $a, b, c, d \in \mathbb{R}$. (3 marks)

Immediately see that $y' = -3y + 2$. [1M]

Then $-4(x' + 6) = x \implies x' = -\frac{1}{4}x - 6$. [1M]

So $a = -\frac{1}{4}$, b=-6, $c = -3$ and $d = 2$. [1A]

iii. Hence, or otherwise, determine a sequence of transformations that would transform the graph of $y = 5e^{\frac{x}{2}-3} + 4$ to the graph of $y = -3e^{-4(x+6)} + 2$.

Give your answer in an order where translations are applied last. (3 marks)

Can do all the transformations in **part d.i** followed by the transformations in **part d.ii** and simplify.
It is probably simpler to just do it from scratch.
Solve $\frac{y-4}{5} = \frac{y'-2}{-3} \implies y' = \frac{22}{5} - \frac{3y}{5}$ **[1M]**
Solve $\frac{x}{2} - 3 = -4(x'+6) \implies x' = -\frac{21}{8} - \frac{x}{8}$. **[1M]**
Therefore a sequence of transformations is:

- A reflection in the x axis
- A dilation by factor $\frac{3}{5}$ from the x-axis
- A reflection in the y-axis.
- A dilation by factor $\frac{1}{8}$ from the y-axis
- A translation $\frac{22}{5}$ units up
- A translation $\frac{21}{4}$ units to the left.

[1A] (order of reflections and dilations can be mixed around.)

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Question 2 (9 marks)

Let f be the function that is given by $f(x) = \frac{ax^2+b}{x^2-c}$, and that has the following properties:

1. $\lim_{x\to\infty} f(x) = 1$

2. $\lim_{x\to 3^-} f(x) = -\infty$

3. $f'(1) = -1$

Where $a, b, c \in R$.

a. Show that f is an even function. (1 mark)

We require that $f(x) = f(-x)$ for all valid x for an even function.
$f(-x) = \frac{a(-x)^2+b}{(-x)^2+c} = \frac{ax^2+b}{x^2-c} = f(x)$.
So is an even function. **[1A]**

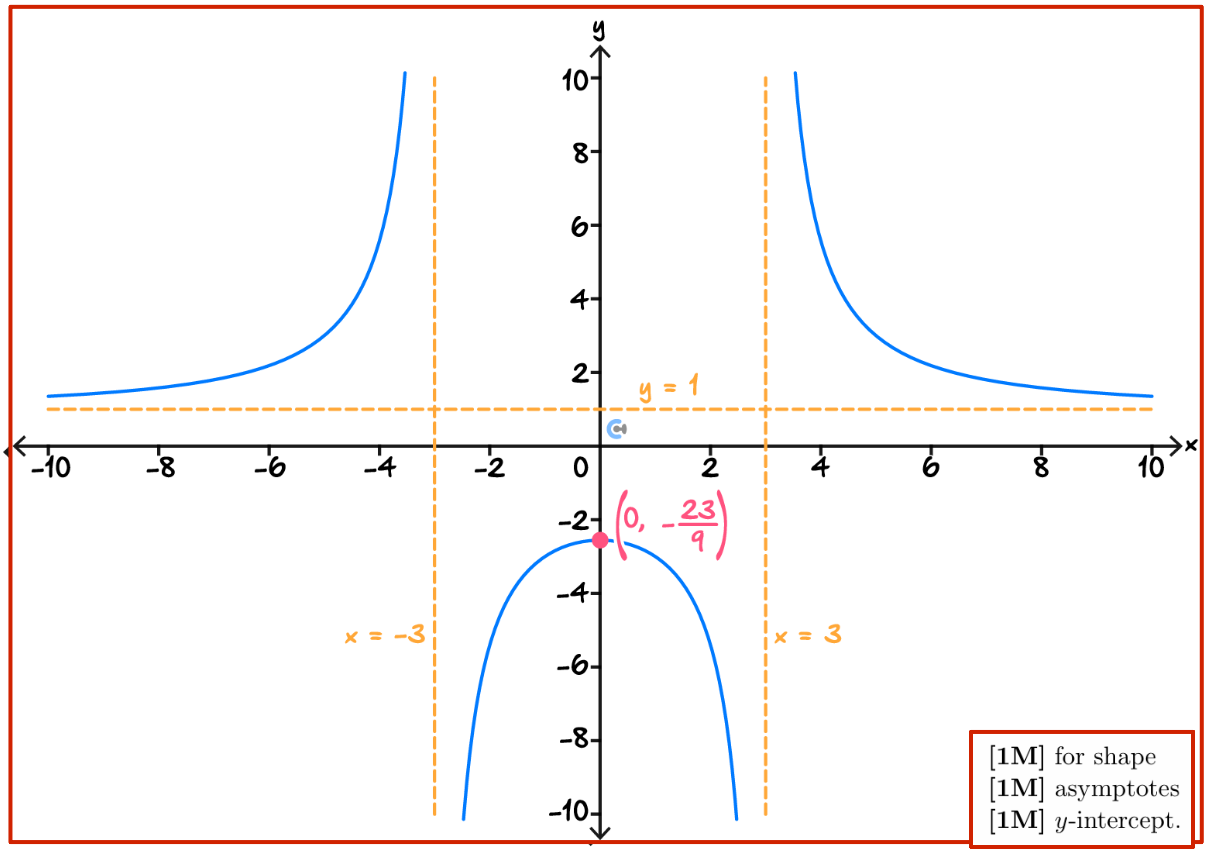
b. Show that $a = 1,\ b = 23$ and $c = 9$. (3 marks)

We can write $f(x) = \frac{ac+b}{x^2-c} + a$. So $\lim_{x\to\infty} = a = 1$. **[1M]**
Vertical asymptote when denominator is zero. Thus $x^2 - c = 0$ when $x = 3$. So $c = 9$. **[1M]**
Then we use $f'(1) = -1$ to find $b = 23$. **[1M]** .

c. Write an equation for each vertical and horizontal asymptote of the graph of f. (2 marks)

Vertical asymptotes: $x = \pm 3$ [1M]
Horizontal asymptote: $y = 1$ [1M]

d. Sketch the graph of $y = f(x)$ on the axes below. Label any asymptotes with equations and axial intercepts with coordinates. (3 marks)



[1M] for shape
[1M] asymptotes
[1M] y-intercept.

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Question 3 (14 marks)

An amusement park wants to open a new ride. Sarah, an architect working for the company, pitched a roller coaster using a model. The design of her ride is aimed towards thrill seekers who are chasing an adrenaline rush.

a. The ascent has two distinct sections. The first section is modelled by the following:

$$a_s : [-4, c], \qquad a_s(x) = 2^{3x+d}$$

Where x is an arbitrary position, whilst $a_s(x)$ is the height above the ground, and c and d are real constants. The units (or exact scale) can be ignored.

i. Write $a_s(x)$ in the form of $e^{m(nx+p)}$, where $m, n, p \in \mathbb{R}$. (2 marks)

Note that $2 = e^{\ln(2)}$. **[1M]**
Thus,

$$a_s(x) = e^{(3x+d)\ln(2)} \quad \textbf{[1A]}$$

ii. Find the rule for the inverse of $a(x) = 2^{3x+d}$. Provide your answer in the form of $\frac{\log_e(x)}{n\log_e(p)} + q$, where $n, p, q \in \mathbb{R}$. (2 marks)

To find the inverse we solve $a(y) = x$ for y. **[1M]**

$$a^{-1}(x) = \frac{\log_e(x)}{3\log_e(2)} - \frac{d}{3} \quad \textbf{[1A]}$$

iii. The latter half of the roller coaster ascent is modelled by the following:

$$a_l : \left[c, \frac{2}{\log_e(2)}\right] \to \mathbb{R}, \qquad a_l(x) = a^{-1}(x)$$

Given that the two sections join **smoothly**, find the value of c and d. (3 marks)

To join smoothly we require that $a_s(c) = a_l(c)$ **[1M]** and that $a_s'(c) = a_l'(c)$

Since they are inverses we can solve $a_s(c) = c$ and $a_s'(c) = 1$ simultaneously **[1M]**

We get $c = \frac{1}{\ln(8)}$ and $d = -\frac{1 + \ln(\ln(8))}{\ln(2)}$ **[1A]**

(Check against approx values of $c \approx 0.48$ and $d \approx -2.5$ to verify your answer)

b. The descent of the roller coaster is a steep drop, modelled by **translations** of the following function:

$$d : [m, n] \to \mathbb{R}, d(x) = -x^3$$

Where $m, n \in \mathbb{R}$.

i. Given that the descent joins continuously with the ascent at the stationary point of the image of $d(x)$, state the sequence of translations that are applied to d. (2 marks)

The descent must join the ascent section at $\left(\frac{2}{\ln(2)}, \frac{\ln(6)+1}{\ln(8)}\right)$. **[1M]**

d has stationary point at $(0, 0)$

So translations are $\frac{2}{\ln(2)}$ units right and $\frac{\ln(6)+1}{\ln(8)}$ units up. **[1A]**

ii. Can the two sections ever join smoothly? Explain your answer. (2 marks)

No. The ascent section always has a gradient > 0 and the descent section has gradient ≤ 0, so they cannot joint smoothly.

[1A for no, 1M for explanation]

iii. Given that the descent ends when it hits the ground (x-axis intercept), find the coordinates of the **endpoints** of $d(x)$ correct to 2 decimal places. (3 marks)

$d(x)$ must start at its stationary point, so one endpoint is $(0, 0)$. **[1A]**

$d(x)$ ends when it has gone $\dfrac{\ln(6)+1}{\ln(8)} \approx 1.34$ units down. **[1M]**

Solve $d(x) = -1.34 \implies x = 1.10$.

Therefore other endpoint is $(1.10, -1.34)$. **[1A]**

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Question 4 (13 marks)

Consider the piecewise function:

$$g(x) = \begin{cases} x^2 \text{ for } -\infty < x < 0 \\ \sqrt{x} \text{ for } 0 \leq x < \infty \end{cases}$$

Where $g_1(x) = x^2$ for $-\infty < x < 0$, and $g_2(x) = \sqrt{x}$ for $0 \leq x < \infty$.

a.

i. Show that $g(x)$ is continuous at $x = 0$. (2 marks)

To show continuity at $x = 0$, we need to verify the following:

- $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x)$
- $g(0)$ is defined and equals the above limit.

We evaluate:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \sqrt{x} = 0 \quad \textbf{[1M]}$$

Since $g(0) = \sqrt{0} = 0$, the function is continuous at $x = 0$. **[1A]**

ii. Show that $g(x)$ is not smooth at $x = 0$. (2 marks)

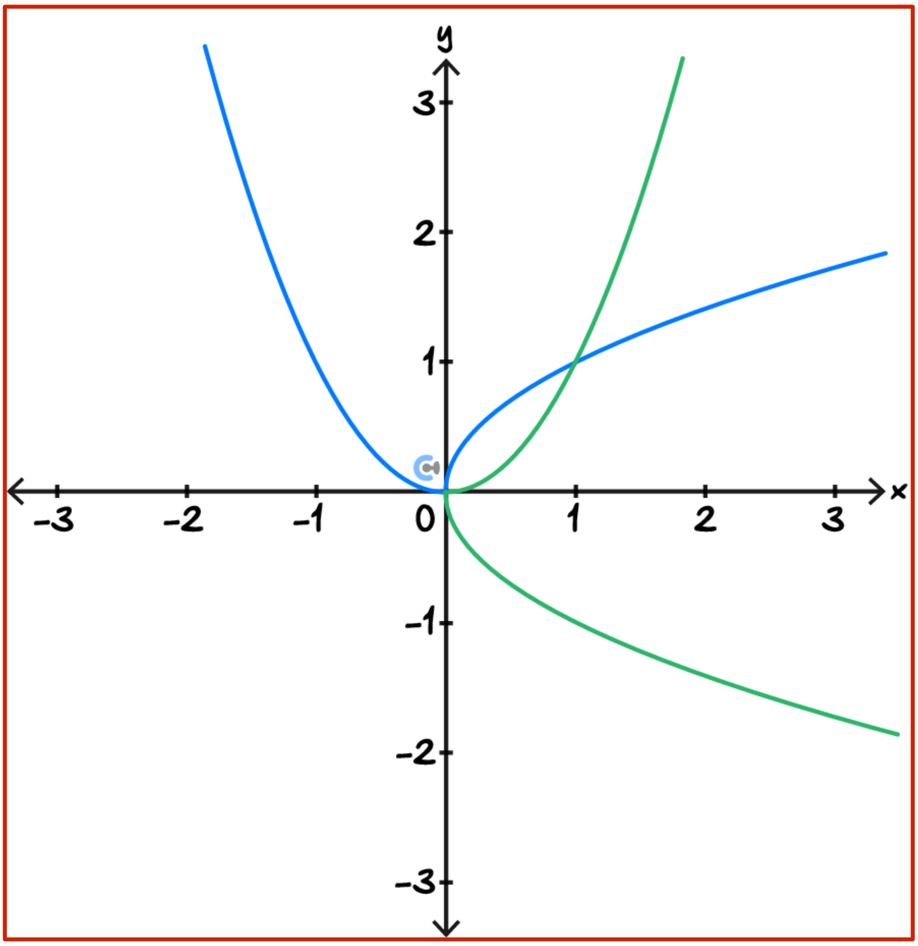
To show that $g(x)$ is not smooth at $x = 0$, we evaluate the left and right derivatives. We have that

$$g'(x) = \begin{cases} 2x, & -\infty < x < 0 \\ \dfrac{1}{2\sqrt{x}}, & 0 < x < \infty \end{cases} \quad \textbf{[1M]}$$

Then $\lim_{x \to 0^-} = \lim_{x \to 0} 2x = 0$
and $\lim_{x \to 0^+} g'(x) = \lim_{x \to 0} \dfrac{1}{2\sqrt{x}}$ which is undefined.
Thus not smooth at $x = 0$ because the left and right limits differ. **[1A]**

b.

i. Sketch the **inverse** of $g(x)$ on the axes provided. $g(x)$ has been drawn for you. (2 marks)



ii. State why g^{-1} is not a function. (1 mark)

g^{-1} is not a function because it fails the vertical line test (or not 1:1); for some values of x, there are two possible y-values in the inverse relation. **[1M]**

c.

i. By investigating compositions of $g_1(x)$ and $g_2(x)$, find the **rule** for $g(g(x))$. (4 marks)

We have that dom $g_1 = \mathbb{R}^-$ and ran $g_1 = \mathbb{R}^+$.
and that dom $g_2 = \mathbb{R}^+ \cup \{0\}$ and ran $g_2 = \mathbb{R}^+ \cup \{0\}$

Then we see that $g_1 \circ g_1$ does not exist and $g_1 \circ g_2$ does not exist.
$g_2 \circ g_1$ does exist and $g_2 \circ g_2$ does exist.
[1M determine which functions exist.]

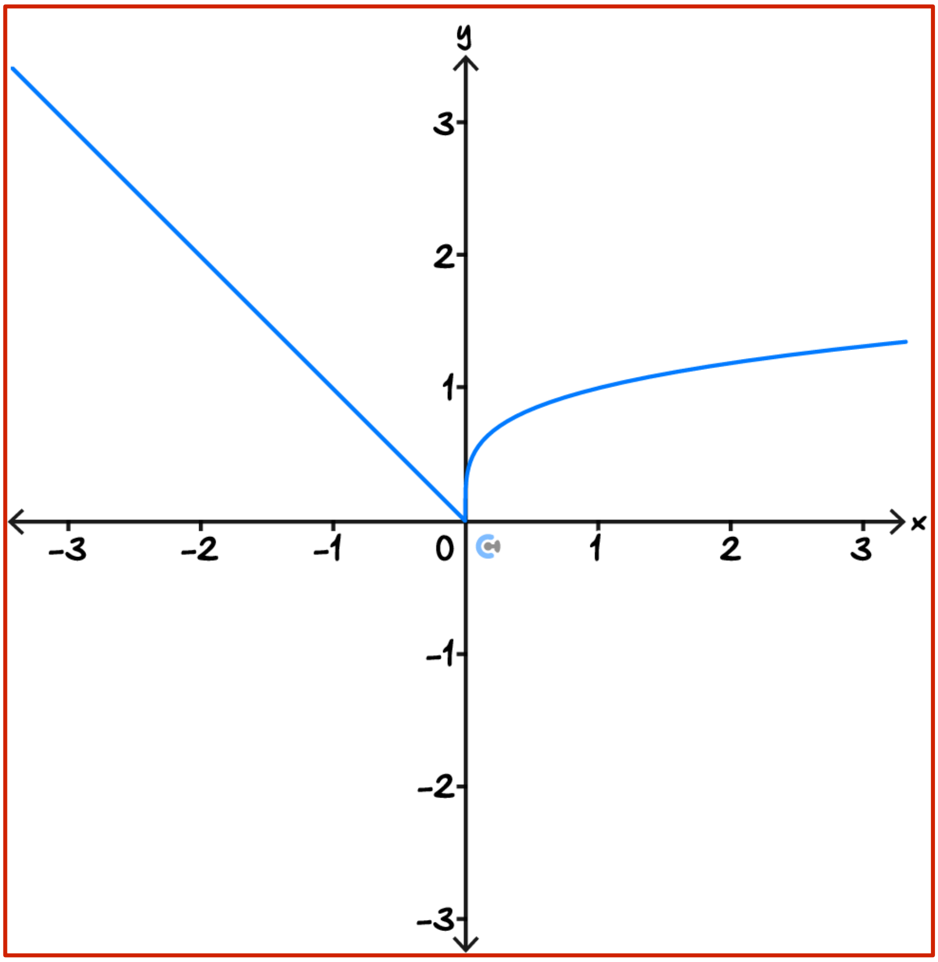
$(g_2 \circ g_1)(x) = \sqrt{x^2}$ with domain $(-\infty, 0)$ **[1M]**
$(g_2 \circ g_2)(x) = \sqrt{x^{1/2}} = x^{1/4}$ with domain $[0, \infty]$ **[1M]**

So

$$g(g(x)) = \begin{cases} -x, & -\infty < x < 0 \\ x^{1/4}, & 0 \leq x < \infty \end{cases}$$

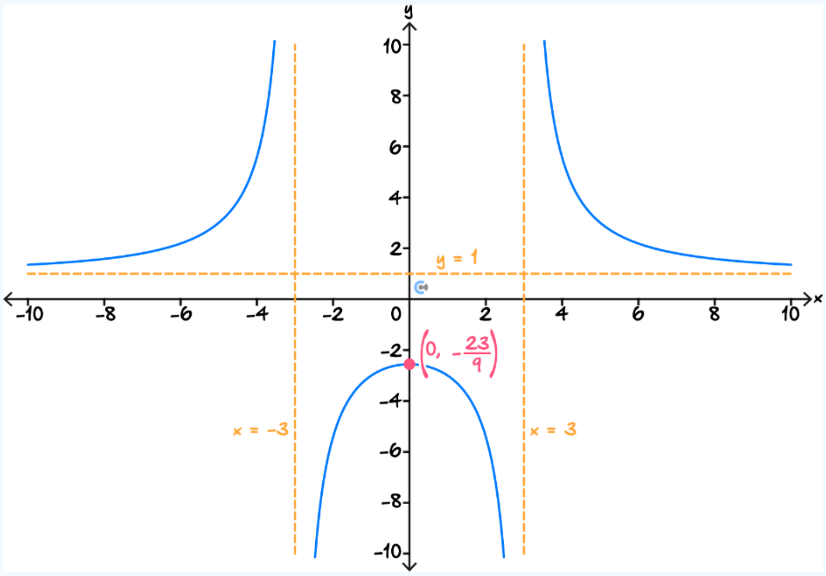
[1A]

ii. Hence, sketch $g(g(x))$ on the axes provided. (2 marks)

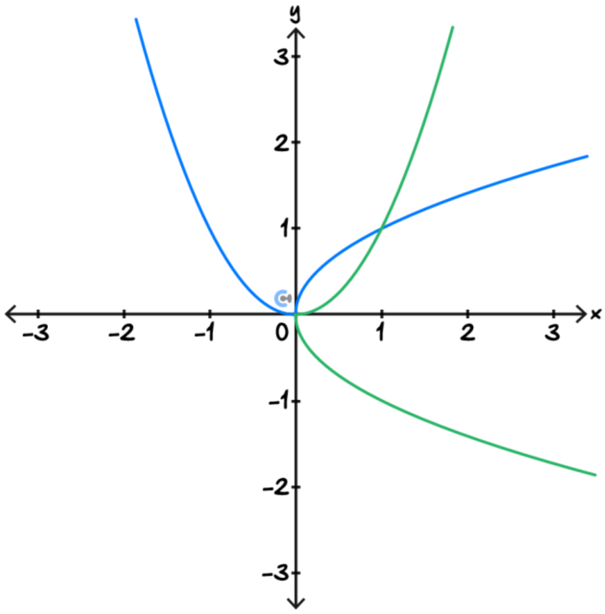


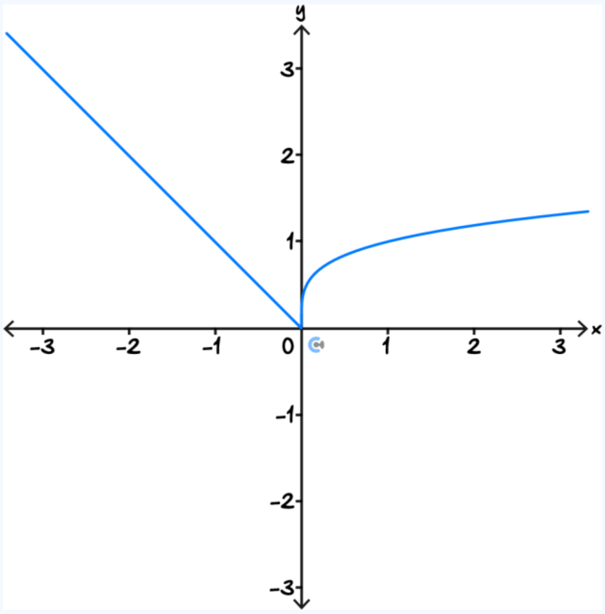
Section C: Marking Scheme

Question Number	Solutions
1a	We apply the transformations to the point (x, y) in order: $(x,y) \to \left(x, \frac{1}{2}y\right) \to \left(x, -\frac{1}{2}y\right) \to \left(-\frac{1}{2}y, x\right) \to (-4y, x) \to (4y, x)$ **[1M for swapping x and y when reflecting in y=x, 1M for clear logical sequence]**
1b	A reflection in the line $y = x$ **[1M]** followed by a dilation by factor 4 from the y-axis **[1M]**
1ci	We have that $x' = 4y \implies y = \frac{1}{4}x'$ and $y' = x \implies x = y'$. **[1M]** Substitute into the equation and remove the dashes to get: $$\frac{1}{4}x = \frac{y^3}{(y+2)^2}$$ $$x = \frac{4y^3}{(y+2)^2} \quad \textbf{[1A]}$$
1cii	The pre-image is **not** a 1:1 function. **[1M]** The transformation involves reflecting in the line $y = x$ which produces the inverse of the pre-image, but the pre-image is not 1:1 so its image will not be a function. **[1M]** .
1di	• A translation 4 units down (or a dilation by factor $\frac{1}{5}$ from the x-axis) • A dilation by factor $\frac{1}{5}$ from the x-axis (a translation $\frac{4}{5}$ units down) **[1M]** Then we have $x' = \frac{x}{2} - 3 = \frac{1}{2}(x-6)$, so • A translation 6 units to the left (or a dilation by factor $\frac{1}{2}$ from the y-axis) • A dilation by factor $\frac{1}{2}$ from the y-axis (a translation 3 units left) **[1M]** There could be many different combinations.
1dii	Immediately see that $y' = -3y + 2$. **[1M]** Then $-4(x' + 6) = x \implies x' = -\frac{1}{4}x - 6$. **[1M]** So $a = -\frac{1}{4}, b = 6, c = -3$ and $d = 2$. **[1A]**

1diii	Can do all the transformations in **part d.i** followed by the transformations in **part d.ii** and simplify. It is probably simpler to just do it from scratch. Solve $\frac{y-4}{5} = \frac{y'-2}{-3} \implies y' = \frac{22}{5} - \frac{3y}{5}$ **[1M]** Solve $\frac{x}{2} - 3 = -4(x'+6) \implies x' = -\frac{21}{8} - \frac{x}{8}$. **[1M]** Therefore a sequence of transformations is: • A reflection in the x axis • A dilation by factor $\frac{3}{5}$ from the x-axis • A reflection in the y-axis. • A dilation by factor $\frac{1}{8}$ from the y-axis • A translation $\frac{22}{5}$ units up • A translation $\frac{21}{4}$ units to the left. **[1A]** (order of reflections and dilations can be mixed around.)
2a	We require that $f(x) = f(-x)$ for all valid x for an even function. $f(-x) = \frac{a(-x)^2+b}{(-x)^2+c} = \frac{ax^2+b}{x^2-c} = f(x)$. So is an even function. **[1A]**
2b	We can write $f(x) = \frac{ac+b}{x^2-c} + a$. So $\lim_{x\to\infty} = a = 1$. **[1M]** Vertical asymptote when denominator is zero. Thus $x^2 - c = 0$ when $x = 3$. So $c = 9$. **[1M]** Then we use $f'(1) = -1$ to find $b = 23$. **[1M]** .
2c	Vertical asymptotes: $x = \pm 3$ **[1M]** Horizontal asymptote: $y = 1$ **[1M]**
2d	 **[1M]** for shape **[1M]** asymptotes **[1M]** y-intercept.

3ai	Note that $2 = e^{\ln(2)}$. **[1M]** Thus, $$a_s(x) = e^{(3x+d)\ln(2)} \quad \textbf{[1A]}$$
3aii	To find the inverse we solve $a(y) = x$ for y. **[1M]** $$a^{-1}(x) = \frac{\log_e(x)}{3\log_e(2)} - \frac{d}{3} \quad \textbf{[1A]}$$
3aiii	To join smoothly we require that $a_s(c) = a_l(c)$ **[1M]** and that $a_s'(c) = a_l'(c)$ Since they are inverses we can solve $a_s(c) = c$ and $a_s'(c) = 1$ simultaneously **[1M]** We get $c = \frac{1}{\ln(8)}$ and $d = -\frac{1+\ln(\ln(8))}{\ln(2)}$ **[1A]** (Check against approx values of $c \approx 0.48$ and $d \approx -2.5$ to verify your answer)
3bi	The descent must join the ascent section at $\left(\frac{2}{\ln(2)}, \frac{\ln(6)+1}{\ln(8)}\right)$. **[1M]** d has stationary point at $(0,0)$ So translations are $\frac{2}{\ln(2)}$ units right and $\frac{\ln(6)+1}{\ln(8)}$ units up. **[1A]**
3bii	No. The ascent section always has a gradient > 0 and the descent section has gradient ≤ 0, so they cannot joint smoothly. **[1A for no, 1M for explanation]**
3biii	$d(x)$ must start at its stationary point, so one endpoint is $(0,0)$. **[1A]** $d(x)$ ends when it has gone $\frac{\ln(6)+1}{\ln(8)} \approx 1.34$ units down. **[1M]** Solve $d(x) = -1.34 \implies x = 1.10$. Therefore other endpoint is $(1.10, -1.34)$. **[1A]**
4ai	To show continuity at $x = 0$, we need to verify the following: • $\lim_{x\to0^-} g(x) = \lim_{x\to0^+} g(x)$ • $g(0)$ is defined and equals the above limit. We evaluate: $$\lim_{x\to0^-} g(x) = \lim_{x\to0^-} x^2 = 0 \quad \text{and} \quad \lim_{x\to0^+} g(x) = \lim_{x\to0^+} \sqrt{x} = 0 \quad \textbf{[1M]}$$ Since $g(0) = \sqrt{0} = 0$, the function is continuous at $x = 0$. **[1A]**

4aii	To show that $g(x)$ is not smooth at $x=0$, we evaluate the left and right derivatives. We have that $$g'(x)=\begin{cases}2x, & -\infty<x<0\\ \frac{1}{2\sqrt{x}}, & 0<x<\infty\end{cases} \quad \textbf{[1M]}$$ Then $\lim_{x\to 0^-} = \lim_{x\to 0} 2x = 0$ and $\lim_{x\to 0^+} g'(x) = \lim_{x\to 0} \frac{1}{2\sqrt{x}}$ which is undefined. Thus not smooth at $x=0$ because the left and right limits differ. **[1A]**
4bi	
4bii	g^{-1} is not a function because it fails the vertical line test (or not 1:1); for some values of x, there are two possible y-values in the inverse relation. **[1M]**
4ci	We have that dom $g_1=\mathbb{R}^-$ and ran $g_1=\mathbb{R}^+$. and that dom $g_2=\mathbb{R}^+\cup\{0\}$ and ran $g_2=\mathbb{R}^+\cup\{0\}$ Then we see that $g_1\circ g_1$ does not exist and $g_1\circ g_2$ does not exist. $g_2\circ g_1$ does exist and $g_2\circ g_2$ does exist. **[1M determine which functions exist.]** $(g_2\circ g_1)(x)=\sqrt{x^2}$ with domain $(-\infty,0)$ **[1M]** $(g_2\circ g_2)(x)=\sqrt{x^{1/2}}=x^{1/4}$ with domain $[0,\infty]$ **[1M]** So $$g(g(x))=\begin{cases}-x, & -\infty<x<0\\ x^{1/4}, & 0\le x<\infty\end{cases}$$ **[1A]**

4cii	

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Section D: Mathematica Solutions

Question Number	Solutions

1	**Q1.**
	d.
	In[2]:= f[x_] := 5 Exp[x / 2 - 3] + 4
	In[3]:= 1 / 5 (f[x] - 4)
	Out[3]= $e^{-3+\frac{x}{2}}$
	In[5]:= 1 / 5 f[x] - 4 / 5 // FullSimplify
	Out[5]= $e^{-3+\frac{x}{2}}$
	In[6]:= f1[x_] := $e^{-3+\frac{x}{2}}$
	In[12]:= f1[2 x + 6] // FullSimplify
	Out[12]= e^{x}
	In[14]:= f1[2 (x + 3)]
	Out[14]= e^{x}
	In[16]:= q[x_] := Exp[x]
	In[18]:= -3 q[-4 (x + 6)] + 2
	Out[18]= $2 - 3 e^{-4 (6+x)}$
	In[15]:= g[x_] := -3 Exp[-4 (x + 6)] + 2
	In[24]:= Solve[$\frac{y-4}{5} == \frac{y1-2}{-3}$, y1] // Expand
	Out[24]= $\{\{y1 \to \frac{22}{5} - \frac{3 y}{5}\}\}$
	In[22]:= Solve[$\frac{x}{2} - 3 == -4 (x1 + 6)$, x1] // Expand
	Out[22]= $\{\{x1 \to -\frac{21}{4} - \frac{x}{8}\}\}$
	In[25]:= -3 / 5 f[-8 (x + 21 / 4)] + 22 / 5 // FullSimplify
	Out[25]= $2 - 3 e^{-4 (6+x)}$

2

Q2.

```
In[39]:= f[x_] := (a x^2 + b)/(x^2 - c)

In[40]:= f[x] == f[-x]

Out[40]= True

In[48]:= Apart[f[x]]
```

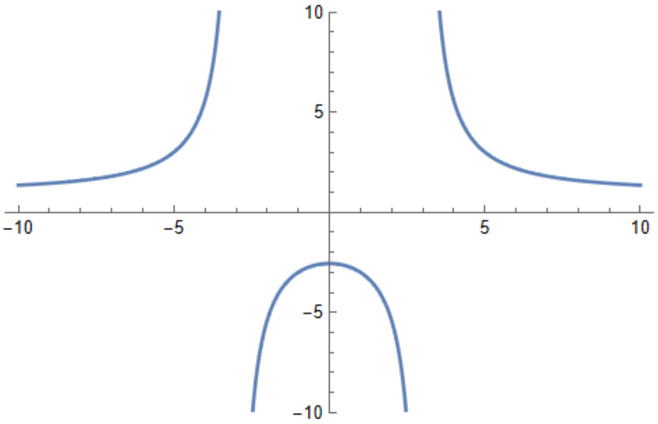
$$\text{Out[48]= } a + \frac{b + a c}{-c + x^2}$$

```
In[50]:= f'[1]
```

$$\text{Out[50]= } -\frac{2\,(a + b)}{(1 - c)^2} + \frac{2\,a}{1 - c}$$

```
In[51]:= Solve[-(2 (a + b))/(1 - c)^2 + (2 a)/(1 - c) == -1 /. {a → 1, c → 9}]

Out[51]= {{b → 23}}

(* Confirm that the properties are satisfied*)

In[52]:= f1[x_] := (x^2 + 23)/(x^2 - 9)

In[54]:= Limit[f1[x], x → Infinity]

Out[54]= 1

In[56]:= Limit[f1[x], x → 3, Direction → "FromBelow"]

Out[56]= -∞

In[57]:= f1'[1]

Out[57]= -1

In[58]:= Plot[f1[x], {x, -10, 10}, PlotRange → {-10, 10}]
```

Out[58]=




```
In[59]:= f1[0]
```

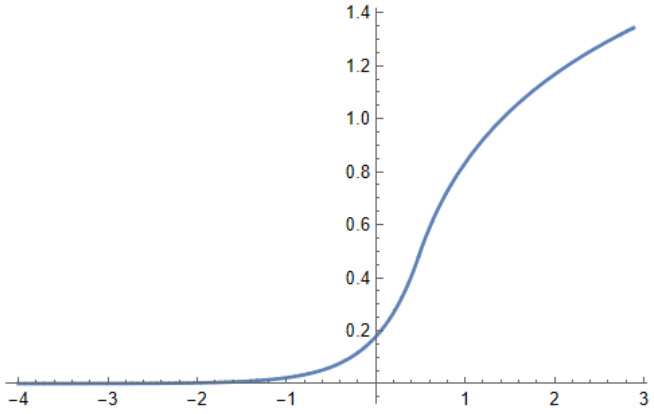
$$\text{Out[59]= } -\frac{23}{9}$$

3

Q3.

a

```
In[60]:= a[x_] := 2^(3 x+d)

In[63]:= Exp[(3 x + d) Log[2]]

Out[63]= 2^(d+3 x)

In[65]:= Solve[a[y] == x, y, Reals]

Out[65]= {{y → -d/3 + Log[x]/(3 Log[2]) if x > 0}}

In[68]:= Solve[a[c] == c && a'[c] == 1] // FullSimplify

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete soluti

Out[68]= {{c → 1/Log[8], d → -(1 + Log[Log[8]])/Log[2]}}

(* Note that we dont get exact solution if we try to
 brute force by using the inverse*)

In[69]:= a1[x_] := -d/3 + Log[x]/(3 Log[2])

In[73]:= Solve[a[c] == a1[c] && a'[c] == a1'[c], Reals] // FullSimplify

Out[73]= {{c → 0.481..., d → -(1 + Log[9 Log[2]^2 0.481...])/Log[2]}}

In[75]:= b[x_] := 2^(3 x - (1+Log[Log[8]])/Log[2])

In[76]:= b1[x_] := Log[x]/(3 Log[2]) + 1/3 * (1 + Log[Log[8]])/Log[2]

In[94]:= af[x_] := Piecewise[{{b[x], -4 ≤ x ≤ 1/Log[8]}, {b1[x], 1/Log[8] < x < 2/Log[2]}}]

In[95]:= Plot[af[x], {x, -4, 2/Log[2]}]

Out[95]=
```



b.

```
In[97]:= b1[2/Log[2]] // FullSimplify

Out[97]= (1 + Log[Log[64]/Log[2]])/Log[8]

In[101]:= FullSimplify[Log[Log[64]/Log[2]]/Log[8]]

Out[101]= Log[6]/Log[8]

In[104]:= d[x_] := -x^3

In[110]:= NSolve[d[x] == -(1 + Log[Log[64]/Log[2]])/Log[8], Reals]

Out[110]= {{x → 1.10317}}

In[112]:= 2 / Log[2] // N

Out[112]= 2.88539

In[111]:= d[1.1031733472385987]

Out[111]= -1.34255
```

4

Q4.

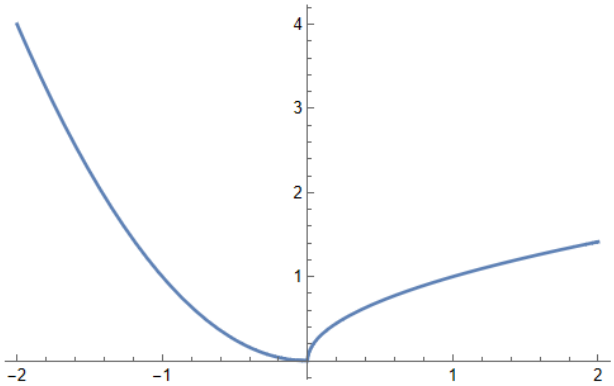
```
In[143]:= g1[x_] := x^2
In[144]:= g2[x_] := √x
In[145]:= g[x_] := Piecewise[{{g1[x], -Infinity < x < 0}, {g2[x], 0 ≤ x < Infinity}}]
In[146]:= g[x]
```

Out[146]=

$$\begin{cases} x^2 & -\infty < x < 0 \\ \sqrt{x} & 0 \le x < \infty \\ 0 & \text{True} \end{cases}$$

```
In[147]:= Plot[g[x], {x, -2, 2}]
```

Out[147]=




```
In[148]:= FunctionRange[{g1[x], x > 0}, x, y]
```

Out[148]= $y > 0$

```
In[149]:= FunctionRange[g2[x], x, y]
```

Out[149]= $y \ge 0$

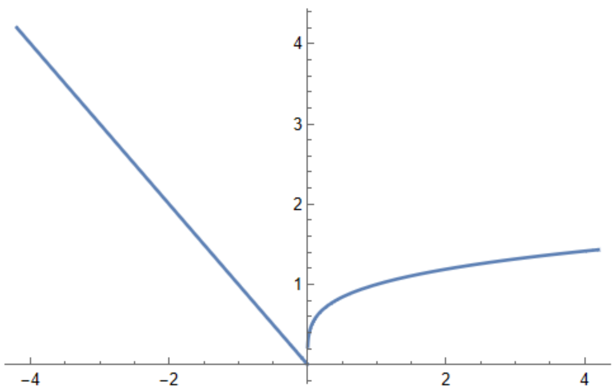
```
In[150]:= g[g[x]] // Simplify
```

Out[150]=

$$\begin{cases} x & \sqrt{x} \in \mathbb{R} \,\&\&\, \sqrt{x} < 0 \,\&\&\, x \ge 0 \\ x^{1/4} & \sqrt{x} \in \mathbb{R} \,\&\&\, x \ge 0 \,\&\&\, \sqrt{x} \ge 0 \\ \sqrt{x^2} & \left(\sqrt{x} \in \mathbb{R} \,\&\&\, \sqrt{x} \ge 0\right) \,||\, x < 0 \\ 0 & \text{True} \end{cases}$$

```
In[151]:= Plot[Piecewise[{{x^(1/4), x ≥ 0}, {√(x^2), x < 0}}, 0], {x, -4.2, 4.2}]
```

Out[151]=



Section E: Casio Solutions

Question Number	Solutions
1	Part d. Define $f(x)=5e^{x/2-3}+4$ → done $\frac{1}{5}(f(x)-4)$ → $e^{\frac{x}{2}-3}$ Define $f1(x)=e^{\frac{x}{2}-3}$ → done $f1(2x+6)$ → $e^{\frac{2\cdot x+6}{2}-3}$ simplify(ans) → e^{x} $f1(2(x+3))$ → e^{x} Define $q(x)=e^{x}$ → done $-3q(-4(x+6))+2$ → $-3\cdot e^{-4\cdot(x+6)}+2$ simplify(ans) → $-3\cdot e^{-4\cdot x-24}+2$ solve$\left(\frac{y-4}{5}=\frac{yone-2}{-3}, yone\right.$ → $\left\{yone=\frac{-3\cdot y}{5}+\frac{22}{5}\right\}$ solve$\left(\frac{x}{2}-3=-4(xone+6), xone\right.$ → $\left\{xone=\frac{-x}{8}-\frac{21}{4}\right\}$ $-3/5f(-8(x+21/4))+22/5$ → $\frac{-3\cdot\left(5\cdot e^{-4\cdot\left(x+\frac{21}{4}\right)-3}+4\right)}{5}+\frac{22}{5}$ simplify(ans) → $-3\cdot e^{-4\cdot x-24}+2$

<table><tr><td>2</td><td>

Define $f(x)=\dfrac{a*x\hat{}2+b}{x\hat{}2-c}$

done

$f(-x)$

$\dfrac{a\cdot x^2+b}{x^2-c}$

$\left\{\begin{array}{l}\lim\limits_{x\to\infty}(f(x))=1\\ x\hat{}2-c=0\mid x=3\\ \dfrac{d}{dx}(f(x))=-1\mid x=1\end{array}\right|a,b,c$

$\{a=1, b=23, c=9\}$

$f(x)\mid\{a=1, b=23, c=9\}$

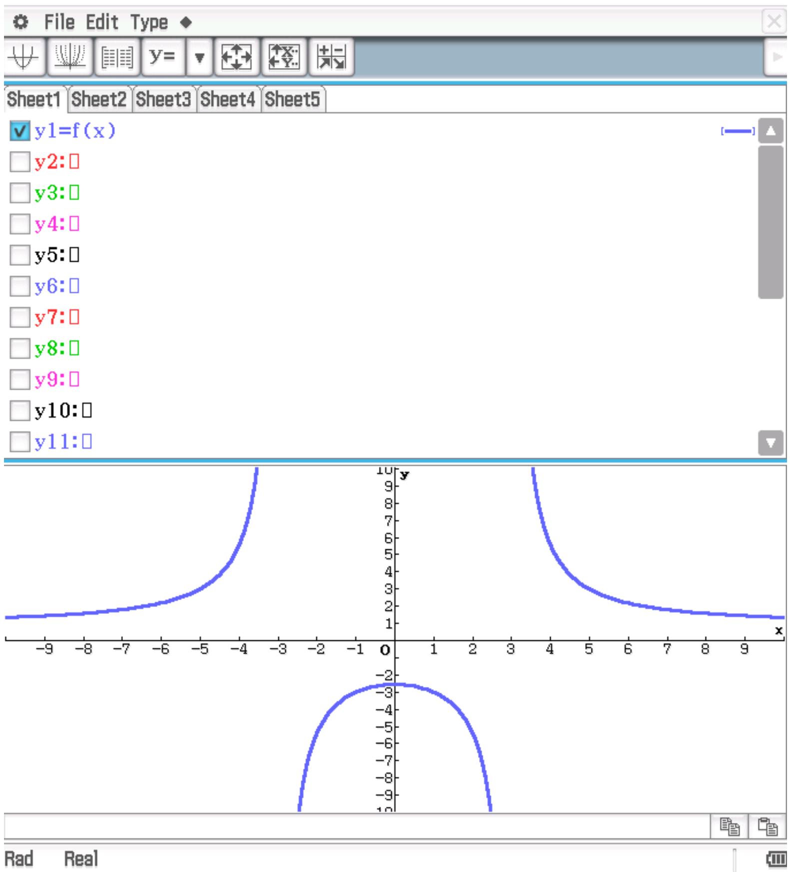
$\dfrac{x^2+23}{x^2-9}$

propFrac(ans)

$\dfrac{32}{x^2-9}+1$

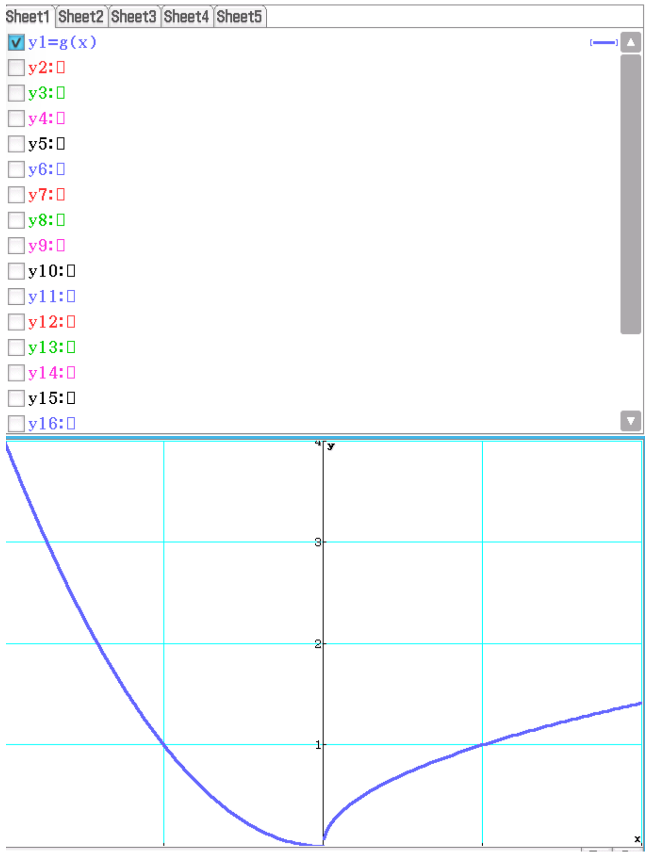
Define $f(x)=\dfrac{x^2+23}{x^2-9}$

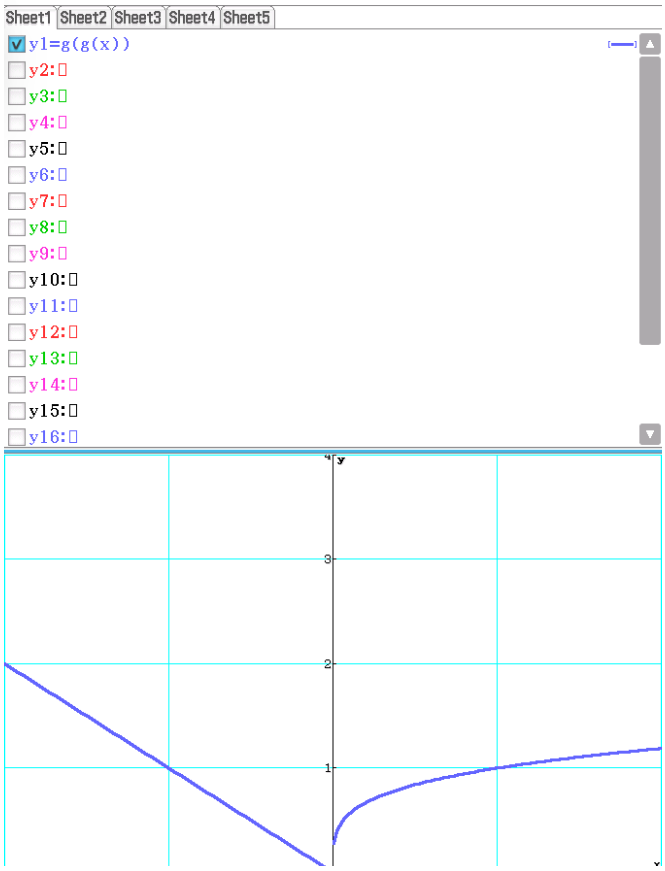
done



</td></tr></table>

3	Define $\text{a1}(x)=2^{3x+d}$ done simplify(a1(x)) $2^{3\cdot x+d}$ solve(a1(y)=x, y $\left\{y=\frac{\ln(x)}{3\cdot\ln(2)}-\frac{d}{3}\right\}$ Define $\text{a2}(x)=\frac{\ln(x)}{3\cdot\ln(2)}-\frac{d}{3}$ done $\left.\begin{cases}\text{a1}(x)=x\\ \frac{d}{dx}(\text{a1}(x))=1\end{cases}\right\|_{x,\,d}$ $\left\{x=2^{\frac{-\ln(\ln(2))}{\ln(2)}-\frac{\ln(3)}{\ln(2)}}, d=\frac{-\ln(\ln(2))}{\ln(2)}-\frac{\ln(3)}{\ln(2)}-3\cdot 2^{\frac{-\ln(\ln(2))}{\ln(2)}-\frac{\ln(3)}{\ln(2)}}\right\}$ simplify(ans) $\left\{x=\frac{1}{3\cdot\ln(2)}, d=\frac{-(\ln(\ln(2))+\ln(3)+1)}{\ln(2)}\right\}$ $\text{a2}(2/\ln(2))\|d=\frac{-(\ln(\ln(2))+\ln(3)+1)}{\ln(2)}$ $\frac{\ln(\ln(2))+\ln(3)+1}{3\cdot\ln(2)}+\frac{-\ln(\ln(2))+\ln(2)}{3\cdot\ln(2)}$ simplify(ans) $\frac{\ln(6)+1}{3\cdot\ln(2)}$ solve$\left(-x\text{^}3=-\frac{\ln(6)+1}{3\cdot\ln(2)}, x\right.$ {x=1.103173347} $-\frac{\ln(6)+1}{3\cdot\ln(2)}$ −1.342552514
4	Define g1(x)=x^2 done Define $\text{g2}(x)=\sqrt{x}$ done Define $g(x)=\begin{cases}x\text{^}2, x<0\\ \sqrt{x}, x\geq 0\end{cases}$ done g(x) Undefined g(g(x)) Undefined CASIO struggles – do by hand!


Sheet1 Sheet2 Sheet3 Sheet4 Sheet5
y1=g(x)
y
x
1
2
3


Sheet1 Sheet2 Sheet3 Sheet4 Sheet5
y1=g(g(x))
y
1
2
3

Section F: TI solutions

Question Number	Solutions
1	d) The transform program can be used to check your transformations are correct. In the current version, it is not yet able to find the transformations. i) Define $f(x)=5\cdot e^{\frac{x}{2}-3}+4$ *Done* $\text{◂hods\_func\transform}\left(f(x),x,\left\{y-4,\frac{y}{5},x-6,\frac{x}{2}\right\}\right)$ ▸ Translation −4 units along the neg. y−dir. $5\cdot e^{\frac{x}{2}-3}$ ▸ Dilation by factor of $\frac{1}{5}$ from the x−axis $e^{\frac{x}{2}-3}$ ▸ Translation 6 units along the neg. x−dir. $e^{\frac{x}{2}}$ ▸ Dilation by factor of $\frac{1}{2}$ from the y−axis e^{x} ii) $\text{◂thods\_func\transform}\left(e^{x},x,\left\{\frac{-x}{4}+6,-3\cdot y+2\right\}\right)$ ▸ Dilation by factor of $\frac{1}{4}$ from the y−axis $e^{4\cdot x}$ ▸ Reflection in the y−axis $e^{-4\cdot x}$

	▸ Translation 6 units along the pos. x−dir. $e^{24-4\cdot x}$ ▸ Dilation by factor of 3 from the x−axis $3\cdot e^{24-4\cdot x}$ ▸ Reflection in the x−axis $-3\cdot e^{24-4\cdot x}$ ▸ Translation 2 units along the pos. y−dir. $2-3\cdot e^{24-4\cdot x}$
2	b) Define $f(x)=\frac{a\cdot x^2+b}{x^2-c}$ *Done* Define $df(x)=\frac{d}{dx}(f(x))$ *Done* expand$(f(x))$ $\frac{a\cdot c}{x^2-c}+\frac{b}{x^2-c}+a$ Define $a=1$ *Done* Define $c=9$ *Done* $df(1)$ $\frac{-(b+9)}{32}$ solve$(df(1)=-1,b)$ $b=23$ Define $b=23$ *Done*

	c,d) $methods\_func\backslash analyse(f(x),x)$ ▸ Start Point: $[-\infty \quad 1]$ ▸ End Point: $[\infty \quad 1]$ ▸ Maximal Domain: $x \neq -3$ and $x \neq 3$ and $-\infty < x < \infty$ ▸ No x –Intercepts Found ▸ Vertical Intercept: $\left[0 \quad \frac{-23}{9}\right]$ ▸ Derivative: $\frac{-64 \cdot x}{(x^2-9)^2}$ ▸ Asymptotes: (3) $x=-3$ (Vertical) $x=3$ (Vertical) $y=1$ (Horizontal)
3	a) ii) Note that the third argument in the inverse program specifies any point in the domain of the function. Since the exponential is defined and one-to-one for all real numbers, we can choose 0. In general, one must take care when choosing this point. Define $as(x)=2^{3\cdot x+d}$ *Done* $methods\_func\backslash inverse(as(x),x,0)$ $\left\{\frac{\ln(x)-d\cdot \ln(2)}{3\cdot \ln(2)}, x>0\right.$ *Done* $\text{expand}\left(\frac{\ln(x)-d\cdot \ln(2)}{3\cdot \ln(2)}\right)$ $\frac{\ln(x)}{3\cdot \ln(2)}-\frac{d}{3}$

	iii) We find the value of d where the function a_s touches the line $y = x$ smoothly. The point where it touches is exactly the point c. $methods\_diffcalc\backslash solve\_touch(as(x),x,x,d)$ ▸ Derivative 1: $3\cdot\ln(2)\cdot 2^{d}\cdot 8^{x}$ ▸ Derivative 2: 1 ▸ Equating functions and derivatives. ▸ Solutions: $x=\frac{1}{3\cdot\ln(2)}$ and $d=\frac{-(\ln(3\cdot\ln(2))+1)}{\ln(2)}$ b) i) Define $al(x)=\frac{\ln(x)}{3\cdot\ln(2)}-\frac{d}{3}\mid d=\frac{-(\ln(3\cdot\ln(2))+1)}{\ln(2)}$ *Done* $al\left(\frac{2}{\ln(2)}\right)$ $\frac{\ln(6)+1}{3\cdot\ln(2)}$ ii) $\text{solve}\left(-x^{3}=\frac{-(\ln(6)+1)}{3\cdot\ln(2)},x\right)$ $x=1.10317$ $-(1.10317)^{3}$ -1.34254
4	c) The range of g is $[0, \infty)$ so only the g_2 branch of the outer g is required when computing the composition. Define $g1(x)=x^{2}$ *Done* Define $g2(x)=\sqrt{x}$ *Done* $g2(g1(x))\mid x<0$ $-x$ $g2(g2(x))\mid x\geq 0$ $x^{\frac{1}{4}}$

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