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VCE Mathematical Methods ¾
AOS 2 Revision [0.15]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________________ Notes:	Pg / Q #: ____________________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
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Section A: Cheat Sheets

Cheat Sheet

[2.1.1] - Find the Instantaneous Rate of Change and Average Rate of Change

- The average rate of change of a function $f(x)$ over $x \in [a, b]$ is given by:
  - $$\text{Average rate of change} = \boxed{\frac{f(b) - f(a)}{b - a}}$$
- It is the gradient of the line joining the two points.
- **Instantaneous Rate of Change** is a gradient of a graph at a single point/moment
- **First Principles** derivative definition:
  - $$f'(x) = \boxed{\lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)}$$
- **The Product Rule**
  - The derivative of $h(x) = f(x) \times g(x)$ is given by:

    $$h'(x) = \boxed{f'(x) \cdot g(x) + g'(x) \cdot f(x)}$$
- **The Quotient Rule**
  - The derivative of a $h(x) = \frac{f(x)}{g(x)}$ is given by:

    $$h'(x) = \boxed{\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}}$$
  - Always differentiate the top function first.

- **The Chain Rule**

  $$y = f(g(x))$$

  $$\frac{dy}{dx} = \boxed{f'(g(x))g'(x)}$$
- The **process** for finding derivatives of **composite functions.**

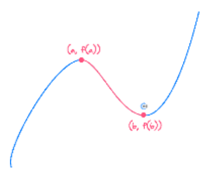
[2.1.2] – Identify the Nature of Stationary Points and Trends (Strictly Increasing and Decreasing)

- Point where the $\boxed{\text{gradient}}$ of the function is zero.

  $$f'(x) = 0, \frac{dy}{dx} = 0$$
- We can identify the nature of a stationary point by using the sign table.

x	Less than a	a	Bigger than a
$f'(x)$	Negative	0	Positive
Shape	∩ - Decreasing curve	Stationary Point	∪ - Increasing curve

- Find the gradient of the neighboring points.
- **Strictly Increasing and Strictly Decreasing Functions**



Strictly Increasing: $x \in \boxed{(-\infty, a] \cup [b, \infty)}$

Strictly Decreasing: $\boxed{x \in [a, b]}$

- Steps:
  1. Find the $\boxed{\text{turning points}}$.
  2. Consider the sign of the $\boxed{\text{derivative}}$ between/outside the turning points.

Cheat Sheet

[2.1.3] - Graph Derivative Functions

- **Steps on Sketching the Derivative Function:**

  1. Plot x-intercept at the same x-value as the stationary point of the original.
  2. Consider the trend of the original function and sketch the derivative.
     - Original is increasing → Derivative is above the x-axis.
     - Original is decreasing → Derivative is below the x-axis

[2.2.1] - Evaluate Limits and Find Points Where the Function is Not Continuous

- **Limit Definition:**

$$\lim_{x \to a} f(x) = L$$

"The function $f(x)$ approaches L as x approaches a"

- **Validity of Limit:**

$$\lim_{x \to a} f(x) \text{ exists if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

  - Limit is defined when the left limit equals the right it.

- **Continuity:**
  - **A function f is said to be continuous at a point $x = a$ if:**
    1. $f(x)$ is defined at $x = a$
    2. $\lim_{x \to a} f(x)$ exists.
    3. $\lim_{x \to a} f(x) = f(a)$.

[2.2.2] - Apply Differentiability to Find Points Where Functions are Not Differentiable, Domain of the Derivative and Unknowns of a Function

- **Differentiability:**
  - **A function f is said to be differentiable at a point $x = a$ if:**
    1. $f(x)$ is continuous at $x = a$.
    2. $\lim_{x \to a} f'(x)$ exists.
       - Limit exists when the left and right limits are the same.
       - Gradient on the LHS and RHS must be the same.

- We **cannot** differentiate:
  1. Discontinuous Points
  2. Sharp Points
  3. Endpoints

- **Finding the Derivative of Hybrid Functions**
  1. Simply derive each function.
  2. Reject the values for x that are not differentiable from the domain.

Cheat Sheet

[2.2.3] – Identify Concavity and Find Inflection Points

- **Second Derivatives**
  - The derivative of the derivative.
  - To get the second derivative, we can **differentiate the original function twice**.

$$\frac{d^2y}{dx^2} = f''(x)$$

- **Concavity**
  - Concave up is when the gradient is increasing.

$$f''(x) > 0 \rightarrow \text{Concave Up}$$

  - Concave down is when the gradient is decreasing.

$$f''(x) < 0 \rightarrow \text{Concave Down}$$

  - Zero concavity is when the gradient is neither increasing nor decreasing.

$$f''(x) = 0 \rightarrow \text{Zero Concavity}$$

- **Points of Inflection**
  - A point at which a curve changes concavity is called a **point of inflection**.
- **The Second Derivative Test**
  - Suppose that $f'(a) = 0$ and hence, f has a stationary point at $x = a$. The second derivative test states:
    - **Concave up gives us local minimum.**

$$f''(x) > 0 \rightarrow \text{Local Minimum}$$

    - **Concave down gives us local maximum.**

$$f''(x) < 0 \rightarrow \text{Local Maximum}$$

    - **Zero concavity gives us stationary point of inflection.**

$$f''(x) = 0 \rightarrow \text{Stationary Point of Inflection}$$

[2.3.1] – Find Derivatives with Functional Notation

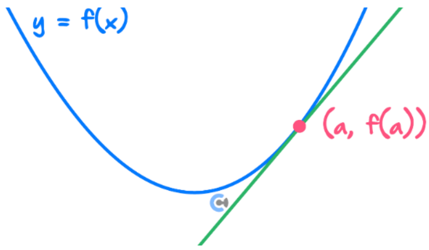
- To derive composite functions like $\sin(f(x))$, apply the chain rule.

[2.3.2] – Apply Differentiability to Join Two Functions Smoothly

- When two functions join smoothly at a point, the value and derivative of each function are both equal at that point.

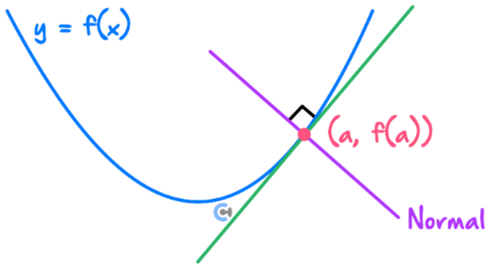
[2.4.1] – Find Tangents and Normals

- A **tangent** is a linear line which **just** touches the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$\text{At } (a, f(a)): \quad m_{tangent} = \boxed{f'(a)}$$

- **Normals**
  - A **normal** is a linear line which is perpendicular to the tangent.
  - The gradient of a normal line has to equal to the negative reciprocal of the gradient of the curve at the intersection.



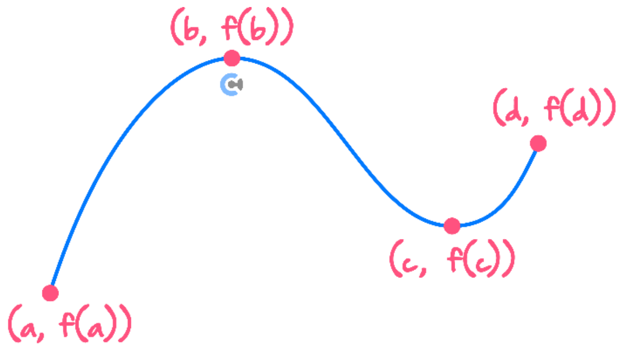
$$\text{At } (a, f(a)): \quad m_{normal} = \boxed{-\frac{1}{f'(a)}}$$

Cheat Sheet

[2.4.2] - Find Minimum and Maximum

➤ Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** ___ y values ___ for the given domain.
- **They occur at either** ___ End points ___ or ___ Turning points ___.



Absolute Min: $f(a)$

Absolute Max: $f(b)$

➤ Steps

1. Find ___ stationary ___ points and ___ end ___ points
2. Find the ___ largest/lowest ___ y value for max/min.

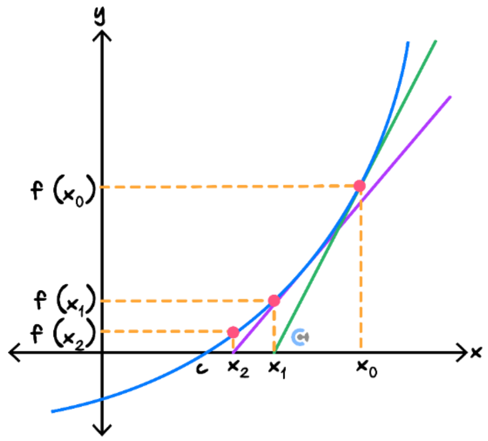
➤ Steps for optimisation

1. Construct a ___ function ___ for the subject you want to find the maximum or minimum of.
2. Find its ___ domain ___ if appropriate.
3. Find its ___ endpoints ___ and ___ turning ___ points.
4. Identify ___ maximum ___ or ___ minimum ___ value.

[2.4.3] - Apply Newton's Method to Find the Approximation of a Root and Its Limitations

➤ Newton's Method

- It is a method of approximating the x-intercept using ___ tangents ___.



$$x_{n+1} = \boxed{x_n - \frac{f(x_n)}{f'(x_n)}}$$

➤ Steps

1. Find the ___ tangent ___ at the x value given.
2. Find the ___ x-intercept ___ of the tangent using iterative formula.
3. Find the next tangent at the $x =$ ___ x-intercept ___ of the previous tangent.
4. Repeat until the value doesn't change by much.

➤ **Tolerance**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\textit{We stop when } |x_{n+1} - x_n| < \boxed{\text{Tolerance}}.$$

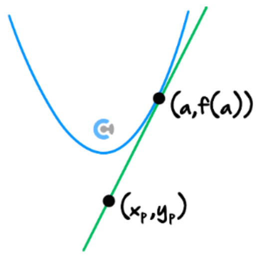
➤ Limitation of Newton's Method

- **Terminating Sequence**: Occurs when we hit a ___ Stationary point ___
- **Approximating a Wrong Root**: Occurs when we start on the ___ Wrong side ___.
- **Oscillating Sequence**: Occurs when we ___ oscillate ___ between two values without getting closer to the real root.

Cheat Sheet

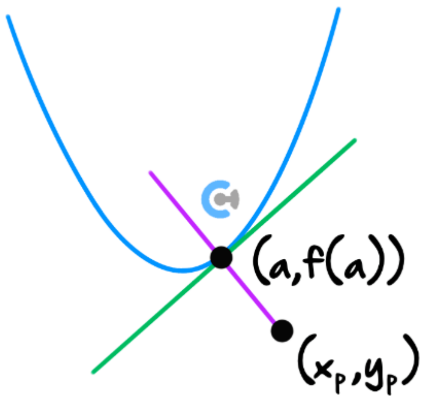
[2.5.1] – Advanced Tangents and Normal Questions

- Finding tangents/normals to functions, which also pass through a given point
- Tangent of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$f'(a) = \boxed{\frac{f(a) - y_p}{a - x_p}}$$

- Normal of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$-\frac{1}{f'(a)} = \boxed{\frac{f(a) - y_p}{a - x_p}}$$

[2.5.2] – Advanced Maximum/Minimum Questions

- To find the maximum/minimum instantaneous rate of change, we find the turning point of the ___ $\boxed{\text{derivative}}$ ___ function.

[2.6.1] – Advanced Maximum/Minimum Questions

- **Families of Functions:** Functions with an unknown.
- They involve understanding and using transformations.
- They involve the use of sliders/manipulate on CAS/technology.

[2.6.2] – Find Unknowns for Number of Solutions

- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

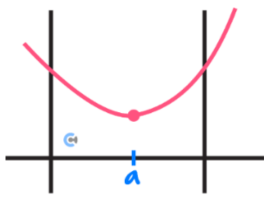
  - With the same gradient.

$$f'(a) = \boxed{m}$$

- We solve these ___ $\boxed{\text{simultaneously}}$ ___.

[2.6.3] – Find Unknowns for Minimum and Maximum

- **Minimum/maximum at a turning point:**

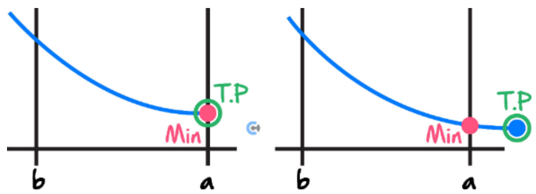


  - To achieve minimum/maximum at $x = a$.

$$f'(a) = 0$$

  - This is only when $x = a$ is not an ___ $\boxed{\text{end point}}$ ___.

- **Minimum/maximum at an endpoint:**



  - Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

  - Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.
  - We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = \boxed{0}$$

Cheat Sheet

[2.7.1] - Evaluate Pseudocode with Conditional Statements and Loops

- **Assigning Variables:**
  - To construct algorithms for more mathematical/complex problems, assigning variables will be useful.

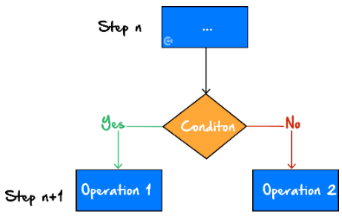
    $A \leftarrow 3$ assigns the **value** 3 to the **variable** $\mathbf{A}$.

  - We can also update our variables using the arrow.

    $A \leftarrow A + 3$ assigns the value $\mathbf{A+3}$ to the **variable** $\mathbf{A}$.

    - Since the value of A was already 3, Its new value will be 6.

- **Selections:**
  - Selections allow us to perform different operations at a given step, depending on a certain condition.



  - We are selectively performing an operation.
  - "If-then"

```
if condition then
    operation
end if
```

    - **Allows** us to perform an operation only when a certain condition is met.

  - "Else"

```
if condition then
    operation 1
else
    operation 2
end if
```

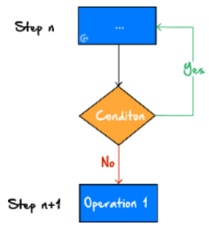
    - Provides an opportunity to perform an operation only when a certain condition is met.

  - "Else-If"

```
if condition 1 then
    operation 1
else if condition 2 then
    operation 2
else
    operation 3
end if
```

    - Provides an opportunity to add multiple pathways, each with different conditions.

- **Iteration (Loops):**
  - Iteration (a.k.a. looping) allows us to repeat steps in a Controlled way.
  - It is controlled by the condition
    - E.g., we only loop when a condition is met.



  - **For loops:**

```
for variable from lower bound to upper bound
                 condition
    operation
end for
```

    - Loops for which a variable increases by one each time it loops.
    - The variable gets moved from the lower bound to the upper bound by 1.

  - **While loops:** Loops that do **not** change the value of any variable by default.

```
while condition
    operation
end while
```

Cheat Sheet

[2.7.2] - Evaluate and Understand the Pseudocode for Different Implementations of Newton's Method

- A key component of Newton's method is the recursive relationship.

$$x_{n+1} = x_n - \boxed{\frac{f(x_n)}{f'(x_n)}}$$

- Newton's method requires an input function $f(x)$, the derivative $f'(x)$ and an initial value x_0.

- The number of iterations that Newton's method performs can be limited in our pseudocode.

- The pseudocode can also specify a tolerance for Newton's method where the algorithm terminates if

$$|\boxed{x_{n+1} - x_n}| < Tolerance$$

Section B: Exam 1 Questions (32 Marks)

INSTRUCTION:

- Regular: 26 Marks. 4 Minutes Reading. 36 Minutes Writing.
- Extension: 32 Marks. 4 Minutes Reading. 36 Minutes Writing.

Question 1 (6 marks)

a. Let $f(x) = \sqrt{\tan(e^{2x})}$, Find $f'(x)$. (2 marks) **[2.1.1]**

$$f'(x) = \frac{1}{2\sqrt{\tan e^{2x}}} \times 2e^{2x}\sec^2(e^{2x}) = \frac{(e^{2x}\sec^2(e^{2x}))}{\sqrt{[\tan(e^{2x})]}} \quad (1A)$$

$(1M\ for\ chain\ rule)$

b. Let $g(x) = \frac{\log_e(3x+1)}{3x+1}$. Find $g'(0)$. (2 marks) **[2.1.1]**

$$g'(x) = \frac{(3x+1) \times \frac{3}{3x+1} - 3\log_e(3x+1)}{(3x+1)^2} = \frac{3 - 3\log_e(3x+1)}{(3x+1)^2} \text{ (1M)}$$

$$g'(0) = \frac{3}{1} = 3 \text{ (1A)}$$

c. Find the equation of the normal to $y = x(x-3)^2$ at the origin $(0,0)$. (2 marks) **[2.4.1]**

Let $h(x) = x(x-3)^2$ then $h'(x) = (x-3)^2 + 2x(x-3)$ and $h'(0) = 9$ (1M).
So normal has gradient $-\frac{1}{9}$ and passes through $(0,0)$, therefore equation of the normal is

$$y = -\frac{1}{9}x \quad (1A)$$

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Question 2 (4 marks)

A tech company is designing a reward system for users who participate in an online task. The reward (in points) that a user receives depends on how long they work on the task, measured in hours. However, to balance effort with efficiency, the company sets the reward function as:

$$R(x) = x \cdot 2^{1-x},$$

where x is the number of hours worked (with $x > 0$), and $R(x)$ is the reward in points

a. Show that the derivative of the reward function is given by:

$$R'(x) = 2^{1-x}(1 - x\log_e(2)).$$

HINT: First write 2^{1-x} in terms of the exponential function. (2 marks) **[2.1.1]**

$R(x) = x \cdot e^{(1-x)\log_e(2)}$ (1M).

$$\begin{aligned} R'(x) &= e^{(1-x)\log_e(2)} - x\log_e(2) \cdot e^{(1-x)\log_e(2)} \\ &= e^{(1-x)\log_e(2)}(1 - x\log_e(2)) \\ &= 2^{1-x}(1 - x\log_e(2)) \end{aligned}$$

(1A clear steps leading to correct conclusion)

b. Determine the value of x that maximises the reward. What is the maximum reward value? (2 marks) **[2.4.2]**

We require $R'(x) = 0$. Note that $2^{1-x} \neq 0$, thus we may just solve

$$1 - x\log_e(2) = 0 \implies x = \frac{1}{\log_e(2)} \text{ (1A)}$$

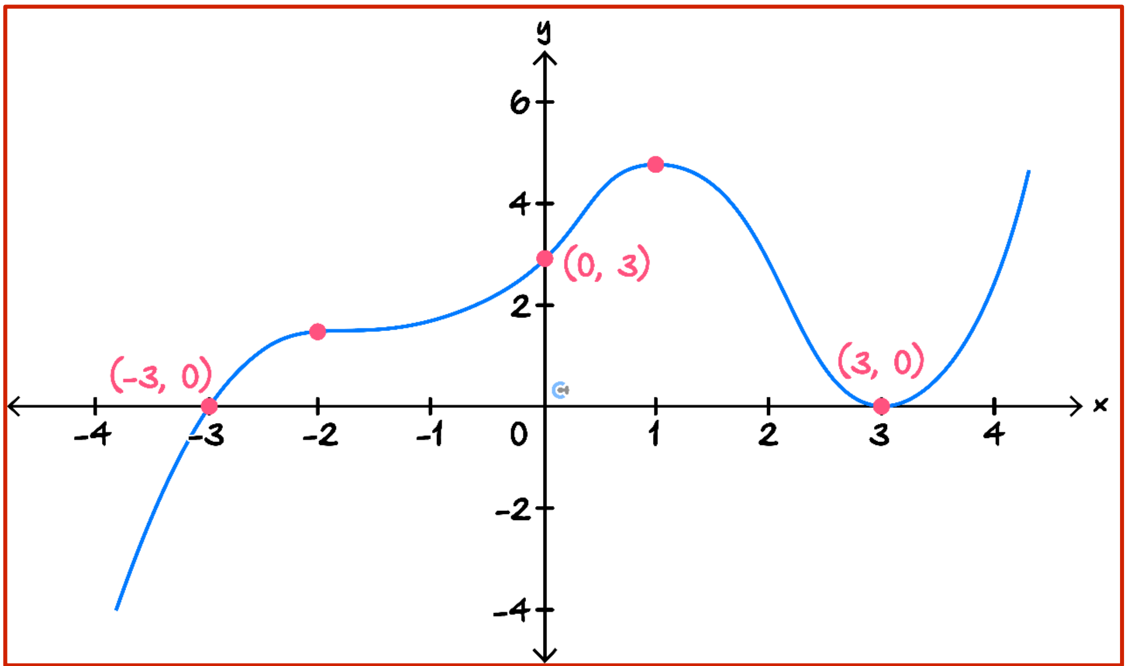
Max value of $R\left(\frac{1}{\log_e(2)}\right) = \frac{1}{\log_e(2)} \times 2 \times 2^{-(\log_e(2))^{-1}} = \frac{2}{e\log_e(2)}$ (1A).

Question 3 (4 marks) **[2.1.3]**

The graph of $f(x)$, where $x \in R$, has the following properties.

- $f(0) = 3, f(-3) = 0, f(3) = 0$
- $f'(-2) = 0, f'(1) = 0, f'(3) = 0$
- $f'(x) < 0$ for $x \in (1, 3)$ and $f'(x) > 0$ for $x \in (-\infty, -2) \cup (-2, 1) \cup (3, \infty)$.

Sketch a possible graph of $y = f(x)$ on the axes below. Label any axial intercepts with their coordinates.



1M for correct interpretation of first dot point, 1M for correct interpretation of second dot point.
1M for stationary point of inflection when $x = -2$, 1M for local max at $x = 1$ and local min at $x = 3$.

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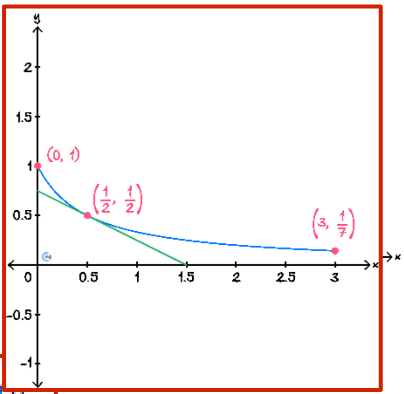
Question 4 (6 marks)

A skateboard ramp has been built at Contour Park. The cross-section of the ramp is modelled by the function,

$$f: [0, 3] \to R, f(x) = \frac{1}{1 + 2x}$$

a. Sketch the curve of the ramp on the axes below. Label endpoints with coordinates. (2 marks)



1A shape, 1A endpoints labelled.

b. The ramp is supported by a rail which touches the curve at one point and has a gradient of $-\frac{1}{2}$.

i. Find the coordinates of the point where the rail meets the ramp. (2 marks) **[2.1.1]**

$f'(x) = -\frac{2}{(1+2x)^2}$ so we solve

$$-\frac{2}{(1+2x)^2} = -\frac{1}{2} \quad \text{(1M)}$$
$$4 = (1+2x)^2$$
$$1 + 2x = \pm 2$$
$$x = \frac{1}{2}, -\frac{3}{2}$$

Only $x = \frac{1}{2}$ is valid. $f\left(\frac{1}{2}\right) = \frac{1}{2}$.

So point is $\left(\frac{1}{2}, \frac{1}{2}\right)$ (1A)

ii. Find the equation that models the rail and sketch the rail on the axes given in **part a.** (2 marks) **[2.4.1]**

Line through $\left(\frac{1}{2}, \frac{1}{2}\right)$ with gradient $-\frac{1}{2}$.

$$y = -\frac{1}{2}x + \frac{3}{4} \quad \text{(1A)}$$

(1A, for accurate sketch)

Question 5 (8 marks)

Consider the function $f(x) = x(x-1)(3-x)$.

a. Find the x-coordinates of any stationary points of f. (2 marks) **[2.1.2]**

$f(x) = x(4x - x^2 - 3) = -x^3 + 4x^2 - 3x$.
We solve $f'(x) = -3x^2 + 8x - 3 = 0$ (1M)

$$x = \frac{-8 \pm \sqrt{64-36}}{-6} = \frac{-8 \pm \sqrt{28}}{-6} = \frac{-8 \pm 2\sqrt{7}}{-6} = \frac{-4 \pm \sqrt{7}}{-3}.$$

So stationary points at $x = \dfrac{4-\sqrt{7}}{3}$ and $x = \dfrac{4+\sqrt{7}}{3}$ (1A)

b. State the nature of the stationary points with x-values as found in **part a**. (1 mark) **[2.1.2]**

f is a negative cubic.
So local minimum when $x = \dfrac{4-\sqrt{7}}{3}$ and local maximum when $x = \dfrac{4+\sqrt{7}}{3}$

c. Find the equations of the two tangents to f that pass through the origin. (3 marks) **[2.5.1]**

Consider the tangent at $x = a$. $f'(a) = -3a^2 + 8a - 3$ and passes through $(a, -a^3 + 4a^2 - 3a)$.

$$y - 4a^2 + 3a + a^3 = (-3a^2 + 8a - 3)(x - a)$$
$$y = (-3a^2 + 8a - 3)x + 4a^2 - 3a - a^3 + 3a^3 - 8a^2 + 3a$$
$$y = (-3a^2 + 8a - 3)x + 2a^3 - 4a^2$$

(1M for tangent in terms of unknown OR 1M for equating $f'(a) = \frac{f(a)}{a}$)
The tangent passes through the origin $\implies 2a^3 - 4a^2 = 0 \implies 2a^2(a-2) = 0$.
Thus $a = 0, 2$ (1A).
When $a = 0$, tangent is $y = -3x$. When $a = 2$, $y = x$.
The two tangents passing through the origin are $y = -3x, \quad y = x$. (1A)

d. **Extension.** The acute angle θ, made by the two tangents to f that pass through the origin, in radians, is given by $\theta = p\pi - \tan^{-1}(q)$, where p is a positive rational number, and q is a positive integer.

Find the values of p and q. (2 marks) **[2.5.1]**

$\tan^{-1}(1) = \frac{\pi}{4}$.
$\theta = (\pi - \tan^{-1}(-3)) - \frac{\pi}{4} = \frac{3\pi}{4} - \tan^{-1}(3)$.
(1A for any correct expression for θ).
Thus $p = \frac{3}{4}$ and $q = 3$ (1A)

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Question 6 (4 marks) **Extension.**

Let $f(x) = (x+1)e^{2x}$. Let $f^{(n)}(x)$ denote the n^{th} derivative of f with respect to x, where $n \geq 1$.

Find a formula for $f^{(n)}(x)$ in terms of x and n.

Note that $f(x) = xe^{2x} + e^{2x}$ and $\dfrac{d^n}{dx^n}\left(e^{2x}\right) = 2^n e^{2x}$ (1M).

Let $g(x) = xe^{2x}$, then $g'(x) = e^{2x} + 2xe^{2x}$

and $g''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$ (1M)

Then $g'''(x) = 8e^{2x} + 4e^{2x} + 2^3 xe^{2x} = 4 \times 3e^{2x} + 2^3 xe^{2x}$

Then $g^{(4)}(x) = 8e^{2x} + 24e^{2x} + 2^4 xe^{2x} = 8 \times 4e^{2x} + 2^4 xe^{2x}$

Find the pattern $g^{(n)}(x) = 2^n xe^{2x} + n \times 2^{n-1}e^{2x}$ (1M)
Thus

$$f^{(n)}(x) = 2^n e^{2x} + n \times 2^{n-1}e^{2x} + 2^n xe^{2x} = 2^{n-1}e^{2x}(n+2+2x) \quad (1A)$$

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Section C: Tech-Active Exam Skills

Calculator Commands: Finding Derivatives

- **Mathematica**

```
f'[x]
```

$\boxed{\texttt{D[f[x], x]}}$

- **TI**
  - Shift Minus

$$\frac{d}{dx}(f(x))$$

- **Casio**
  - Math 2

$$\frac{d}{dx}(f(x))$$

Calculator Commands: Finding Second Derivatives

- **Mathematica**

```
f''[x]
D[f[x], {x, 2}]
```

- **TI**
  - Shift Minus

$$\frac{d^2}{dx^2}(f(x))$$

- **Casio**
  - Math 2

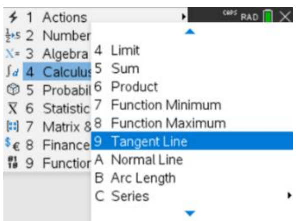
$$\frac{d^2}{dx^2}(f(x))$$

Calculator Commands: Finding Tangents on CAS

- **Mathematica**

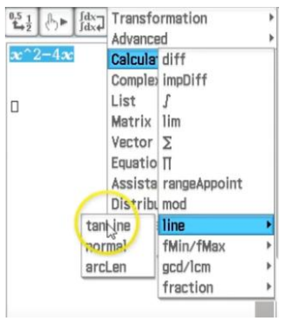
```
<< SuiteTools`
TangentLine[f[x], x, a]
```

- **TI-Nspire**
  - Menu 4 9



$$\text{tangentLine}(f(x),x,a)$$

- **Casio Classpad**



$$\text{tangentLine}(f(x),x,a)$$

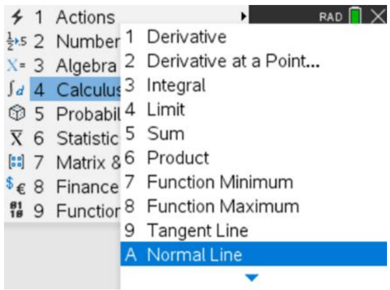
Calculator Commands: Finding Normals on CAS

Mathematica

```
<< SuiteTools`

NormalLine[f[x], x, a]
```

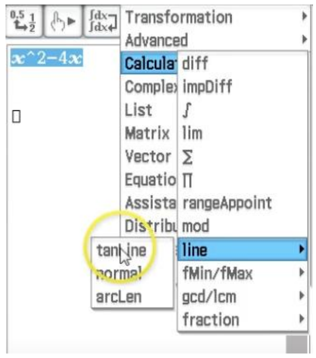
TI-Nspire

Menu 4 A



normalLine$(f(x),x,a)$

Casio Classpad



normalLine$(f(x),x,a)$

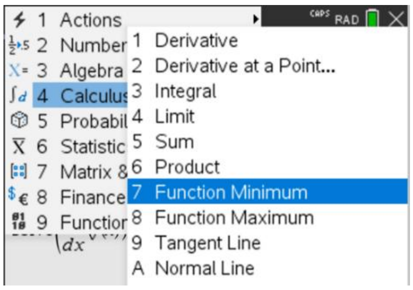
Calculator Commands: Finding Absolute Max and Min for $x \in [a, b]$

Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]

Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

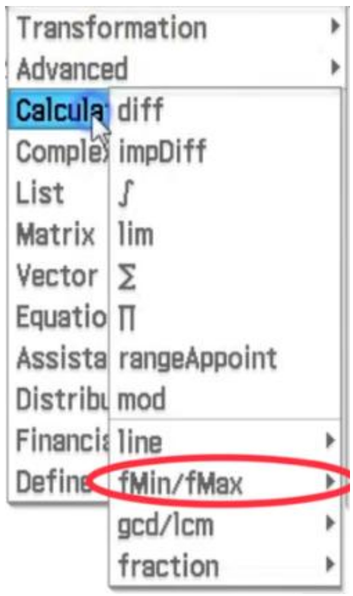
Menu 4 7 and Menu 4 8



fMax$(f(x),x,a,b)$

fMin$(f(x),x,a,b)$

Casio Classpad



fMax$(f(x),x,a,b)$

fMin$(f(x),x,a,b)$

Calculator Commands: Newton's Method on Technology

- Consider finding a root to $f(x) = x^3 - 2$ with initial value $x_0 = 1$.

- **Mathematica.**

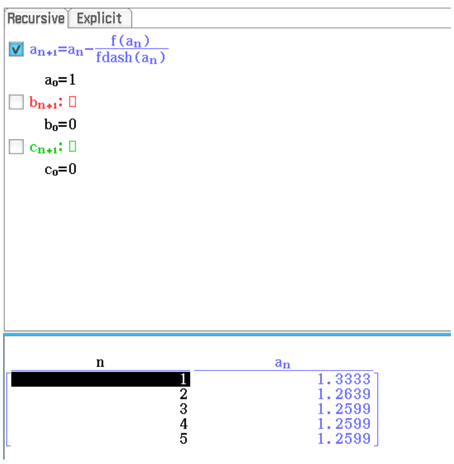
```
In[531]:= f[x_] := x^3 - 2

In[533]:= n[x_] := x - f[x]/f'[x]

In[534]:= n[x] // Together

Out[534]= 2 (1 + x^3)/(3 x^2)

In[537]:= For[i = 1; x = 1, i < 5, i++, x = 2 (1.0 + x^3)/(3 x^2); Print[x]]
          1.33333
          1.26389
          1.25993
          1.25992
```

- **TI.** Define the $n(x)$ function then keep iterating by putting your previous value back into $n(x)$.

Define $f(x)=x^3-2$	*Done*
$x-\dfrac{f(x)}{\frac{d}{dx}(f(x))}$	$\dfrac{2\cdot(x^3+1)}{3\cdot x^2}$
Define $n(x)=\dfrac{2\cdot(x^3+1)}{3\cdot x^2}$	*Done*
$n(1)$	1.33333
$n(1.3333333333333)$	1.26389
$n(1.2638888888889)$	1.25993

- **Classpad.**

  - Use the same method as TI. OR, under Sequences.



Calculator Commands: Joining Smoothly

Mathematica

```
f[x_] := One Function

g[x_] := Another Function

Solve[f[x value] == g[x value] && f'[x value] == g'[x value]]
```

TI and Casio

- Define each branch as $f(x)$ and $g(x)$.
- TI: Define its derivative as $df(x)$ and $dg(x)$.

  Casio: Define them as different names.
- Solve $f(a) = g(a)$ and $df(a) = dg(a)$ simultaneously.

Calculator Commands: Using Sliders/Manipulate on CAS

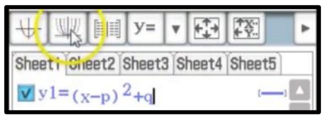
Mathematica

```
Manipulate[Plot[function, {x, xmin, xmax}],

    {unknown, lowerbound, upperbound}]
```

- **NOTE:** The function **must** be typed out instead of using its saved name.

TI-Nspire

Casio Classpad



Space for Personal Notes

Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	done
$\text{solve}(\frac{d}{dx}(f(x))=0, x)$	$\{x=1\}$
$f(1)$	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]

In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]

Out[5]= {{x → 1, y → e}}
```

Calculator Commands: Finding Tangents/Normals Which Pass Through a Point

- Suppose we want to find the equation of a tangent/normal to the graph of $f(x)$ that passes through the point $P(x_1, y_1)$.

- Steps:

  1. Find the equation of the tangent to $f(x)$ at arbitrary point $x = a$.
  2. Let this tangent line be $t(x)$.
  3. Solve the equation $t(x_1) = y_1$ to find possible value(s) of a.
  4. Find the equation of the tangent at $x = a$.

- A similar procedure for the normal line.

- **Example:** Find the equation of a tangent to $f(x) = x^3 - 2x$ that passes through the point $(0,2)$.

```
In[564]:= f[x_] := x^3 - 2 x

In[565]:= TangentLine[f[x], {x, a}]
Out[565]= -2 a^3 + (-2 + 3 a^2) x

In[566]:= t[x_] := -2 a^3 + (-2 + 3 a^2) x

In[568]:= Solve[t[0] == 2, a, Reals]
Out[568]= {{a → -1}}

In[570]:= t[x] /. a → -1
Out[570]= 2 + x

In[571]:= TangentLine[f[x], {x, -1}]
Out[571]= 2 + x
```

Space for Personal Notes

Section D: Exam 2 Questions (33 Marks)

INSTRUCTION:

- Regular: 30 Marks. 5 Minutes Reading. 40 Minutes Writing.
- Extension: 34 Marks. 5 Minutes Reading. 36 Minutes Writing.

Question 7 (1 mark) **[2.3.1]**

The derivative of $g(x)e^{2x}$, with respect to x is:

A. $2e^{2x}(g(x) + g'(x))$

B. $e^{2x}(2g(x) + g'(x))$

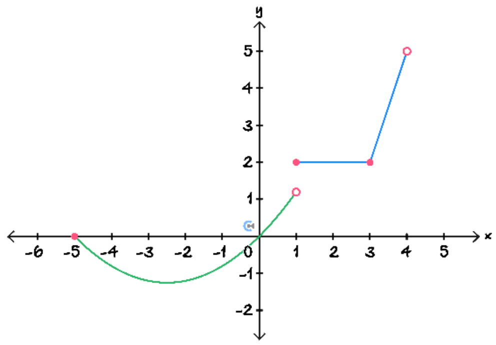
C. $g'(x)e^{2x}$

D. $xg'(x) + e^{2x}g(x)$

```
In[4]:= D[g[x] * Exp[2 x], x] // Factor
Out[4]= e^(2 x) (2 g[x] + g'[x])
```

Question 8 (1 mark) **[2.2.2]**

The graph of the hybrid function $y = h(x)$ is shown below.



Hence, $h(x)$ is:

Not differentiable at endpoints/points of discontinuity.

A. Not differentiable at $x = 1$ and $x = 4$ but is differentiable at $x = -5$ and $x = 3$.

B. Not differentiable at $x = 1$, $x = 3$ and $x = 4$ but is differentiable at $x = -5$.

C. Not differentiable at $x = -5$, $x = 1$ and $x = 4$ but is differentiable at $x = 3$.

D. Not differentiable at $x = -5$, $x = 1$, $x = 3$ and $x = 4$.

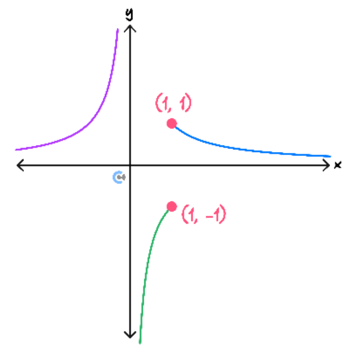
Question 9 (1 mark) **[2.1.3]**

Consider the hybrid function:

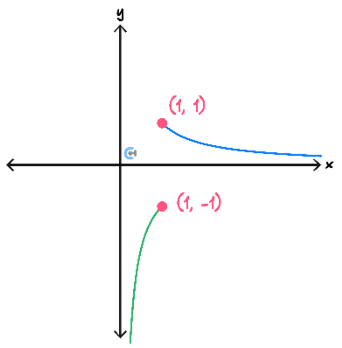
$$f(x) = \begin{cases} \log_e(x), & x > 1 \\ -\log_e(x), & x < 1 \end{cases}$$

The graph of $f'(x)$ could be :

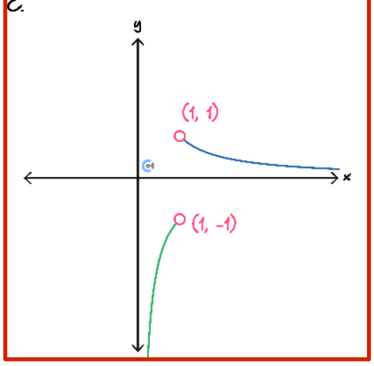
A.



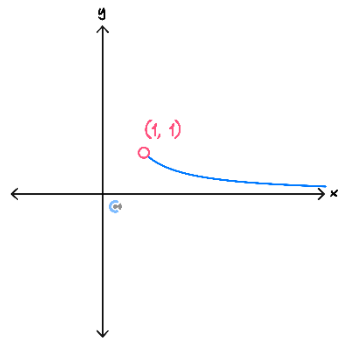
B.



C.



D.



Question 10 (1 mark) **[2.1.1]**

Consider the hybrid function:

$$h(x) = \begin{cases} \dfrac{\sin(3x)}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$$

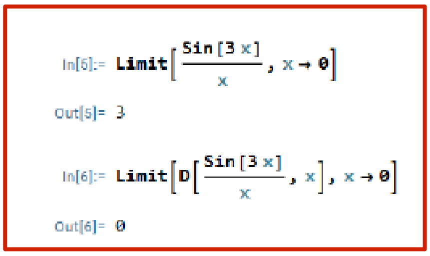
Which of the following statements is true about the continuity and differentiability of $h(x)$ at $x = 0$?

A. $h(x)$ is continuous but not differentiable at $x = 0$.

B. $h(x)$ is continuous and differentiable at $x = 0$.

C. $h(x)$ is not continuous at $x = 0$.

D. $h(x)$ is not continuous but is differentiable at $x = 0$.


```
In[5]:= Limit[Sin[3 x]/x, x → 0]
Out[5]= 3
In[6]:= Limit[D[Sin[3 x]/x, x], x → 0]
Out[6]= 0
```

Question 11 (1 mark)

Let $f(x) = e^{-b^2x^2}$, where $b > 0$. $f(x)$ is concave down for:

A. $-b < x < b$

B. $-\frac{1}{\sqrt{b}} < x < \frac{1}{\sqrt{b}}$

C. $-\frac{1}{\sqrt{2}b} < x < \frac{1}{\sqrt{2}b}$

D. $-\sqrt{b} < x < \sqrt{b}$

```
In[61]:= f[x_] := Exp[-b^2 x^2]
In[62]:= Solve[f''[x] == 0]
Out[62]= {{b → 0}, {x → -1/(√2 b)}, {x → 1/(√2 b)}}
In[63]:= Reduce[f''[x] < 0]
Out[63]= (b ∈ ℝ && b ≠ 0 && x == 0) || (b ∈ ℝ && b ≠ 0 && -√(1/b^2)/√2 < x < √(1/b^2)/√2)
```

Question 12 (1 mark)

Consider the hybrid function:

$$h(x) = \begin{cases} -ax + 3, & x \leq 1 \\ x^2 - bx + 4, & x > 1 \end{cases}$$

Where $a, b \in R, h$ is smooth continuous for all $x \in R$ if:

A. $a = 2$ and $b = 4$

B. $a = -2$ and $b = 4$

C. $a = -2$ and $b = -4$

D. $a = 1$ and $b = 4$

```
In[80]:= h1[x_] := -a x + 3
In[81]:= h2[x_] := x^2 - b x + 4
In[82]:= Solve[h1[1] == h2[1] && h1'[1] == h2'[1]]
Out[82]= {{b → 2 + a}}
```

Question 13 (1 mark) **[2.6.2]**

Let $a \in R$. The graphs of $y = e^{-x} + a$ and $y = 2 - x$, intersect exactly once when:

A. $a = -2$

B. $a = -1$

C. $a = 1$

D. $a = 2$

```
In[86]:= Solve[f[x] == g[x] && f'[x] == g'[x], Reals]
Out[86]= {{a → 1, x → 0}}
```

Question 14 (1 mark) **[2.6.3]**

The function $f(x) = x^3 \log_e(x-k)$ has a local minimum when $x = -2$. The value of k is closest to:

A. -3.12

B. -3.54

C. -2.53

D. -3.71

```
In[90]:= f[x_] := x^3 Log[x - k]
In[91]:= Solve[f'[-2] == 0, k] // N
Out[91]= {{k → -3.5412}}
```

Question 15 (1 mark) **[2.7.1] [2.7.2]**

An implementation of Newton's method is shown below.

```
define newton(f(x), x0, n, tol):
    df(x) ← the derivative of f(x)
    i ← 0
    xn ← x0
    while i < n do
        if df(xn) = 0 then
            return "Error: Division by zero"
        end if
        xn+1 ← xn − f(xn)/df(xn)
        if −tol < xn+1 − xn < tol then
            return xn+1
        end if
        xn ← xn+1
        i ← i + 1
    end while
    return xn
```

What is the value of i when the function Newton$(x^2 - 5, 5, 10, 0.001)$ finishes running?

A. 2

B. 3

C. 4

D. 5

The algorithm terminates in its 4th iteration. $i = 4$
Mathematica code used found here: https://pastebin.com/SXKL89qE

```
In[106]:= NewtonMethod[x^2 - 5, 5, 10, 0.001]
```

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	5.0000000	3.000000	2.000000
1	3.000000	2.333333	0.666667
2	2.333333	2.238095	0.095238
3	2.238095	2.23607	0.00203
4	2.23607	2.23607	$0. \times 10^{-6}$

Question 16 (1 mark) **[2.1.3]**

Consider the polynomial function that is continuous and smooth for all $x \in R$ and has the following features:

- $f'(x) = 0, x \in \{2, 7, 10\}$
- $f'(x) < 0, x \in (-\infty, 2) \cup (2, 7) \cup (10, \infty)$
- $f'(x) > 0, x \in (7, 10)$

Which of the following statements is true about $f(x)$?

Stationary point at $x = 2$ but sign of derivative does not change, hence stationary point of inflection.
Stationary point at $x = 7$ and derivative sign changes from negative to positive therefore local minimum.
Stationary point at $x = 10$ and derivative sign changes from positive to negative hence local maximum.

A. $f(x)$ has a … nimum at $x = 10$.

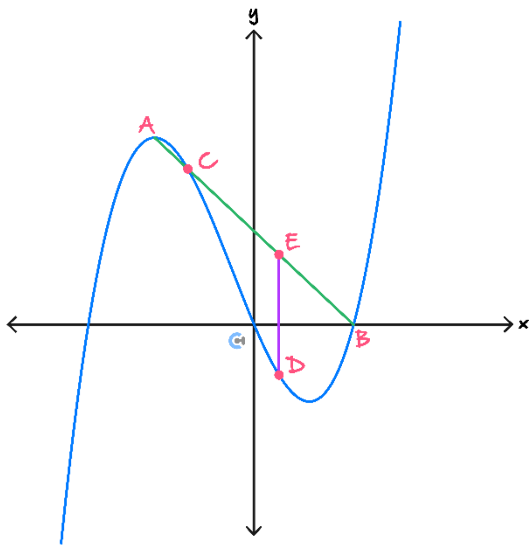
B. $f(x)$ has …

C. $f(x)$ has a stationary point of inflection at $x = 2$, a local minimum at $x = 7$, and a local maximum at $x = 10$.

D. $f(x)$ has x-intercepts at $x = 2$ and $x = 7$, and a local maximum between $x = 7$ and $x = 10$.

Question 17 (14 marks)

The graph of $f(x) = \frac{9x^3}{16} + \frac{3x^2}{4} - \frac{15x}{4}$ is shown below:



a. Write down the coordinates of point B. (1 mark)

$(2, 0)$

```
In[111]:= f[x_] := 9 x^3 / 16 + 3 x^2 / 4 - 15 x / 4
In[112]:= Solve[f[x] == 0, x]
Out[112]= {{x → -10/3}, {x → 0}, {x → 2}}
```

b. Write down the coordinates of the turning point at point A. (1 mark) **[2.1.2]**

$(-2, 6)$

```
In[114]:= Solve[f'[x] == 0 && y == f[x]]
```

$$\text{Out[114]= } \left\{\{x \to -2, y \to 6\}, \left\{x \to \frac{10}{9}, y \to -\frac{200}{81}\right\}\right\}$$

c. Let L be the line joining points A and B.

i. Show that the equation of the line L is $2y + 3x = 6$. (2 marks)

Gradient $= -\frac{6}{4} = -\frac{3}{2}$ (1M)

Line passes through $(2, 0)$. Thus

$$y - 0 = -\frac{3}{2}(x - 2)$$

$$\implies 2y + 3x = 6 \quad \text{(1A)}$$

ii. The line L passes through the graph of $f(x)$ at point C. Write down the coordinates of point C. (1 mark)

$\left(-\frac{4}{3}, 5\right)$

```
In[116]:= Solve[c[x] == f[x] && y == f[x]]
```

$$\text{Out[116]= } \left\{\{x \to -2, y \to 6\}, \left\{x \to -\frac{4}{3}, y \to 5\right\}, \{x \to 2, y \to 0\}\right\}$$

iii. Find the distance AC. (1 mark)

$\frac{\sqrt{13}}{3}$

```
In[117]:= EuclideanDistance[{-4/3, 5}, {-2, 6}]
```

$$\text{Out[117]= } \frac{\sqrt{13}}{3}$$

iv. **Extension.** The line L undergoes a transformation T such that it still passes through the point B, but is now tangent to the graph of f at a point P, where P has an x-coordinate less than zero.
Give the coordinates of point P and describe the transformation T. (3 marks)

Since the point B is fixed, the transformation that we look for should be a dilation by factor k from the x-axis.
We find that the tangent to f when $x = -\frac{5}{3}$ passes through B (1M).
So P has coordinates $\left(-\frac{5}{3}, \frac{275}{48}\right)$ (1A)
The transformation that L undergoes is a dilation by factor $\frac{25}{24}$ from the x-axis. (1A)

In[25]:= TangentLine[f[x], x, a]
Out[25]= $-\frac{3}{8}a^2(2+3a)+\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)x$

In[26]:= $-\frac{3}{8}a^2(2+3a)+\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)x$ /. x → 2
Out[26]= $-\frac{3}{8}a^2(2+3a)+2\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)$

In[27]:= Solve[$-\frac{3}{8}a^2(2+3a)+2\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)$ == 0, a]
Out[27]= $\{\{a \to -\frac{5}{3}\}, \{a \to 2\}, \{a \to 2\}\}$

In[28]:= f[-5/3]
Out[28]= $\frac{275}{48}$

In[29]:= m = $\frac{-275/48}{5/3+2}$
Out[29]= $-\frac{25}{16}$

In[37]:= $\frac{-25/16}{-3/2}$
Out[37]= $\frac{25}{24}$

The vertical line segment DE joins the graph of $f(x)$ and the line joining points A and B. We wish to maximise the length of the line segment DE.

d. Write down an expression for the length DE in terms of x. (1 mark) **[2.5.2]**

$$\left(3 - \frac{3x}{2}\right) - f(x) = -\frac{9x^3}{16} - \frac{3x^2}{4} + \frac{9x}{4} + 3$$

In[118]:= c[x] - f[x]
Out[118]= $3+\frac{9x}{4}-\frac{3x^2}{4}-\frac{9x^3}{16}$

e. Determine the value of x, correct to two decimal places, for which the length DE is a maximum and determine the maximum length of the line segment DE, correct to two decimal places. (You do not have to verify that this value gives the maximum length for the line DE). (2 marks) **[2.5.2]**

$x \approx 0.79$ (1A)
Max distance ≈ 4.03 (1A)

In[119]:= Maximize[{c[x] - f[x], -2 ≤ x ≤ 2}, x] // N
Out[119]= {4.03211, {x → 0.792837}}

f. Newton's method is used to find a solution to the equation $f(x) = 2$, with $x_0 = 3$. Complete the table below for the values of x_1, x_2 and x_3. Give your answers correct to three decimal places. (2 marks) **[2.4.3]**

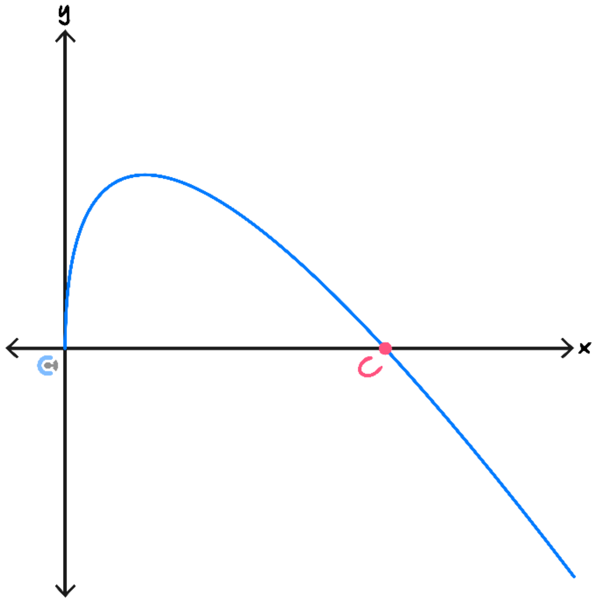
```
In[183]:= g[x_] := f[x] - 2

In[184]:= n[x_] := x - g[x]/g'[x]

In[185]:= n[3.0]
Out[185]= 2.4549

In[186]:= n[n[3.0]]
Out[186]= 2.29296

In[187]:= n[n[n[3.0]]]
Out[187]= 2.27825
```

	3
	2.455
	2.293
	2.278

Question 18 (10 marks)

Consider the family of functions $f_a: [0, \infty) \to R$ which is defined by $f_a(x) = 6a\sqrt{x} - x$, where a is a real number and $a > 0$.

Part of the graph of f_a is shown below.



a. Find c in terms of a, where $f_a(c) = 0$ and $c \neq 0$. (2 marks) **[2.6.1]**

We must solve $6a\sqrt{c} - c = 0$ (1M)
$c = 36a^2$ (1A)

```
In[31]:= f[x_] := 6 a √x - x

In[32]:= Solve[f[c] == 0, c]
   Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[32]= {{c → 0}, {c → 36 a^2}}
```

b. Determine the interval over which f_a is strictly decreasing. (2 marks) **[2.1.2]**

We find the turning point.
Solve $f_a'(x) = \frac{3a}{\sqrt{x}} - 1 = 0$ (1M)
$\implies x = 9a^2$.
Therefore strictly decreasing on $[9a^2, \infty)$ (1A)

```
In[33]:= f'[x]
Out[33]= -1 + 3 a/√x
In[34]:= Solve[f'[x] == 0, x]
Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[34]= {{x → 9 a^2}}
```

c. Show that the equation of the tangent to the graph f_a at the point $(c, 0)$ is : (2 marks) **[2.4.1]**

$$y = -\frac{1}{2}x + 18a^2$$

$f_a'(c) = \frac{3a}{\sqrt{c}} - 1 = \frac{3a}{6a} - 1 = -\frac{1}{2}$ (1M).
Tangent passes through $(36a^2, 0)$ and with gradient $-\frac{1}{2}$.
Thus $y = -\frac{1}{2}(x - 36a^2) = -\frac{1}{2}x + 18a^2$ (1A)

```
In[87]:= TangentLine[f[x], x, 36 a^2] // Expand
Out[87]= 18 a √(a^2) - x + √(a^2) x/(2 a)
In[88]:= Assuming[a > 0, Refine[18 a √(a^2) - x + √(a^2) x/(2 a)]]
Out[88]= 18 a^2 - x/2
```

d. The function $f_a(x)$ is transformed to form $g_a(x)$, where $g_a(x)$ is defined as

$$g_a(x) = f_a(x) - b$$

Find the value of b, in terms of a, such that the tangent drawn to the curve of $g_a(x)$ at $x = c$ passes through the origin. (2 marks) **[2.6.1]**

$g_a'(c) = f_a'(c) = -\frac{1}{2}$.
$g_a(c) = f_a(c) - b = -b$.
The tangent is $y + b = -\frac{1}{2}x + 18a^2$ (1M)
To pass through $(0, 0)$ must have $b = 18a^2$. (1A)

```
In[94]:= Assuming[a > 0, Refine[TangentLine[f[x] - b, x, 36 a^2]]] // Expand
Out[94]= 18 a^2 - b - x/2
In[96]:= Solve[0 == 18 a^2 - b - x/2 /. x → 0, b]
Out[96]= {{b → 18 a^2}}
```

Let $h_a: [0, d] \to R, h_a(x) = f(x)$, where d is chosen as large as possible and such that h_a is a one-to-one function and $a > 0$.

e. State the coordinates of any points of intersection between h_a and its inverse function h_a^{-1}. Give your answer in terms of a where appropriate. (1 mark)

$d = 9a^2$

$(0, 0)$ and $(9a^2, 9a^2)$ (1A)

f. Find the angle made by tangents to h_a and h_a^{-1}at points where the respective curves intersect each other. (1 mark) **[2.5.1]**

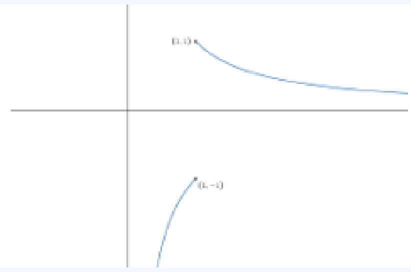
The tangents are vertical/horizontal lines. Thus $90°$ or $\frac{\pi}{2}$ (1A)

Space for Personal Notes

<u>Section E:</u> Tech-Active Solutions - Mathematica

<table>
<tr><th><u>Question Number</u></th><th><u>Solutions</u></th></tr>
<tr><td>7</td><td>$e^{2x}(2g(x) + g'(x))$

<code>In[4]:= D[g[x] * Exp[2 x], x] // Factor</code>
<code>Out[4]= e^(2 x) (2 g[x] + g'[x])</code></td></tr>
<tr><td>8</td><td>Not differentiable at $x = -5$, $x = 1$, $x = 3$ and $x = 3$.

Not differentiable at endpoints/points of discontinuity.</td></tr>
<tr><td>9</td><td>C. </td></tr>
<tr><td>10</td><td>$h(x)$ is continuous and differentiable at $x = 0$.

<code>In[5]:= Limit[Sin[3 x]/x, x → 0]</code>
<code>Out[5]= 3</code>
<code>In[6]:= Limit[D[Sin[3 x]/x, x], x → 0]</code>
<code>Out[6]= 0</code></td></tr>
<tr><td>11</td><td>$-\frac{1}{\sqrt{2}b} < x < \frac{1}{\sqrt{2}b}$

<code>In[61]:= f[x_] := Exp[-b^2 x^2]</code>
<code>In[62]:= Solve[f''[x] == 0]</code>
<code>Out[62]= {{b → 0}, {x → -1/(√2 b)}, {x → 1/(√2 b)}}</code>
<code>In[63]:= Reduce[f''[x] < 0]</code>
<code>Out[63]= (b ∈ R && b ≠ 0 && x == 0) || (b ∈ R && b ≠ 0 && -√(1/b^2)/√2 < x < √(1/b^2)/√2)</code></td></tr>
</table>

12

$a = 2$ and $b = 4$

```
In[80]:= h1[x_] := -a x + 3
In[81]:= h2[x_] := x^2 - b x + 4
In[82]:= Solve[h1[1] == h2[1] && h1'[1] == h2'[1]]
Out[82]= {{b → 2 + a}}
```

13

$a = 1$

```
In[86]:= Solve[f[x] == g[x] && f'[x] == g'[x], Reals]
Out[86]= {{a → 1, x → 0}}
```

14

-3.54

```
In[90]:= f[x_] := x^3 Log[x - k]
In[91]:= Solve[f'[-2] == 0, k] // N
Out[91]= {{k → -3.5412}}
```

15

4

The algorithm terminates in its 4th iteration. $i = 4$
Mathematica code used found here: https://pastebin.com/SXKL89qE

```
In[106]:= NewtonMethod[x^2 - 5, 5, 10, 0.001]
```

Iteration	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	5.0000000	3.000000	2.000000
1	3.000000	2.333333	0.666667
2	2.333333	2.238095	0.095238
3	2.238095	2.23607	0.00203
4	2.23607	2.23607	$0. \times 10^{-6}$

16

$f(x)$ has a stationary point of inflection at $x = 2$, a local minimum at $x = 7$, and a local maximum at $x = 10$.

Stationary point at $x = 2$ but sign of derivative does not change, hence stationary point of inflection.
Stationary point at $x = 7$ and derivative sign changes from negative to positive therefore local minimum.
Stationary point at $x = 10$ and derivative sign changes from positive to negative hence local maximumum.

17 (a)	$(2,0)$
	`In[111]:= f[x_] := 9 x^3 / 16 + 3 x^2 / 4 - 15 x / 4`
	`In[112]:= Solve[f[x] == 0, x]`
	`Out[112]=` $\left\{\left\{x \to -\frac{10}{3}\right\}, \{x \to 0\}, \{x \to 2\}\right\}$

17 (b)	$(-2,6)$ `In[114]:= Solve[f'[x] == 0 && y == f[x]]` `Out[114]=` $\left\{\{x \to -2, y \to 6\}, \left\{x \to \frac{10}{9}, y \to -\frac{200}{81}\right\}\right\}$
17 (c)(i)	Gradient $= -\frac{6}{4} = -\frac{3}{2}$ (1M) Line passes through $(2,0)$. Thus $y - 0 = -\frac{3}{2}(x-2)$ $\implies 2y + 3x = 6 \quad (1A)$
17(c)(ii)	$\left(-\frac{4}{3}, 5\right)$ `In[116]:= Solve[c[x] == f[x] && y == f[x]]` `Out[116]=` $\left\{\{x \to -2, y \to 6\}, \left\{x \to -\frac{4}{3}, y \to 5\right\}, \{x \to 2, y \to 0\}\right\}$
17(c)(iii)	$\frac{\sqrt{13}}{3}$ `In[117]:= EuclideanDistance[{-4 / 3, 5}, {-2, 6}]` `Out[117]=` $\frac{\sqrt{13}}{3}$

17(c)(iv)

Since the point B is fixed, the transformation that we look for should be a dilation by factor k from the x-axis.

We find that the tangent to f when $x = -\frac{5}{3}$ passes through B (1M).

So P has coordinates $\left(-\frac{5}{3}, \frac{275}{48}\right)$ (1A)

The transformation that L undergoes is a dilation by factor $\frac{25}{24}$ from the x-axis. (1A)

In[25]:= TangentLine[f[x], x, a]

Out[25]= $-\frac{3}{8}a^2(2+3a)+\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)x$

In[26]:= $-\frac{3}{8}a^2(2+3a)+\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)x$ /. x → 2

Out[26]= $-\frac{3}{8}a^2(2+3a)+2\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right)$

In[27]:= Solve$\left[-\frac{3}{8}a^2(2+3a)+2\left(-\frac{15}{4}+\frac{3a}{2}+\frac{27a^2}{16}\right) == 0, a\right]$

Out[27]= $\left\{\left\{a \to -\frac{5}{3}\right\}, \{a \to 2\}, \{a \to 2\}\right\}$

In[28]:= f[-5 / 3]

Out[28]= $\frac{275}{48}$

In[29]:= m = $\frac{-275/48}{5/3+2}$

Out[29]= $-\frac{25}{16}$

In[37]:= $\frac{-25/16}{-3/2}$

Out[37]= $\frac{25}{24}$

17(d)

$$\left(3-\frac{3x}{2}\right)-f(x) = -\frac{9x^3}{16}-\frac{3x^2}{4}+\frac{9x}{4}+3$$

In[118]:= c[x] - f[x]

Out[118]= $3+\frac{9x}{4}-\frac{3x^2}{4}-\frac{9x^3}{16}$

17(e)

$x \approx 0.79$ (1A)

Max distance ≈ 4.03 (1A)

In[119]:= Maximize[{c[x] - f[x], -2 ≤ x ≤ 2}, x] // N

Out[119]= {4.03211, {x → 0.792837}}

17(f)

x_0	0
x_1	2.455
x_2	2.293
x_3	2.278

	In[183]:= `g[x_] := f[x] - 2` In[184]:= `n[x_] := x - g[x]/g'[x]` In[185]:= `n[3.0]` Out[185]= `2.4549` In[186]:= `n[n[3.0]]` Out[186]= `2.29296` In[187]:= `n[n[n[3.0]]]` Out[187]= `2.27825`
18(a)	We must solve $6a\sqrt{c} - c = 0$ (1M) $c = 36a^2$ (1A) In[31]:= `f[x_] := 6 a √x - x` In[32]:= `Solve[f[c] == 0, c]` Solve: There may be values of the parameters for which some or all solutions are not valid. Out[32]= `{{c → 0}, {c → 36 a^2}}`
18(b)	We find the turning point. Solve $f_a'(x) = \frac{3a}{\sqrt{x}} - 1 = 0$ (1M) $\implies x = 9a^2$. Therefore strictly decreasing on $[9a^2, \infty)$ (1A) In[33]:= `f'[x]` Out[33]= `-1 + 3 a/√x` In[34]:= `Solve[f'[x] == 0, x]` Solve: There may be values of the parameters for which some or all solutions are not valid. Out[34]= `{{x → 9 a^2}}`

18(c)	$f_a'(c) = \dfrac{3a}{\sqrt{c}} - 1 = \dfrac{3a}{6a} - 1 = -\dfrac{1}{2}$ (1M). Tangent passes through $(36a^2, 0)$ and with gradient $-\dfrac{1}{2}$. Thus $y = -\dfrac{1}{2}(x - 36a^2) = -\dfrac{1}{2}x + 18a^2$ (1A) `In[87]:= TangentLine[f[x], x, 36 a^2] // Expand` `Out[87]=` $18\,a\,\sqrt{a^2} - x + \dfrac{\sqrt{a^2}\,x}{2\,a}$ `In[88]:=` Assuming$\left[a > 0, \text{Refine}\left[18\,a\,\sqrt{a^2} - x + \dfrac{\sqrt{a^2}\,x}{2\,a}\right]\right]$ `Out[88]=` $18\,a^2 - \dfrac{x}{2}$
18(d)	$g_a'(c) = f_a'(c) = -\dfrac{1}{2}$. $g_a(c) = f_a(c) - b = -b$. The tangent is $y + b = -\dfrac{1}{2}x + 18a^2$ (1M) To pass through $(0, 0)$ must have $b = 18a^2$. (1A) `In[94]:= Assuming[a > 0, Refine[TangentLine[f[x] - b, x, 36 a^2]]] // Expand` `Out[94]=` $18\,a^2 - b - \dfrac{x}{2}$ `In[96]:=` Solve$\left[0 == 18\,a^2 - b - \dfrac{x}{2}\ /.\ x \to 0, b\right]$ `Out[96]=` $\{\{b \to 18\,a^2\}\}$
18(e)	$d = 9a^2$ $(0, 0)$ and $(9a^2, 9a^2)$ (1A)
18(f)	The tangents are vertical/horizontal lines. Thus 90° or $\dfrac{\pi}{2}$ (1A)

Section F: Tech-Active Solutions – Casio

Question Number	Solutions
7	$\frac{d}{dx}\left(g(x) \times e^{2x}\right)$ $\frac{d}{dx}(g(x)) \cdot e^{2 \cdot x} + 2 \cdot e^{2 \cdot x} \cdot g(x)$ factor(ans) $\left(\frac{d}{dx}(g(x)) + 2 \cdot g(x)\right) \cdot e^{2 \cdot x}$
8	Not differentiable at $x = -5$, $x = 1$, $x = 3$ and $x = 3$.
9	C
10	$h(x)$ is continuous and differentiable at $x = 0$. $\lim_{x \to 0}\left(\frac{\sin(3x)}{x}\right)$ 3 $\lim_{x \to 0}\left(\frac{d}{dx}\left(\frac{\sin(3x)}{x}\right)\right)$ 0
11	$\text{solve}\left(\frac{d^2}{dx^2}\left(e^{-b^2 \times x^2}\right) = 0, x\right)$ $\left\{x = \frac{-\sqrt{2}}{2 \cdot b}, x = \frac{\sqrt{2}}{2 \cdot b}\right\}$ $\text{solve}\left(\frac{d^2}{dx^2}\left(e^{-b^2 \times x^2}\right) < 0 \mid b > 0\right.$ $\left\{(4 \cdot b^4 \cdot x^2 - 2 \cdot b^2) \cdot e^{-b^2 \cdot x^2} < 0\right\}$ Inequality does not work :(

12	Define $f(x)=-a\times x+3$ done Define $g(x)=x\hat{}2-b\times x+4$ done Define $m(x)=\frac{d}{dx}(f(x))$ done Define $n(x)=\frac{d}{dx}(g(x))$ done $\begin{cases} f(1)=g(1) \\ m(1)=n(1) \end{cases}\Big\vert_{a,b}$ $\{a=b-2, b=b\}$
13	Define $f(x)=e^{-x}+a$ done Define $g(x)=2-x$ done $\begin{cases} f(x)=g(x) \\ \frac{d}{dx}(f(x))=\frac{d}{dx}(g(x)) \end{cases}\Big\vert_{x,a}$ $\{x=0, a=1\}$
14	-3.54 $\text{solve}\left(\frac{d}{dx}(f(x))=0 \mid x=-2, k\right)$ $\{k=-3.541202191\}$
15	Define $f(x)=x\hat{}2-5$ done Define $n(x)=x-\frac{f(x)}{\frac{d}{dx}(f(x))}$ done n(5) 3 n(3) 2.333333333 n(2.333333333) 2.238095238 n(2.238095238) 2.236068896

16	$f(x)$ has a stationary point of inflection at $x = 2$, a local minimum at $x = 7$, and a local maximum at $x = 10$. Stationary point at $x = 2$ but sign of derivative does not change, hence stationary point of inflection. Stationary point at $x = 7$ and derivative sign changes from negative to positive therefore local minimum. Stationary point at $x = 10$ and derivative sign changes from positive to negative hence local maximumum.
17 (a)	Define f(x)=9x^3/16+3x^2/4▶ done solve(f(x)=0, x $\left\{x=0, x=2, x=-\frac{10}{3}\right\}$
17 (b)	$\left\{\begin{array}{l}\frac{d}{dx}(f(x))=0\\ y=f(x)\end{array}\right\|_{x,y}$ $\left\{\{x=-2, y=6\}, \left\{x=\frac{10}{9}, y=-\frac{20(}{81}\right.\right.$▶
17 (c)(i)	Gradient $= -\frac{6}{4} = -\frac{3}{2}$ (1M) Line passes through $(2, 0)$. Thus $y - 0 = -\frac{3}{2}(x-2)$ $\implies 2y + 3x = 6 \quad (1A)$
17(c)(ii)	$\left\{\begin{array}{l}2y+3x=6\\ y=f(x)\end{array}\right\|_{x,y}$ ◀$6\}, \{x=2, y=0\}, \left\{x=-\frac{4}{3}, y=5\right\}\}$

17(c)(iii)	$\sqrt{(-2-(-4/3))^2+(6-5)^2}$ $\frac{\sqrt{13}}{3}$
17(c)(iv)	tanLine(f(x),x,a) $\frac{9 \cdot a^3}{16}+x \cdot \left(\frac{27 \cdot a^2}{16}+\frac{3 \cdot a}{2}-\frac{15}{4}\right)-a \cdot$ solve(ans=0 | x=2, a) $\left\{a=2, a=-\frac{5}{3}\right\}$ f(−5/3) $\frac{275}{48}$ $m=\frac{-275/48}{5/3+2}$ $m=-\frac{25}{16}$ $-\frac{25}{16}/\frac{-3}{2}$ $\frac{25}{24}$
17(d)	Define $c(x)=3-\frac{3x}{2}$ done c(x)−f(x) $\frac{-9 \cdot x^3}{16}-\frac{3 \cdot x^2}{4}+\frac{9 \cdot x}{4}+3$
17(e)	fMax(c(x)−f(x), x, −2, 2) {MaxValue=4.032107349, x=0 x=0.7928365251}
17(f)	Define $n(x)=x-\frac{f(x)-2}{\frac{d}{dx}(f(x)-2)}$ done n(3) 2.454901961 n(2.454901961) 2.292958821 n(2.292958821) 2.278251116

18(a)	Define $f(x)=6a\times\sqrt{x}-x$ done solve(f(c)=0, c) $\{c=0, c=36\cdot a^2\}$
18(b)	solve($\frac{d}{dx}$(f(x))=0, x $\{x=9\cdot a^2\}$
18(c)	tanLine(f(x), x, 36a^2) $-36\cdot a^2\cdot\left(\frac{a}{2\cdot\lvert a\rvert}-1\right)-36\cdot a^2+36\cdot$ ans|a>0 $18\cdot a^2-\frac{x}{2}$
18(d)	tanLine(f(x)−b, x, 36a^2) $-36\cdot a^2\cdot\left(\frac{a}{2\cdot\lvert a\rvert}-1\right)-36\cdot a^2+36\cdot$ ans|a>0 $18\cdot a^2-\frac{x}{2}-b$ solve(ans=0|x=0, b $\{b=18\cdot a^2\}$
18(e)	$d=9a^2$ $(0,0)$ and $(9a^2, 9a^2)$ (1A)
18(f)	The tangents are vertical/horizontal lines. Thus $90°$ or $\frac{\pi}{2}$ (1A)

Section G: Tech-Active Solutions - TI

Question Number	Solutions
7	$e^{2x}(2g(x)+g'(x))$ ©Shortcut: [shift][−] for derivative $\text{factor}\left(\frac{d}{dx}\left(g(x)\cdot \mathbf{e}^{2\cdot x}\right)\right)$ $\left(\frac{d}{dx}(g(x))+2\cdot g(x)\right)\cdot \mathbf{e}^{2\cdot x}$
8	Not differentiable at $x=-5$, $x=1$, $x=3$ and $x=3$. Not differentiable at endpoints/points of discontinuity.
9	C
10	$h(x)$ is continuous and differentiable at $x=0$. ©Shortcut: [menu][4][4] for limit Define $g(x)=\frac{\sin(3\cdot x)}{x}$ *Done* Define $dg(x)=\frac{d}{dx}(g(x))$ *Done* $\lim_{x\to 0}(g(x))$ 3 $\lim_{x\to 0}(dg(x))$ 0

11	$-\frac{1}{\sqrt{2b}} < x < \frac{1}{\sqrt{2b}}$ ©Tip: Use template button under [del] key for second derivative template. Define $f(x)=\mathbf{e}^{-b^2\cdot x^2}$ *Done* $\text{solve}\left(\frac{d^2}{dx^2}(f(x))<0,x\right)\mid b>0$ $\frac{-\sqrt{2}}{2\cdot b}<x<\frac{\sqrt{2}}{2\cdot b}$ and $b>0$
12	$a = 2$ and $b = 4$ $\backslash solve\_smooth\left(-a\cdot x+3,x^2-b\cdot x+4,x,1,\{a,b\}\right)$ ▸ Left Derivative: $-a$ ▸ Right Derivative: $2\cdot x-b$ $\begin{bmatrix} \text{"At x=1:"} & \text{"Left Func."} & \text{"Right Func."} \\ \text{"Value:"} & 3-a & 5-b \\ \text{"Gradient:"} & -a & 2-b \end{bmatrix}$ ▸ Solutions: $a=\boldsymbol{c1}-2$ and $b=\boldsymbol{c1}$
13	$a = 1$ $methods\_diffcalc\backslash solve\_touch\left(\mathbf{e}^{-x}+a,2-x,x,a\right)$ ▸ Derivative 1: $-\mathbf{e}^{-x}$ ▸ Derivative 2: -1 ▸ Equating functions and derivatives. ▸ Solutions: $x=0$ and $a=1$

14	-3.54 Define $f(x)=x^3\cdot\ln(x-k)$ *Done* Define $df(x)=\frac{d}{dx}(f(x))$ *Done* solve$(df(-2)=0,k)$ $k=-3.5412$
15	4 Number of Iterations: 10 OK Cancel *methods_diffcalc\newtons_method*$(x^2-5,x,5)$ ▸ Derivative: $2\cdot x$ ▸ Iterative Formula: $\frac{x^2+5}{2\cdot x}$ ▸ Number of Iterations: 10 "n" "xn" "f(xn)" "f'(xn)" 0. 5. 20. 10. 1. 3. 4. 6. 2. 2.33333 0.444444 4.66667 3. 2.2381 0.00907 4.47619 4. 2.23607 0.000004 4.47214 5. 2.23607 9.E-13 4.47214 6. 2.23607 0. 4.47214 7. 2.23607 0. 4.47214 Note that on the 4^{th} step, the program computes the 5^{th} x-value and compares to the 4^{th} x-value. Since this different is less than the tolerance, the program terminates on the 4^{th} step.

16	$f(x)$ has a stationary point of inflection at $x = 2$, a local minimum at $x = 7$, and a local maximum at $x = 10$. Stationary point at $x = 2$ but sign of derivative does not change, hence stationary point of inflection. Stationary point at $x = 7$ and derivative sign changes from negative to positive therefore local minimum. Stationary point at $x = 10$ and derivative sign changes from positive to negative hence local maximumum.
17 (a)	Define $f(x)=\frac{9\cdot x^3}{16}+\frac{3\cdot x^2}{4}-\frac{15\cdot x}{4}$ *Done* *methods_func\analyse*$(f(x),x)$ ▸ Start Point: $[-\infty \quad -\infty]$ ▸ End Point: $[\infty \quad \infty]$ ▸ Maximal Domain: $-\infty<x<\infty$ ▸ x −Intercepts: (3) $\left[\frac{-10}{3} \quad 0\right],[0 \quad 0],$ $[2 \quad 0]$ ▸ Vertical Intercept: $[0 \quad 0]$

17 (b)	Note: Analyse program from above continued... ▸ Derivative: $\frac{27\cdot x^2}{16}+\frac{3\cdot x}{2}-\frac{15}{4}$ ▸ Inflection Point: $\begin{bmatrix} \frac{-4}{9} & \frac{143}{81} \end{bmatrix}$ (Decreasing) ▸ Stationary Points: (2) $\begin{bmatrix} -2 & 6 \end{bmatrix}$ (Local max.) $\begin{bmatrix} \frac{10}{9} & \frac{-200}{81} \end{bmatrix}$ (Local min.)
17 (c)(i)	Define $a=\begin{bmatrix} -2 & 6 \end{bmatrix}$ *Done* Define $b=\begin{bmatrix} 2 & 0 \end{bmatrix}$ *Done* *methods_misc\linear_info*(a,b) ▸ Point 1: $\begin{bmatrix} -2 & 6 \end{bmatrix}$ ▸ Point 2: $\begin{bmatrix} 2 & 0 \end{bmatrix}$ ▸ Midpoint: $\begin{bmatrix} 0 & 3 \end{bmatrix}$ ▸ Distance: $2\cdot\sqrt{13}$ ▸ Gradient: $\frac{-3}{2}$ ▸ Perp. Bisector: $y=\frac{2\cdot x}{3}+3$ ▸ Linear Equation: $y=3-\frac{3\cdot x}{2}$ ▸ x−Intercept: $\begin{bmatrix} 2 & 0 \end{bmatrix}$ ▸ y−Intercept: $\begin{bmatrix} 0 & 3 \end{bmatrix}$

17(c)(ii)	Note: We use the linear equation output from the linear_info program above. To quickly define the linear function, start by typing Define l(x)=. Then to up to the linear_info program and highlight the desired part. This can be done by holding shift while pressing the arrow keys. Pressing enter will copy the highlighted selection down to the current line. $\text{Define } l(x)=3-\frac{3\cdot x}{2}$ *Done* $methods\_func\backslash intersect(f(x),l(x),x)$ ▸ Intersection Points: (3) $\begin{bmatrix} -2 & 6 \end{bmatrix}$ $\begin{bmatrix} \frac{-4}{3} & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \end{bmatrix}$
17(c)(iii)	Note: To quickly define the point, start by typing Define c=, then go up to the output of the intersect program and highlight the given point. This can be done by holding shift while pressing the arrow keys. Pressing enter will copy the highlighted selection down to the current line. $\text{Define } c=\begin{bmatrix} \frac{-4}{3} & 5 \end{bmatrix}$ *Done* $methods\_misc\backslash linear\_info(a,c)$ ▸ Point 1: $\begin{bmatrix} -2 & 6 \end{bmatrix}$ ▸ Point 2: $\begin{bmatrix} \frac{-4}{3} & 5 \end{bmatrix}$ ▸ Midpoint: $\begin{bmatrix} \frac{-5}{3} & \frac{11}{2} \end{bmatrix}$ ▸ Distance: $\frac{\sqrt{13}}{3}$

17(c)(iv)	$methods\_diffcalc\backslash solve\_touch(f(x),k\cdot l(x),x,k)$ ▶ Derivative 1: $\frac{27\cdot x^2}{16}+\frac{3\cdot x}{2}-\frac{15}{4}$ ▶ Derivative 2: $\frac{-3\cdot k}{2}$ ▶ Equating functions and derivatives. ▶ Solutions: ▶ Solutions: $x=\frac{-5}{3}$ and $k=\frac{25}{24}$ or $x=2$ and $k=-4$ *Done* $f\left(\frac{-5}{3}\right)$ $\frac{275}{48}$ Since the point B on the x-axis is fixed, we apply a dilation from the x-axis by k and solve for when the line touches the function, that is, it has the same gradient at the point of intersection with the function. Subbing this x-value back into the function gives the y-coordinate.
17(d)	$l(x)-f(x)$ $\frac{-9\cdot x^3}{16}-\frac{3\cdot x^2}{4}+\frac{9\cdot x}{4}+3$

17(e)	$methods\_func\backslash analysed(l(x)-f(x),x,-2,2)$ ▶ Start Point: $\begin{bmatrix} -2. & 0. \end{bmatrix}$ ▶ End Point: $\begin{bmatrix} 2. & 0. \end{bmatrix}$ ▶ Maximal Domain: $-2. \leq x \leq 2.$ ▶ x –Intercepts: (3.) $\begin{bmatrix} -2. & 0. \end{bmatrix}, \begin{bmatrix} -1.33333 & 0. \end{bmatrix},$ $\begin{bmatrix} 2. & 0. \end{bmatrix}$ ▶ Vertical Intercept: $\begin{bmatrix} 0. & 3. \end{bmatrix}$ ▶ Derivative: $-1.6875 \cdot x^2 - 1.5 \cdot x + 2.25$ ▶ Inflection Point: $\begin{bmatrix} -0.444444 & 1.90123 \end{bmatrix}$ (Increasing) ▶ Stationary Points: (2.) $\begin{bmatrix} -1.68173 & -0.229638 \end{bmatrix}$ (Local min.) $\begin{bmatrix} 0.792837 & 4.03211 \end{bmatrix}$ (Local max.)

<table>
<tr><td>17(f)</td><td>
Number of Iterations: 3
OK Cancel

$methods\_diffcalc\backslash newtons\_method(f(x)-2,x,3)$
▶ Derivative:
$\frac{27\cdot x^2}{16}+\frac{3\cdot x}{2}-\frac{15}{4}$
▶ Iterative Formula: $\frac{2\cdot(9\cdot x^3+6\cdot x^2+16)}{3\cdot(x+2)\cdot(9\cdot x-10)}$
▶ Number of Iterations: 3
$\begin{bmatrix} \text{"n"} & \text{"xn"} & \text{"f(xn)"} & \text{"f'(xn)"} \\ 0. & 3. & 8.6875 & 15.9375 \\ 1. & 2.4549 & 1.63597 & 10.1021 \\ 2. & 2.29296 & 0.125924 & 8.56174 \\ 3. & 2.27825 & 0.000997 & 8.42622 \end{bmatrix}$
</td></tr>
<tr><td>18(a)</td><td>
DelVar a — Done
Define $f(x)=6\cdot a\cdot\sqrt{x}-x$ — Done
$\text{solve}(f(x)=0,x)$ — $x=36\cdot a^2$ and $a\geq 0$ or $x=0$
Note: Remember to delete the variable a since we have already used it in the previous question. It is best practice to insert a new problem page [doc][4][1] to avoid conflicting variables.
</td></tr>
</table>

18(b)	Define $df(x)=\frac{d}{dx}(f(x))$ *Done* $\text{solve}(df(x)<0,x)\|a>0$ $x>9\cdot a^2$ and $a>0$
18(c)	$methods\_diffcalc\backslash tangent\_line(f(x),x,36\cdot a^2)$ ▶ Derivative: $\frac{3\cdot a}{\sqrt{x}}-1$ ▶ Gradient: $\frac{\text{sign}(a)}{2}-1$ ▶ Passes Through: $[36\cdot a^2 \quad 36\cdot a\cdot \lvert a\rvert-36\cdot a^2]$ ▶ x −Intercept: $\left[\frac{-36\cdot a\cdot \lvert a\rvert}{\text{sign}(a)-2} \quad 0\right]$ ▶ Vertical Intercept: $[0 \quad 18\cdot a\cdot \lvert a\rvert]$ ▶ Tangent Line: $\frac{(\text{sign}(a)-2)\cdot x}{2}+18\cdot a\cdot \lvert a\rvert$ $\frac{(\text{sign}(a)-2)\cdot x}{2}+18\cdot a\cdot \lvert a\rvert\|a>0$ $18\cdot a^2-\frac{x}{2}$ Note: Since a>0, we can replace sign(a) with 1 in our working out.

18(d)	Define $t(x)=18\cdot a^2-\frac{x}{2}$ *Done* solve$(0=t(0)-b,b)$ $b=18\cdot a^2$
18(e)	solve$(f(x)=x,x)$ $x=9\cdot a^2$ and $a\geq 0$ or $x=0$
18(f)	The tangents are vertical/horizontal lines. Thus 90° or $\frac{\pi}{2}$ (1A)

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VCE Mathematical Methods ¾

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