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VCE Mathematical Methods ¾
Family of Functions & its Exam Skills [0.14]
Workshop Solutions

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________ Notes:	Pg / Q #: ____________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
Pg / Q #: ____________ Notes:	Pg / Q #: ____________ Notes:

Section A: Recap

Families of Functions

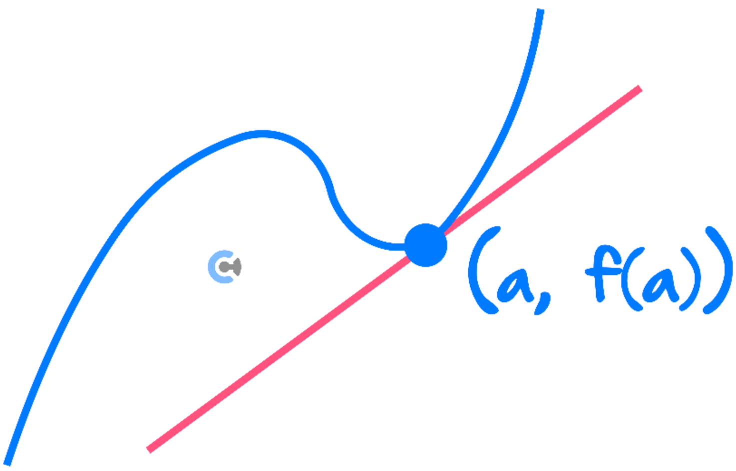
Functions with an unknown.

- They involve understanding and using transformations.
- They involve the use of sliders/manipulate on CAS/technology.

Using Sliders/Manipulate on Technology

1. **Understanding the effect of the unknown on the graph.**
   - As often this is not obvious from transformations.
2. **Checking your answer.**
   - When finding the value(s) of an unknown, check the value smaller and larger than the value obtained to see which side satisfies the condition.

Tangent to a Family of Functions



- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

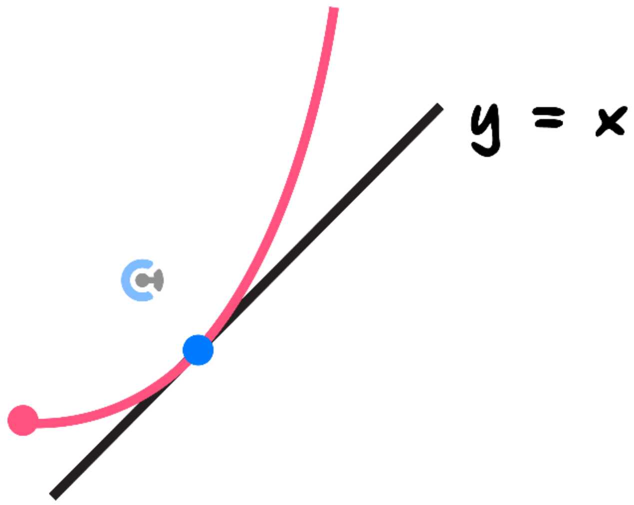
- With the same gradient.

$$f'(a) = m$$

- We just solve these simultaneously.

Definition

Family of Functions and Inverse



- For a function to "touch" $y = x$ as a tangent:
  - They intersect.

$$f(a) = a$$

  - With the same gradient.

$$f'(a) = 1$$

- We just solve these simultaneously.

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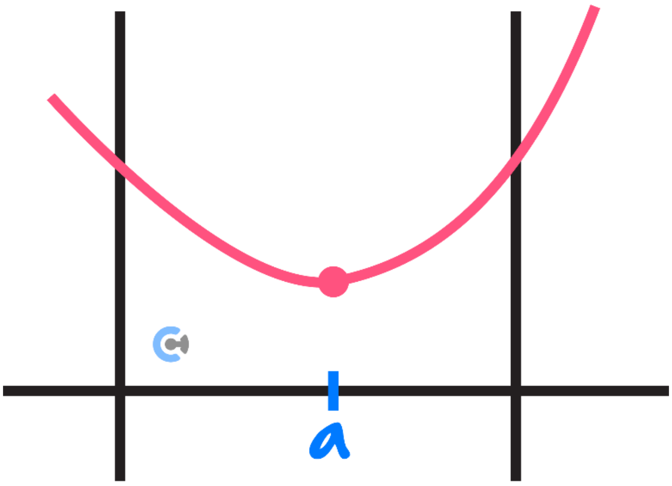
Definition

Minimum/Maximum at a Turning Point



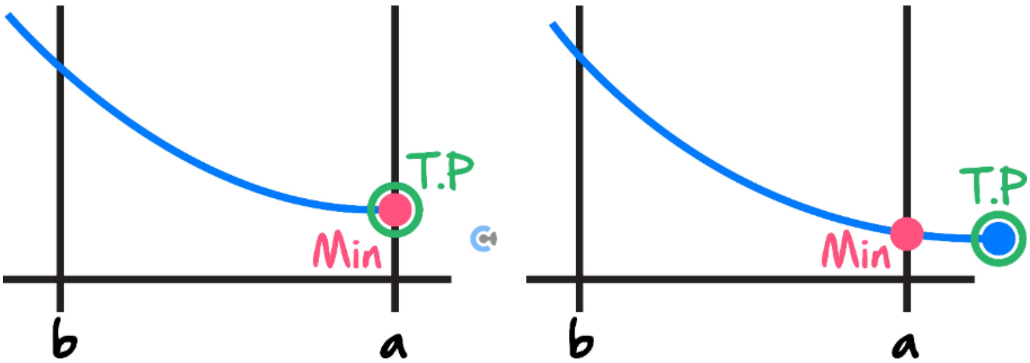
- To achieve minimum/maximum at $x = a$.

$$f'(a) = 0$$

- This is only when $x = a$ is not an end point.

Definition

Minimum/Maximum at an End Point



- Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

- Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.

  We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = 0$$

Section B: Warm Up (11 Marks)

INSTRUCTION:

- Regular: 11 Marks. 11 Minutes Writing.
- Extension: Skip

Question 1 (3 marks)

Consider the function:

$$f(x) = e^{2x} + k$$

Find the value of k such that $f(x)$ hits the line $y = 2x$ only once.

```
In[1]:= f[x_] := Exp[2 x] + k
In[2]:= Solve[f'[x] == 2 && f[x] == x, Reals]
Out[2]= {{k → -1, x → 0}}
```

$k = -1$

Space for Personal Notes

Question 2 (3 marks)

Consider the function $f(x) = \sqrt{x+1} + k$.

Find the value(s) of k such that $f(x)$ and its inverse never intersect.

```
In[3]:= f[x_] := √(x + 1) + k

In[4]:= Solve[f[x] == x && f'[x] == 1]

Out[4]= {{k → -5/4, x → -3/4}}
```

Then a rough sketch gives our answer as $k < -\frac{5}{4}$

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Question 3 (3 marks)

Consider the function $f(x) = 2x \log_e(x)$

Find the value of k such that $f(x - k)$ has a local minimum when $x = 10$.

```
In[5]:= f[x_] := 2 x Log[x]
In[6]:= Solve[f'[x] == 0, x]
Out[6]= {{x → 1/e}}
```

Shift k units right. So local min when $x = 10$ if $k = 10 - \frac{1}{e}$

Question 4 (2 marks)

Consider the family of functions $f(x) = (x - k)^2 + 4, x \in [2, 10]$.

Find the value(s) of k such that the minimum value of the function occurs at $x = 2$

Local min is at $(k, 4)$.
So $x = 2$ is the minimum if $k \leq 2$

Section C: Exam 1 Questions (17 Marks)

INSTRUCTION:

- Regular: 17 Marks. 5 Minutes Reading. 25 Minutes Writing.
- Extension: 17 Marks. 5 Minutes Reading. 17 Minutes Writing.

Question 5 (3 marks)

Consider the function:

$$f(x) = e^{\frac{1}{2}x} + 1$$

Find the value(s) of k such that $f(x-k)$ hits the line $y = \frac{1}{2}x$ twice.

```
f[x_] := E^(1/2 x) + 1

Solve[f[x - k] == x/2 && f'[x - k] == 1/2, {x, k}]
풀이 함수

Solve: Inconsistent or redundant transcendental equation. After
Solve: Inverse functions are being used by Solve, so some solut
{{x → 4, k → 4}}
```

$k > 4$

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Question 6 (4 marks)

Consider the function $f: [a, \infty) \to R, f(x) = (x-3)^2 + k$.

a. Find the minimum value of a such that f^{-1} exists. (1 mark)

$a = 3$

Let $g: D \to R, g(x) = \log_e(2x + 1)$ where D is the maximal domain of g.

b. Find the value(s) of k such that $g(f(x))$ exists. (1 mark)

$k > -\frac{1}{2}$

c. Find the value(s) of k such that f and f^{-1} has two intersections. (2 marks)

```
f[x_] := (x - 3)^2 + k

Solve[f[x] == x && f'[x] == 1, {x, k}]
풀이 함수

{{x → 7/2, k → 13/4}}

Solve[f[3] == 3, k]
풀이 함수

{{k → 3}}

(* [3, 13/4) *)
```

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Question 7 (3 marks)

Let $f: R \to R, f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$, has a y-axis intercept in the interval $y \in [0,2]$.

Find the possible values of k in terms of a.

```
In[22]:= f[x_] := Exp[x] + k

In[23]:= TangentLine[f[x], {x, a}] // Expand
Out[23]= e^a - a e^a + k + e^a x

In[24]:= e^a - a e^a + k + e^a x /. x → 0 // Expand
Out[24]= e^a - a e^a + k

In[25]:= Solve[e^a - a e^a + k == 0, k] // Expand
Out[25]= {{k → -e^a + a e^a}}

In[26]:= Solve[e^a - a e^a + k == 2, k]
Out[26]= {{k → 2 - e^a + a e^a}}
```

$$k \in [ae^a - e^a, ae^a - e^a + 2]$$

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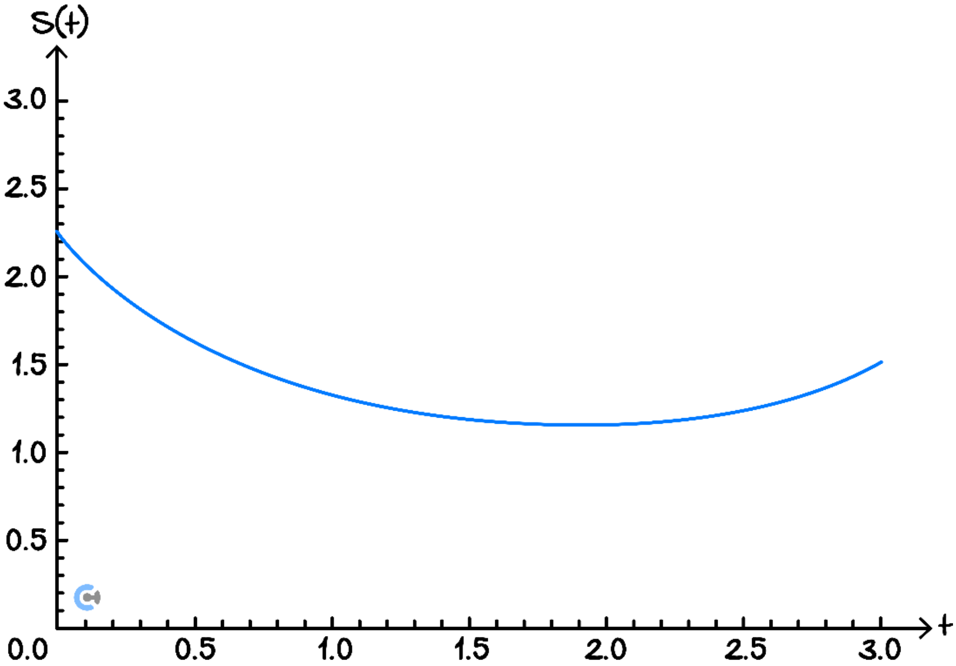
Question 8 (7 marks)

Contour students' stress levels were measured during their MM*34* workshop which lasts for 3 hours.



$$S(t) = \frac{a}{t+1} + \frac{1}{4-t} \text{ where } 0 \le t \le 3, a \in (0, 1000)$$

t is number of hours since the start of the MM*34* workshop.

a. Find the derivative of $S(t)$. (2 marks)

$$\frac{1}{(4-t)^2} - \frac{a}{(1+t)^2}$$

b. For what value(s) of a would students have a minimum stress level at $t = 1$. (2 marks)

```
Solve[f'[1] == 0, a]
```
풀이 함수

$\left\{\left\{a \to \frac{4}{9}\right\}\right\}$

$a \in \left[\frac{4}{9}, 100\right)$

c. For what value(s) of a would students have a minimum stress level at the end of the MM34 workshop. (3 marks)

```
Solve[f'[3] == 0, a]
풀이 함수

{{a → 16}}
```

$a \in [16, 1000)$

Space for Personal Notes

Section D: Tech Active Exam Skills

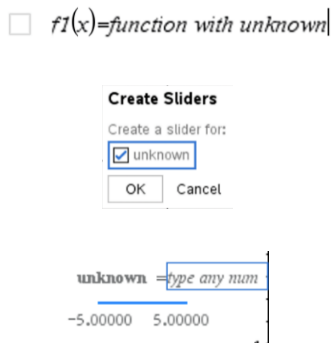
Calculator Commands: Using Sliders/Manipulate on CAS

- **Mathematica**

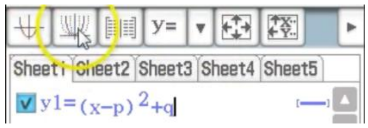
```
Manipulate[Plot[function, {x, xmin, xmax}],

{unknown, lowerbound, upperbound}]
```

- **NOTE:** The function **must** be typed out instead of using its saved name.

- **TI**



- **Casio**



Calculator Commands: Finding k so that f and f^{-1} intersect once

- Step 1: Plot the functions with sliders.
- Step 2: Solve the equations $f(a) = a$ and $f'(a) = 1$ simultaneously on our CAS.
- Step 3: Check that your answer makes sense by using your sliders.
- **Example** Consider the function $f(x) = e^{kx}$, where $k > 0$. Find the exact value of k for which f and f^{-1}, have exactly one point of intersection.
- **Mathematica:**

```
In[29]:= f[x_] := Exp[k x]

In[33]:= Solve[f[x] == x && f'[x] == 1]
```

Out[33]= $\left\{\left\{k \to \frac{1}{e},\ x \to e\right\}\right\}$

- TI:

Define $f(x)=e^{k\cdot x}$	*Done*
Define $df(x)=k\cdot e^{k\cdot x}$	*Done*
⚠ solve$(f(x)=x$ and $df(x)=1,k,x)$	$k=0.367879$ and $x=2.71828$
solve$(f(e)=e,k)$	$k=e^{-1}$
solve$(k\cdot x=\ln(x)$ and $df(x)=1,k,x)$	$k=e^{-1}$ and $x=e$

- Casio:

$$\begin{cases}\exp(k*x)=x\\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{e^{k\cdot x}-x=0,\ k\cdot e^{k\cdot x}-1=0\}$$

$$\begin{cases}k*x = \ln(x)\\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{x=e,\ k=e^{-1}\}$$

NOTE: Sometimes for tricker equations we will not immediately get a solution for the system of equations or it will not be exact, but often altering the equations with some simple algebra will then allow the CAS to solve it correctly.

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Section E: Exam 2 Questions (27 Marks)

INSTRUCTION:

- Regular: 27 Marks. 5 Minutes Reading. 40 Minutes Writing.
- Extension: 27 Marks. 5 Minutes Reading. 27 Minutes Writing.

Question 9 (1 mark)

The graph of $y = kx - 2$ intersects the graph of $y = x^2 + 6x$ at two distinct points for:

A. $k = 8$

B. $k > 6 + 2\sqrt{2}$ or $k < 6 - 2\sqrt{2}$

C. $2 \leq k \leq 6$

D. $4 - 2\sqrt{3} \leq k \leq 4 + 2\sqrt{3}$

```
In[27]:= Solve[k x - 2 == x^2 + 6 x, x]
Out[27]= {{x -> 1/2 (-6 + k - Sqrt[28 - 12 k + k^2])}, {x -> 1/2 (-6 + k + Sqrt[28 - 12 k + k^2])}}

In[29]:= Solve[28 - 12 k + k^2 == 0, k]
Out[29]= {{k -> 2 (3 - Sqrt[2])}, {k -> 2 (3 + Sqrt[2])}}
```

Question 10 (1 mark)

Consider the function $g(x) = e^{x^2 - 4kx + k}$. The value(s) of k for which g has a local minimum when $y = 1$ are:

A. $k = 0$

B. $k = 0, \frac{1}{4}$

C. $k = 0,1$

D. $k = 2$

```
In[40]:= f[x_] := Exp[x^2 - 4 k x + k]
In[41]:= Solve[f'[x] == 0, x] // Quiet
Out[41]= {{x -> 2 k}}

In[42]:= Solve[f[2 k] == 1, k, Reals]
Out[42]= {{k -> 0}, {k -> 1/4}}
```

Space for Personal Notes

Question 11 (1 mark)

Consider the function $f(x) = 2x^3 + x^2 - 7x - 6$. The value(s) of k for which $f(x+k)$ has one positive x-intercept are:

A. $k > 2$

B. $-2 < k \leq 1$

C. $-1 \leq k < 2$

D. $-1 < k < 2$

```
In[45]:= f[x_] := -6 - 7 x + x^2 + 2 x^3
In[46]:= Solve[f[x] == 0, x]
Out[46]= {{x → -3/2}, {x → -1}, {x → 2}}
```

Question 12 (1 mark)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 3x + 4$. The function $g(x) = f(x) + c$, where $c \in R$, has one x-intercept for:

A. $c > -2$

B. $c < -6$

C. $-6 < c < -2$

D. $c < -6$ or $c > -2$

```
In[50]:= f[x_] := x^3 - 3 x + 4
In[51]:= Solve[f'[x] == 0, x]
Out[51]= {{x → -1}, {x → 1}}
In[53]:= f[-1]
Out[53]= 6
In[54]:= f[1]
Out[54]= 2
```

Question 13 (1 mark)

Consider the function:

$$f: [0,25] \to R, f(x) = \frac{2}{x+1} + \frac{k}{30-x}$$

where $k > 0$.

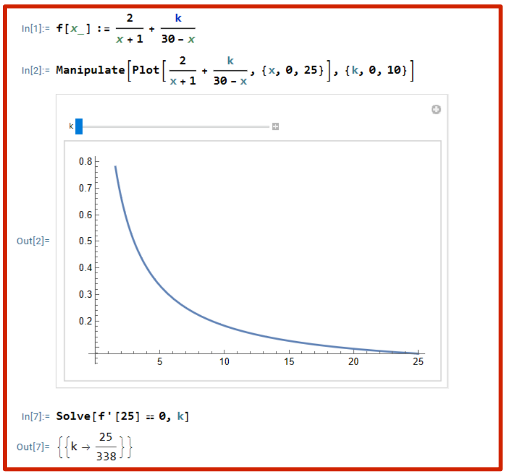
Find the value(s) of k such that the minimum of $f(x)$ occurs at $x = 25$.

A. $\frac{25}{338}$

B. $\left[\frac{25}{338}, \infty\right)$

C. $\left(-\infty, \frac{25}{338}\right]$

D. $\left(0, \frac{25}{338}\right]$



Question 14 (1 mark)

For $f: [2, \infty) \to f(x) = (x - 2)^2 + k$, the value(s) of k such that there is only one intersection between f and f^{-1}.

A. $(-\infty, 2)$

B. $\left(2, \frac{9}{4}\right]$

C. $(-\infty, 2] \cup \left\{\frac{9}{4}\right\}$

D. $(-\infty, 2) \cup \left\{\frac{9}{4}\right\}$

```
In[8]:= f[x_] := (x - 2) ^2 + k

In[10]:= Solve[f[x] == x, x]

Out[10]= {{x → 1/2 (5 - √(9 - 4 k))}, {x → 1/2 (5 + √(9 - 4 k))}}

In[11]:= Solve[9 - 4 k == 0, k]

Out[11]= {{k → 9/4}}
```

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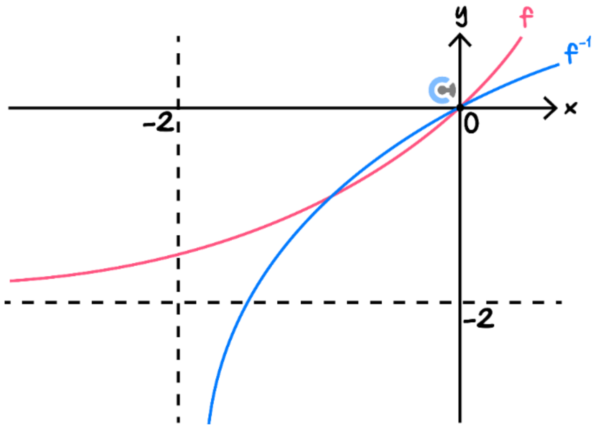
Question 15 (10 marks)

Consider a function $f(x) = 2e^x - 2$.

Part of the graphs of f and f^{-1} are shown below.



a. Find the gradient of f and the gradient of f^{-1} at $x = 0$. (2 marks)

```
In[19]:= f[x_] := 2 Exp[x] - 2
In[20]:= Solve[f[y] == x, y, Reals]
Out[20]= {{y -> Log[(2 + x)/2] if x > -2}}
In[21]:= f1[x_] := Log[(2 + x)/2]
```

```
In[22]:= f'[0]
Out[22]= 2
In[23]:= f1'[0]
Out[23]= 1/2
```

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

b. Find the value of k such that $g_k(x) = f(x)$. (1 mark)

```
In[24]:= g[x_] := 2 Exp[k x] - 2
In[26]:= Solve[f[x] == g[x], k, Reals]
Out[26]= {{k -> 1}}
```

c. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$. (1 mark)

```
In[27]:= Solve[g[y] == x, y, Reals]
Out[27]= {{y -> Log[(2+x)/2]/k if x > -2}}
```

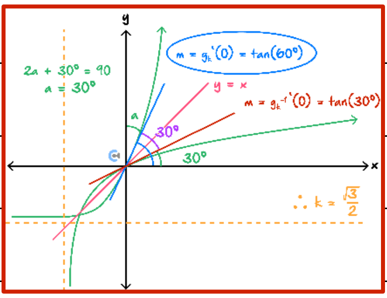
d. Describe the transformation that maps the graph of g_1 onto the graph of g_k. (1 mark)

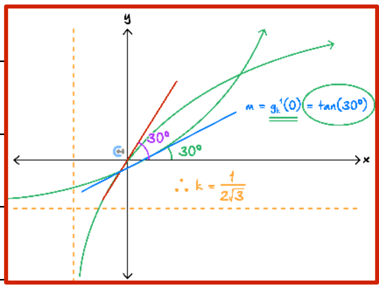
Dilation by factor $\frac{1}{k}$ from the y-axis.

e. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}. (1 mark)

Dilation by factor $\frac{1}{k}$ from the x-axis.

f. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively. Find the value(s) of k for which the angle between L_1 and L_2 is 30°. (2 marks)

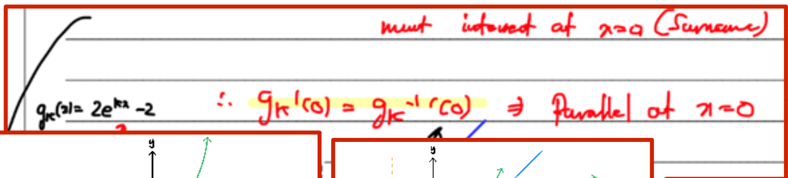


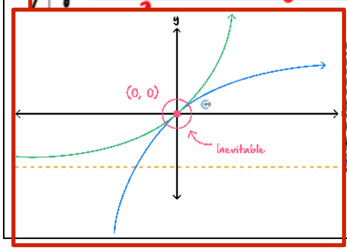


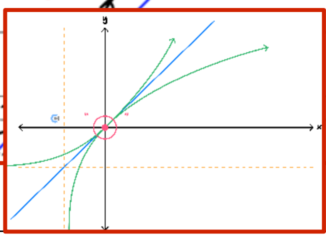
In[32]:= g'[0]
Out[32]= 2 k

In[34]:= Solve[Abs[$\frac{2k - \frac{1}{2k}}{1 + 2k * \frac{1}{2k}}$] == Tan[30 Degree] && k > 0, k]
Out[34]= $\{\{k \to \frac{1}{2\sqrt{3}}\}, \{k \to \frac{\sqrt{3}}{2}\}\}$

g. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution. Find p. (2 marks)





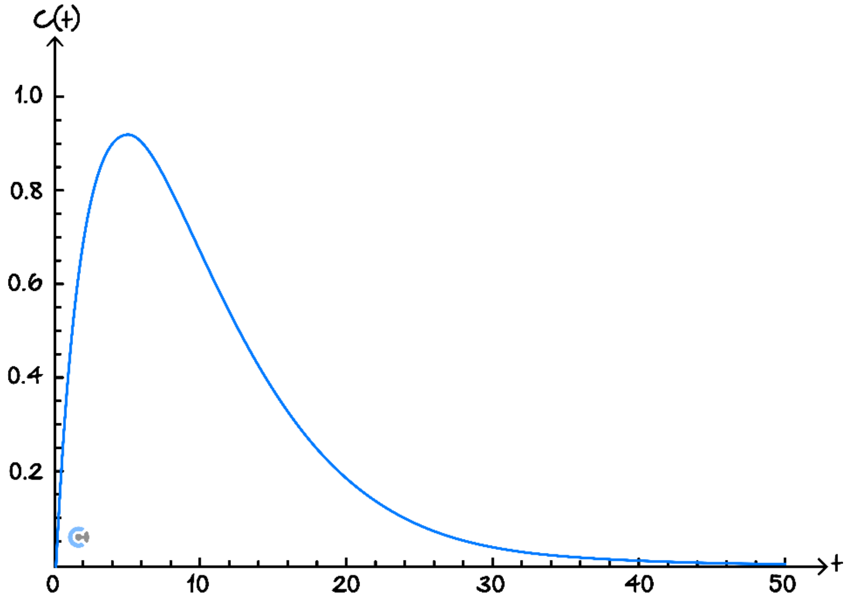


In[41]:= Solve[g[x] == x && g'[x] == 1, Reals]
Out[41]= $\{\{k \to \frac{1}{2}, x \to 0\}\}$

Space for Personal Notes

Question 16 (11 marks)

The function $C(t) = 0.5te^{-0.2t}$ is a reasonable model of the measured blood cyanide concentrations in $\mu g/mL$ after t minutes, which is shown in the figure below.



a. Find the value of t for which the concentration is maximum. (2 marks)

```
In[12]:= c[t_] := 1/2 t Exp[-1/5 t]
In[14]:= c'[t]
Out[14]= e^(-t/5)/2 - 1/10 e^(-t/5) t
In[13]:= Solve[c'[t] == 0, t]
Out[13]= {{t -> 5}}
```

b. Find the inflection point of $C(t)$. Give your answer correct to two decimal places. (2 marks)

```
In[15]:= c''[t]
Out[15]= -1/5 e^(-t/5) + 1/50 e^(-t/5) t
In[16]:= Solve[c''[t] == 0, t]
Out[16]= {{t -> 10}}
In[18]:= {10, c[10]} // N
Out[18]= {10., 0.676676}
```

(10.00,0.68)

Another function $C_1(t) = ate^{-bt}$ is used to measure the concentration of another substance in the blood in $\mu g/mL$ after t minutes.

c. Find the values of a and b if the maximum amount of this substance in the blood was 120 $\mu g/mL$ after 2 hours. (2 marks)

```
Solve[f'[120] == 0 && f[120] == 120]
풀이 함수

Solve: Inconsistent or redundant transcendenta
Solve: Inverse functions are being used by Solv

{{a → e, b → 1/120}}
```

It is known that $C_1(t)$ has a maximum value of 100 $\mu g/mL$.

d. Solve for the value of a in terms of b. (2 marks)

```
Solve[f'[t] == 0, t]
풀이 함수
{{t → 1/b}}

Solve[f[1/b] == 100, a]
풀이 함수
{{a → 100 b e}}
```

e. Solve for the value(s) of b such that $C_1(t)$ is above 80 $\mu g/mL$ for less than 10 minutes.

Give your answer correct to two decimal places. (3 marks)

```
f[t_] := 100 b e^(1-b t) t

NSolve[f[t] == f[t + 10] && f[t] == 80, {t, b}, Reals]
수치 해석                                              실수 영역

{{t → 3.48685, b → 0.135272}, {t → -13.4868, b → -0.135272}}

(* b>0.14: Check using sliders *)
```

Section F: Extension Exam 1 (11 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 11 Marks. 2 Minutes Reading. 15 Minutes Writing.

Question 17 (11 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = e^{-kx} + 2x$, where k is a real number.

a.

i. Find in terms of k, the coordinates for the stationary point of the graph of $y = f(x)$ (when it exists) and specify the range of values of k for which this stationary point exists. (3 marks)

$$f'(x) = -ke^{-kx} + 2 = 0 \implies e^{-kx} = \frac{2}{k} \implies -kx = \log_e((2^{-1}k)^{-1})$$

$$\implies x = \frac{1}{k}\log_e\left(\frac{k}{2}\right)$$

Then

$$\begin{aligned} f\left(\frac{1}{k}\log_e\left(\frac{k}{2}\right)\right) &= e^{-\log_e(k/2)} + \frac{2}{k}\log_e\left(\frac{k}{2}\right) \\ &= \frac{2}{k} + \frac{2}{k}\log_e\left(\frac{k}{2}\right) \\ &= \frac{2}{k}\left(1 + \log_e\left(\frac{k}{2}\right)\right) \end{aligned}$$

So f has a stationary point when $k > 0$ (domain of log), at

$$\left(\frac{1}{k}\log_e\left(\frac{k}{2}\right), \frac{2}{k}\left(1 + \log_e\left(\frac{k}{2}\right)\right)\right)$$

ii. Find the value of k for which the stationary point of f occurs on the x-axis. (1 mark)

We require $1 + \log_e\left(\frac{k}{2}\right) = 0 \implies \frac{k}{2} = \frac{1}{e} \implies k = \frac{2}{e}$

iii. Find the value(s) of k such that the x-coordinate of the stationary point of f is a positive number. (1 mark)

We require $\frac{1}{k}\log_e\left(\frac{k}{2}\right) > 0 \implies \frac{k}{2} > 1 \implies k > 2$

b. For a particular value of k, the tangent to f at $x = -4$ passes through the origin. Find this value of k and the equation of this tangent. (3 marks)

We find the equation of the tangent at $x = -4$ in terms of k:
$f'(-4) = -ke^{4k} + 2$ and $f(-4) = e^{4k} - 8$. Thus the tangent satisfies

$$y - e^{4k} + 8 = (-ke^{4k} + 2)(x + 4)$$
$$y = (2 - ke^{4k})x + e^{4k} - 4ke^{4k}$$

To make this pass through the origin let $x = 0$, then

$$e^{4k}(1 - 4k) = 0 \implies k = \frac{1}{4}.$$

So tangent through origin is $y = \left(2 - \frac{1}{4}e\right)x$, when $k = \frac{1}{4}$.

Consider the function $g(x) = e^{-\frac{x}{2}} + 2x$.

c. State how many real solutions the equation $g(x) = 0$ has. (1 mark)

$g(x)$ is f when $k = 1/2$.
Note that the single stationary point of g is below the x-axis and is a local minimum.
Therefore $g(x) = 0$ has two real solutions.

It is known that the tangent to $g(x)$ when $x = -4$ has equation $y = \left(\frac{4-e^2}{2}\right)x - e^2$.

d. Newton's method is used to approximate a root of g, with $x_0 = -4$. Find the value of x_1 without performing any differentiation. (2 marks)

x_1 is the x-intercept of the the tangent drawn to g at $x_0 = -4$.

Therefore we solve $\left(\frac{4-e^2}{2}\right)x - e^2 = 0 \implies x = \frac{2e^2}{4-e^2}$.

Thus $x_1 = \frac{2e^2}{4-e^2}$.

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Section G: Extension Exam 2 (13 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 13 Marks. 3 Minutes Reading. 18 Minutes Writing.

Question 18 (13 marks)

Let f be the hyperbolic tangent function, that is $f(x) = \tanh(x)$.

The function f may be defined, $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{e^{2x}-1}{e^{2x}+1}$.

a. Define the inverse function of f. Write down the rule for $f^{-1}(x)$ in the form $f^{-1}(x) = \log_e(g(x))$, for some function g. (2 marks)

$f^{-1} : (-1, 1) \to \mathbb{R},\ f^{-1}(x) = \log_e\left(\sqrt{\frac{1+x}{1-x}}\right)$

(1M for domain, 1M for rule in correct form)

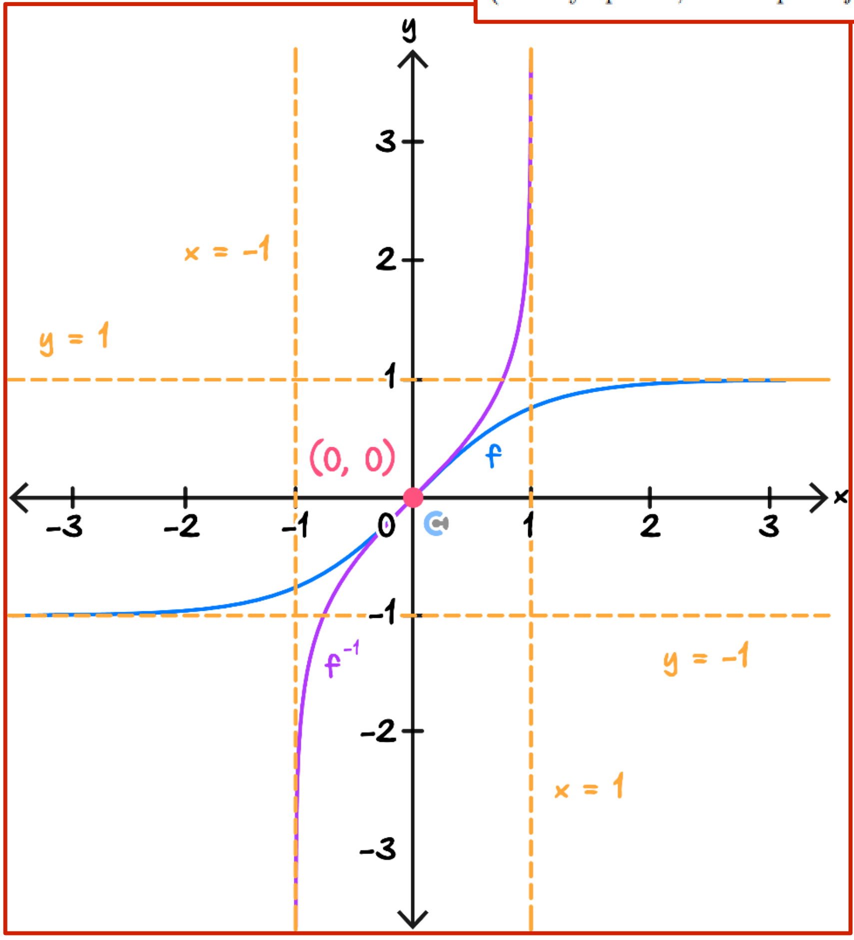
```
In[140]:= f[x_] := (Exp[2 x] - 1)/(Exp[2 x] + 1)

In[141]:= Solve[f[y] == x, y, Reals]

Out[141]= {{y -> 1/2 Log[(-1 - x)/(-1 + x)] if -1 < x < 1}}
```

b. Sketch the graphs of f and f^{-1} on the axes below. Label all asymptotes with equations and points of intersection with coordinates. (3 marks)

(1M asymptotes, 1M shape of f, 1M shape of f^{-1})



Now consider the family of functions $f : \mathbb{R} \to \mathbb{R},\ g_k(x) = \frac{e^{kx}-1}{e^{kx}+1}$. The function g_1 specifies the function g_k when $k = 1$.

c. Find the acute angle between the tangents to g_1 and g_1^{-1}, at their point of intersection. Give your answer in degrees correct to one decimal place. (2 marks)

Intersect at $(0,0)$. $g_1'(0) = \frac{1}{2} \implies (g^{-1})'(0) = 2$ (1M).
Therefore angle is $|\arctan(2) - \arctan(1/2)| = 36.9°$ (1M)

```
In[142]:= g[x_] := (Exp[k x] - 1)/(Exp[k x] + 1)

In[143]:= g'[0] /. k → 1

Out[143]= 1/2

In[144]:= Abs[ArcTan[2] - ArcTan[1/2]] / Degree // N

Out[144]= 36.8699
```

d. Find the value(s) of k such that g_k and g_k^{-1} intersect exactly once. (2 marks)

Always intersect at the origin. Use sliders/our graph from part b.
$-2 < k < 2$ (1M)

```
In[153]:= FindInstance[g[x] == x && g'[x] == 1, {x, k}, Reals]
Out[153]= {{x → 0, k → 2}}

In[156]:= Solve[g'[0] == 1]
Out[156]= {{k → 2}}
```

e. Find the value(s) of k such that g_k and g_k^{-1} intersect once and make an acute angle of 45° at this point of intersection. (2 marks)

Intersect at the origin and $-2 < k < 2$. $g_k'(0) = \frac{k}{2}$

So we solve $\left|\frac{\frac{k}{2} - \frac{2}{k}}{1 + 1}\right| = \tan(45°)$ and $-2 < k < 2$

$\Rightarrow k = 2 + 2\sqrt{2}, 2 - 2\sqrt{2}$

f. Find the value(s) of k such that g_k and g_k^{-1} intersect at a point when $x = -\frac{1}{2}$. (2 marks)

Note that for $k > 2$ intersections occur along the line $y = x$ but for $k < -2$ intersections occur along the line $y = -x$. (1M)
$k = \pm 2\log_e(3)$ (1M)

```
In[163]:= Solve[g[-1/2] == 1/2, Reals]
Out[163]= {{k → -2 Log[3]}}

In[164]:= Solve[g[-1/2] == -1/2, Reals]
Out[164]= {{k → 2 Log[3]}}

In[165]:= Solve[g[y] == x, y, Reals]
Out[165]= {{y → Log[(-1-x)/(-1+x)]/k if -1 < x < 1}}

In[166]:= g1[x_] := 1/k Log[(-1 - x)/(x - 1)]
```

```
In[168]:= Solve[g[-1/2] == g1[-1/2], {x, k}, Reals]
Solve: Equations may not give solutions for all "solve" variables.
Out[168]= {{k → -2.20...}, {k → 2.20...}}

In[172]:= Solve[g[x] == g1[x] /. k → 2 Log[3], x, Reals]
Out[172]= {{x → -1/2}, {x → 0}, {x → 1/2}}

In[173]:= Solve[g[x] == g1[x] /. k → -2 Log[3], x, Reals]
Out[173]= {{x → -1/2}, {x → 0}, {x → 1/2}}
```

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Section H: Tech-Active Solutions

Question Number	Solutions
9	$k > 6 + 2\sqrt{2}$ or $k < 6 - 2\sqrt{2}$ In[27]:= Solve[k x - 2 == x^2 + 6 x, x] Out[27]= $\left\{\left\{x \to \frac{1}{2}\left(-6 + k - \sqrt{28 - 12k + k^2}\right)\right\}, \left\{x \to \frac{1}{2}\left(-6 + k + \sqrt{28 - 12k + k^2}\right)\right\}\right\}$ In[29]:= Solve[$28 - 12k + k^2$ == 0, k] Out[29]= $\{\{k \to 2(3 - \sqrt{2})\}, \{k \to 2(3 + \sqrt{2})\}\}$

10	$k = 0, \frac{1}{4}$
	In[40]:= f[x_] := Exp[x^2 - 4 k x + k]
	In[41]:= Solve[f'[x] == 0, x] // Quiet
	Out[41]= $\{\{x \to 2k\}\}$
	In[42]:= Solve[f[2 k] == 1, k, Reals]
	Out[42]= $\left\{\{k \to 0\}, \left\{k \to \frac{1}{4}\right\}\right\}$

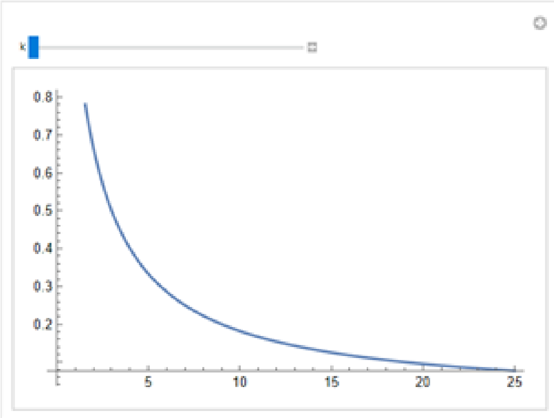
11	$-1 \le k < 2$ In[45]:= f[x_] := $-6 - 7x + x^2 + 2x^3$ In[46]:= Solve[f[x] == 0, x] Out[46]= $\left\{\left\{x \to -\frac{3}{2}\right\}, \{x \to -1\}, \{x \to 2\}\right\}$
12	$c < -6$ or $c > -2$ In[50]:= f[x_] := x^3 - 3 x + 4 In[51]:= Solve[f'[x] == 0, x] Out[51]= $\{\{x \to -1\}, \{x \to 1\}\}$ In[53]:= f[-1] Out[53]= 6 In[54]:= f[1] Out[54]= 2

<table>
<tr><td>13</td><td>

$$\left(0, \frac{25}{338}\right]$$

```
In[1]:= f[x_] := 2/(x + 1) + k/(30 - x)
In[2]:= Manipulate[Plot[2/(x + 1) + k/(30 - x), {x, 0, 25}], {k, 0, 10}]
```

Out[2]=




```
In[7]:= Solve[f'[25] == 0, k]
Out[7]= {{k -> 25/338}}
```

</td></tr>
<tr><td>14</td><td>

$$(-\infty, 2) \cup \left\{\frac{9}{4}\right\}$$

```
In[8]:= f[x_] := (x - 2)^2 + k
In[10]:= Solve[f[x] == x, x]
Out[10]= {{x -> 1/2 (5 - √(9 - 4 k))}, {x -> 1/2 (5 + √(9 - 4 k))}}
In[11]:= Solve[9 - 4 k == 0, k]
Out[11]= {{k -> 9/4}}
```

</td></tr>
<tr><td>15 (a)</td><td>

```
In[19]:= f[x_] := 2 Exp[x] - 2
In[20]:= Solve[f[y] == x, y, Reals]
Out[20]= {{y -> Log[(2 + x)/2] if x > -2}}
In[21]:= f1[x_] := Log[(2 + x)/2]
In[22]:= f'[0]
Out[22]= 2
In[23]:= f1'[0]
Out[23]= 1/2
```

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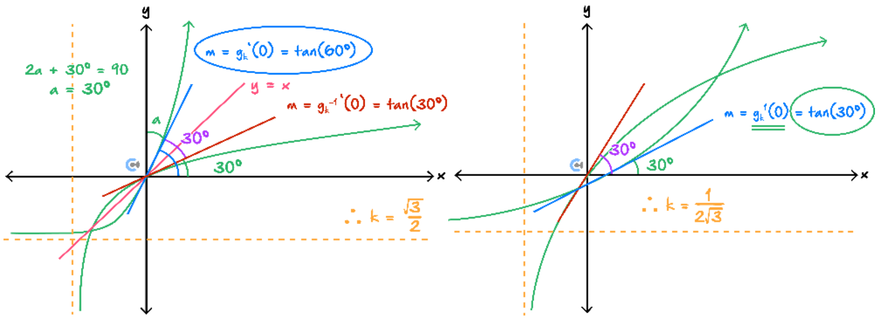
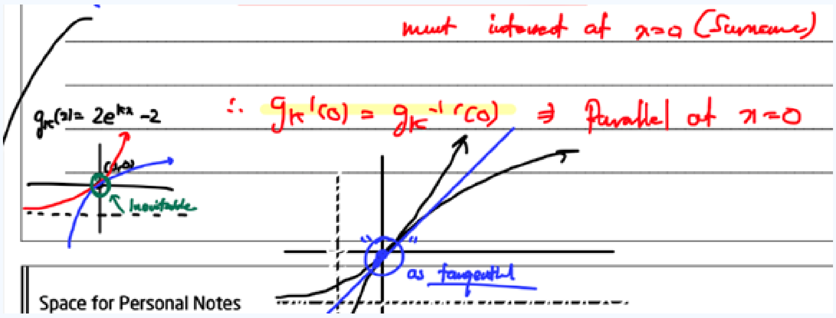
</table>

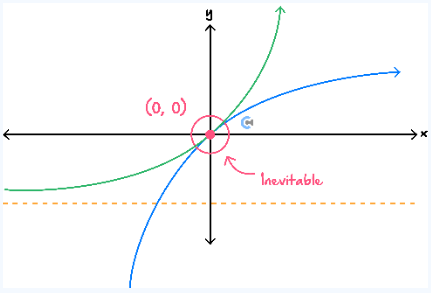
15 (b)	`In[24]:= g[x_] := 2 Exp[k x] - 2`
	`In[26]:= Solve[f[x] == g[x], k, Reals]`
	`Out[26]= {{k → 1}}`

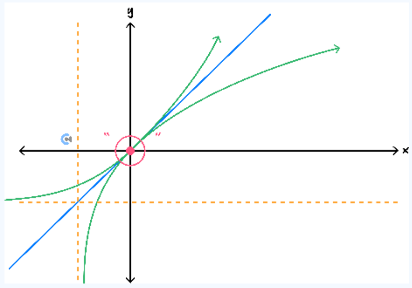
15 (c)	`In[27]:= Solve[g[y] == x, y, Reals]` `Out[27]=` $\left\{\left\{y \to \boxed{\frac{\text{Log}\left[\frac{2+x}{2}\right]}{k} \text{ if } x > -2}\right\}\right\}$
15 (d)	Dilation by factor $\frac{1}{k}$ from the y-axis.
15 (e)	Dilation by factor $\frac{1}{k}$ from the x-axis.

15 (f)	
	`In[32]:= g'[0]`
	`Out[32]= 2 k`
	`In[34]:=` $\text{Solve}\left[\text{Abs}\left[\frac{2k - \frac{1}{2k}}{1 + 2k * \frac{1}{2k}}\right] == \text{Tan}[30\,\text{Degree}]\ \&\&\ k > 0,\ k\right]$
	`Out[34]=` $\left\{\left\{k \to \frac{1}{2\sqrt{3}}\right\}, \left\{k \to \frac{\sqrt{3}}{2}\right\}\right\}$

15 (g)








```
In[41]:= Solve[g[x] == x && g'[x] == 1, Reals]
Out[41]= {{k → 1/2, x → 0}}
```

16 (a)

```
In[12]:= c[t_] := 1/2 t Exp[-1/5 t]
In[14]:= c'[t]
Out[14]= e^(-t/5)/2 - 1/10 e^(-t/5) t
In[13]:= Solve[c'[t] == 0, t]
Out[13]= {{t → 5}}
```

16 (b)

```
In[15]:= c''[t]
Out[15]= -1/5 e^(-t/5) + 1/50 e^(-t/5) t
In[16]:= Solve[c''[t] == 0, t]
Out[16]= {{t → 10}}
In[18]:= {10, c[10]} // N
Out[18]= {10., 0.676676}
```

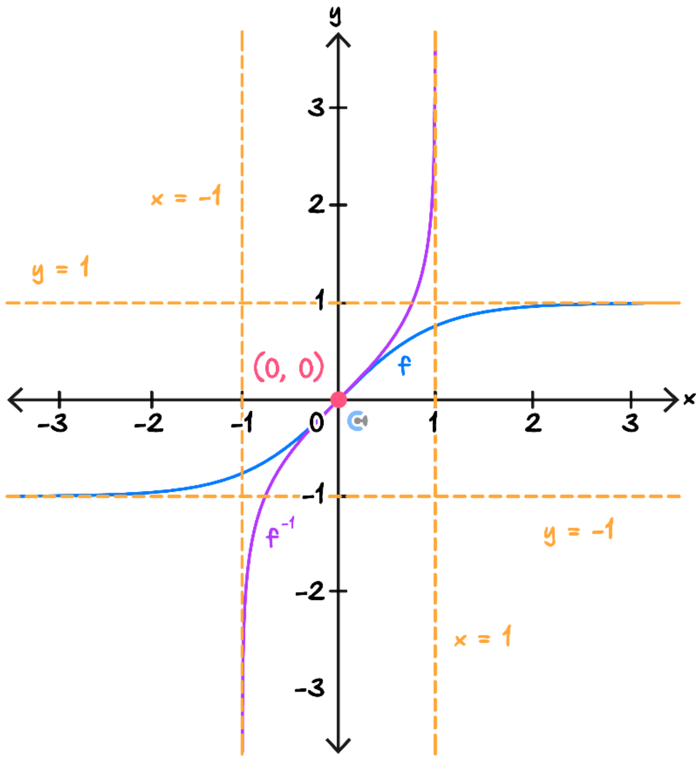
(10.00,0.68)

16 (**c**)	`Solve[f'[120] == 0 && f[120] == 120]` 풀이 함수 Solve: Inconsistent or redundant transcendenta Solve: Inverse functions are being used by Solv $\left\{\left\{a \to e,\ b \to \frac{1}{120}\right\}\right\}$

16 (**d**)	`Solve[f'[t] == 0, t]`
	풀이 함수
	$\left\{\left\{t \to \frac{1}{b}\right\}\right\}$
	`Solve[f[1/b] == 100, a]`
	풀이 함수
	$\{\{a \to 100\,b\,e\}\}$

16 (**e**)	`f[t_] := 100 b e^(1-b t) t`
	`NSolve[f[t] == f[t + 10] && f[t] == 80, {t, b}, Reals]`
	수치 해석 실수 영역
	`{{t → 3.48685, b → 0.135272}, {t → -13.4868, b → -0.135272}}`
	`(* b>0.14: Check using sliders *)`

18 (**a**)	$f^{-1} : (-1, 1) \to \mathbb{R},\ f^{-1}(x) = \log_e\left(\sqrt{\frac{1+x}{1-x}}\right)$ (1M for domain, 1M for rule in correct form) In[140]:= $f[x\_] := \frac{\text{Exp}[2x] - 1}{\text{Exp}[2x] + 1}$ In[141]:= `Solve[f[y] == x, y, Reals]` Out[141]= $\left\{\left\{y \to \frac{1}{2}\text{Log}\left[\frac{-1-x}{-1+x}\right] \text{ if } -1 < x < 1\right\}\right\}$

18 (b)	 (1M asymptotes, 1M shape of f, 1M shape of f^{-1})

18 (c)	Intersect at $(0,0)$. $g_1'(0) = \frac{1}{2} \implies (g^{-1})'(0) = 2$ (1M).
	Therefore angle is $\lvert\arctan(2) - \arctan(1/2)\rvert = 36.9°$ (1M)
	In[142]:= `g[x_] := (Exp[k x] - 1)/(Exp[k x] + 1)`
	In[143]:= `g'[0] /. k → 1`
	Out[143]= $\frac{1}{2}$
	In[144]:= `Abs[ArcTan[2] - ArcTan[1/2]] / Degree // N`
	Out[144]= 36.8699

18 (d)	Always intersect at the origin. Use sliders/our graph from part b. $-2 < k < 2$ (1M) In[153]:= `FindInstance[g[x] == x && g'[x] == 1, {x, k}, Reals]` Out[153]= {{x → 0, k → 2}} In[156]:= `Solve[g'[0] == 1]` Out[156]= {{k → 2}}

<table>
<tr>
<td>18 (e)</td>
<td>Intersect at the origin and $-2 < k < 2$. $g_k'(0) = \frac{k}{2}$
$So\ we\ solve\ \left|\frac{\frac{k}{2} - \frac{2}{k}}{1+1}\right| = \tan(45°)\ and\ -2 < k < 2$
$\Rightarrow k = 2 + 2\sqrt{2}, 2 - 2\sqrt{2}$</td>
</tr>
<tr>
<td>18 (f)</td>
<td>Note that for $k > 2$ intersections occur along the line $y = x$ but for $k < -2$ intersections occur along the line $y = -x$. (1M)
$k = \pm 2\log_e(3)$ (1M)

In[163]:= Solve[g[-1 / 2] == 1 / 2, Reals]
Out[163]= {{k → -2 Log[3]}}

In[164]:= Solve[g[-1 / 2] == -1 / 2, Reals]
Out[164]= {{k → 2 Log[3]}}

In[165]:= Solve[g[y] == x, y, Reals]
Out[165]= $\left\{\left\{y \to \frac{\text{Log}\left[\frac{-1-x}{-1+x}\right]}{k}\ \text{if}\ -1 < x < 1\right\}\right\}$

In[166]:= $\text{g1}[x\_] := \frac{1}{k}\text{Log}\left[\frac{-1-x}{x-1}\right]$

In[168]:= Solve[g[-1 / 2] == g1[-1 / 2], {x, k}, Reals]
Solve: Equations may not give solutions for all "solve" variables.
Out[168]= {{k → -2.20...}, {k → 2.20...}}

In[172]:= Solve[g[x] == g1[x] /. k → 2 Log[3], x, Reals]
Out[172]= $\left\{\left\{x \to -\frac{1}{2}\right\}, \{x \to 0\}, \left\{x \to \frac{1}{2}\right\}\right\}$

In[173]:= Solve[g[x] == g1[x] /. k → -2 Log[3], x, Reals]
Out[173]= $\left\{\left\{x \to -\frac{1}{2}\right\}, \{x \to 0\}, \left\{x \to \frac{1}{2}\right\}\right\}$</td>
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</table>

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