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VCE Mathematical Methods ¾
Family of Functions & its Exam Skills [0.14]
Workshop

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ______________________________ Notes:	Pg / Q #: ______________________________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
Pg / Q #: ______________________________ Notes:	Pg / Q #: ______________________________ Notes:

Section A: Recap

Families of Functions

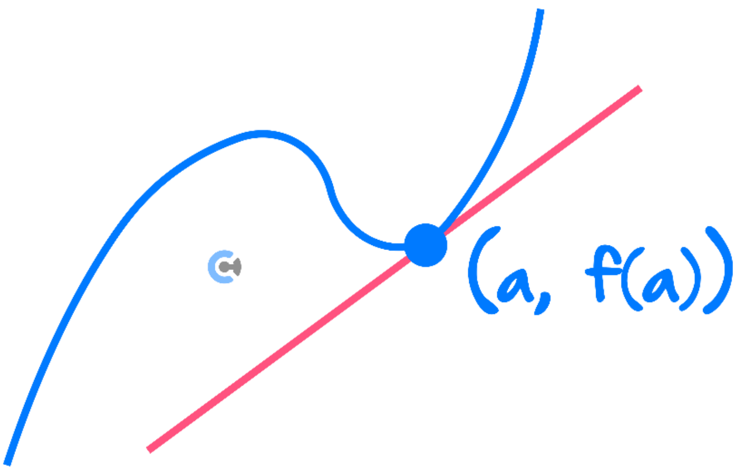
Functions with an unknown.

- They involve understanding and using transformations.
- They involve the use of sliders/manipulate on CAS/technology.

Using Sliders/Manipulate on Technology

1. **Understanding the effect of the unknown on the graph.**
   - As often this is not obvious from transformations.
2. **Checking your answer.**
   - When finding the value(s) of an unknown, check the value smaller and larger than the value obtained to see which side satisfies the condition.

Tangent to a Family of Functions



- For a function to "touch" a line as a tangent:
  - They intersect.

$$f(a) = mx + c$$

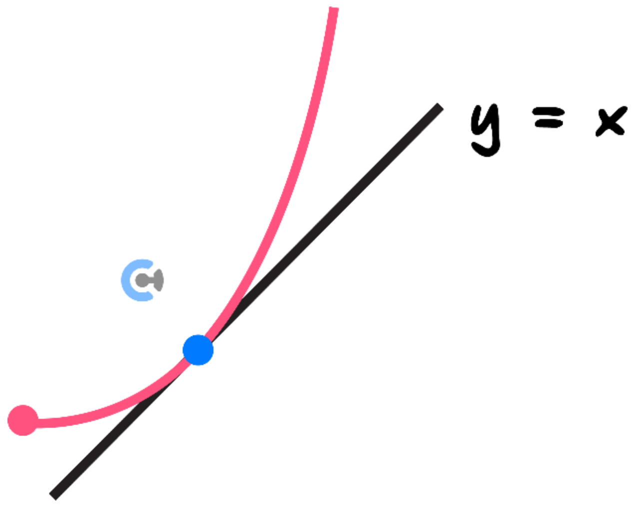
- With the same gradient.

$$f'(a) = m$$

- We just solve these simultaneously.

Definition

Family of Functions and Inverse



- For a function to "touch" $y = x$ as a tangent:
  - They intersect.

$$f(a) = a$$

  - With the same gradient.

$$f'(a) = 1$$

- We just solve these simultaneously.

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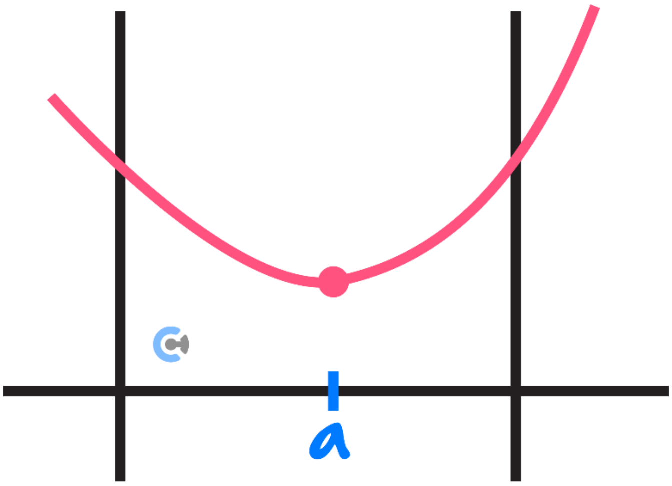
Definition

Minimum/Maximum at a Turning Point



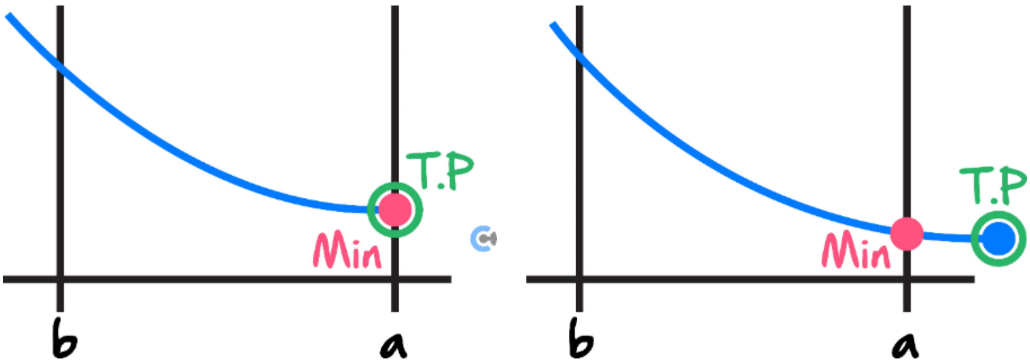
- To achieve minimum/maximum at $x = a$.

$$f'(a) = 0$$

- This is only when $x = a$ is not an end point.

Definition

Minimum/Maximum at an End Point



- Step 1: Find the value of the unknown such that the turning point occurs at $x = a$.

$$f'(a) = 0$$

- Step 2: Find the value of the unknown such that the turning point occurs after/below $x = a$.

  We can determine whether the unknown can be bigger or smaller than the value achieved in step 1.

$$f'(a \pm something) = 0$$

Section B: Warm Up (11 Marks)

INSTRUCTION:

- Regular: 11 Marks. 11 Minutes Writing.
- Extension: Skip

Question 1 (3 marks)

Consider the function:

$$f(x) = e^{2x} + k$$

Find the value of k such that $f(x)$ hits the line $y = 2x$ only once.

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Question 2 (3 marks)

Consider the function $f(x) = \sqrt{x+1} + k$.

Find the value(s) of k such that $f(x)$ and its inverse never intersect.

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Question 3 (3 marks)

Consider the function $f(x) = 2x \log_e(x)$

Find the value of k such that $f(x-k)$ has a local minimum when $x = 10$.

Question 4 (2 marks)

Consider the family of functions $f(x) = (x-k)^2 + 4, x \in [2, 10]$.

Find the value(s) of k such that the minimum value of the function occurs at $x = 2$

Section C: Exam 1 Questions (17 Marks)

INSTRUCTION:

- Regular: 17 Marks. 5 Minutes Reading. 25 Minutes Writing.
- Extension: 17 Marks. 5 Minutes Reading. 17 Minutes Writing.

Question 5 (3 marks)

Consider the function:

$$f(x) = e^{\frac{1}{2}x} + 1$$

Find the value(s) of k such that $f(x-k)$ hits the line $y = \frac{1}{2}x$ twice.

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Question 6 (4 marks)

Consider the function $f: [a, \infty) \to R, f(x) = (x-3)^2 + k$.

a. Find the minimum value of a such that f^{-1} exists. (1 mark)

Let $g: D \to R, g(x) = \log_e(2x + 1)$ where D is the maximal domain of g.

b. Find the value(s) of k such that $g(f(x))$ exists. (1 mark)

c. Find the value(s) of k such that f and f^{-1} has two intersections. (2 marks)

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Question 7 (3 marks)

Let $f: R \to R, f(x) = e^x + k$, where k is a real number. The tangent to the graph of f at the point where $x = a$, has a y-axis intercept in the interval $y \in [0,2]$.

Find the possible values of k in terms of a.

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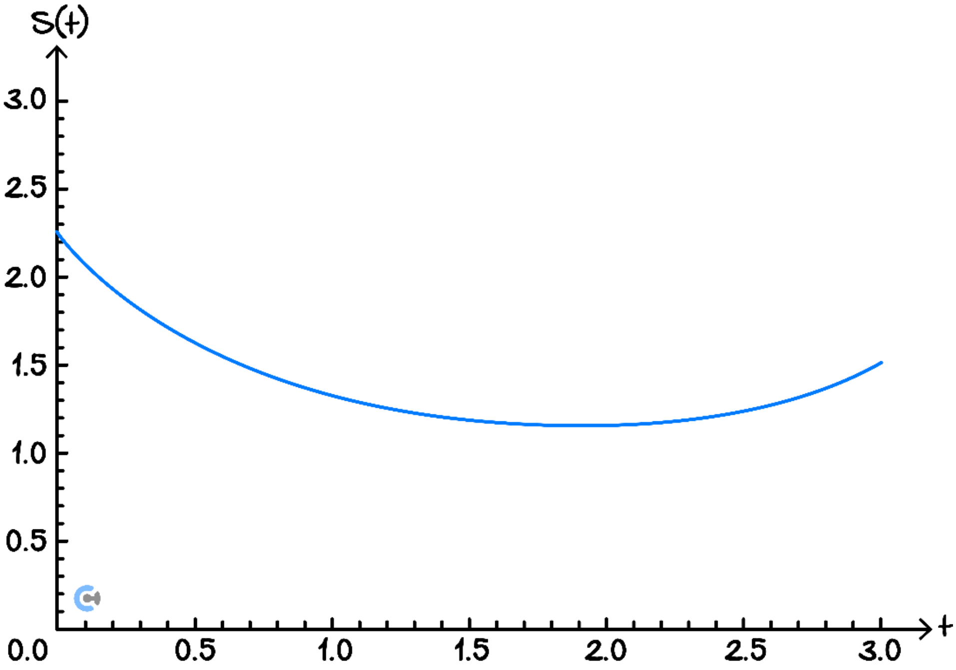
Question 8 (7 marks)

Contour students' stress levels were measured during their MM*34* workshop which lasts for 3 hours.



$$S(t) = \frac{a}{t+1} + \frac{1}{4-t} \text{ where } 0 \le t \le 3, a \in (0, 1000)$$

t is number of hours since the start of the MM*34* workshop.

a. Find the derivative of $S(t)$. (2 marks)

b. For what value(s) of a would students have a minimum stress level at $t = 1$. (2 marks)

c. For what value(s) of a would students have a minimum stress level at the end of the MM34 workshop. (3 marks)

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Section D: Tech Active Exam Skills

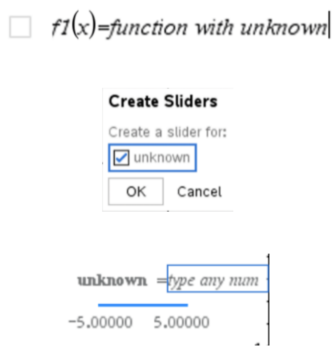
Calculator Commands: Using Sliders/Manipulate on CAS

- **Mathematica**

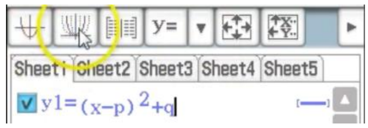
```
Manipulate[Plot[function, {x, xmin, xmax}],
{unknown, lowerbound, upperbound}]
```

- **NOTE:** The function **must** be typed out instead of using its saved name.

- **TI**



- **Casio**



Calculator Commands: Finding k so that f and f^{-1} intersect once

- Step 1: Plot the functions with sliders.
- Step 2: Solve the equations $f(a) = a$ and $f'(a) = 1$ simultaneously on our CAS.
- Step 3: Check that your answer makes sense by using your sliders.
- **Example** Consider the function $f(x) = e^{kx}$, where $k > 0$. Find the exact value of k for which f and f^{-1}, have exactly one point of intersection.
- **Mathematica:**

```
In[29]:= f[x_] := Exp[k x]

In[33]:= Solve[f[x] == x && f'[x] == 1]

Out[33]= {{k → 1/e, x → e}}
```

- TI:

Define $f(x)=e^{k\cdot x}$	*Done*
Define $df(x)=k\cdot e^{k\cdot x}$	*Done*
⚠ solve$(f(x)=x$ and $df(x)=1,k,x)$	$k=0.367879$ and $x=2.71828$
solve$(f(e)=e,k)$	$k=e^{-1}$
solve$(k\cdot x=\ln(x)$ and $df(x)=1,k,x)$	$k=e^{-1}$ and $x=e$

- Casio:

$$\begin{cases}\exp(k*x)=x \\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{e^{k\cdot x}-x=0,\ k\cdot e^{k\cdot x}-1=0\}$$

$$\begin{cases}k*x = \ln(x) \\ k*\exp(k*x)=1\end{cases}\Bigg|_{x,\,k}$$

$$\{x=e,\ k=e^{-1}\}$$

NOTE: Sometimes for tricker equations we will not immediately get a solution for the system of equations or it will not be exact, but often altering the equations with some simple algebra will then allow the CAS to solve it correctly.

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Section E: Exam 2 Questions (27 Marks)

INSTRUCTION:

- Regular: 27 Marks. 5 Minutes Reading. 40 Minutes Writing.
- Extension: 27 Marks. 5 Minutes Reading. 27 Minutes Writing.

Question 9 (1 mark)

The graph of $y = kx - 2$ intersects the graph of $y = x^2 + 6x$ at two distinct points for:

A. $k = 8$

B. $k > 6 + 2\sqrt{2}$ or $k < 6 - 2\sqrt{2}$

C. $2 \leq k \leq 6$

D. $4 - 2\sqrt{3} \leq k \leq 4 + 2\sqrt{3}$

Question 10 (1 mark)

Consider the function $g(x) = e^{x^2 - 4kx + k}$. The value(s) of k for which g has a local minimum when $y = 1$ are:

A. $k = 0$

B. $k = 0, \frac{1}{4}$

C. $k = 0,1$

D. $k = 2$

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Question 11 (1 mark)

Consider the function $f(x) = 2x^3 + x^2 - 7x - 6$. The value(s) of k for which $f(x + k)$ has one positive x-intercept are:

A. $k > 2$

B. $-2 < k \leq 1$

C. $-1 \leq k < 2$

D. $-1 < k < 2$

Question 12 (1 mark)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 3x + 4$. The function $g(x) = f(x) + c$, where $c \in R$, has one x-intercept for:

A. $c > -2$

B. $c < -6$

C. $-6 < c < -2$

D. $c < -6$ or $c > -2$

Question 13 (1 mark)

Consider the function:

$$f: [0,25] \to R, f(x) = \frac{2}{x+1} + \frac{k}{30-x}$$

where $k > 0$.

Find the value(s) of k such that the minimum of $f(x)$ occurs at $x = 25$.

A. $\frac{25}{338}$

B. $\left[\frac{25}{338}, \infty\right)$

C. $\left(-\infty, \frac{25}{338}\right]$

D. $\left(0, \frac{25}{338}\right]$

Question 14 (1 mark)

For $f: [2, \infty) \to f(x) = (x-2)^2 + k$, the value(s) of k such that there is only one intersection between f and f^{-1}.

A. $(-\infty, 2)$

B. $(2, \frac{9}{4}]$

C. $(-\infty, 2] \cup \left\{\frac{9}{4}\right\}$

D. $(-\infty, 2) \cup \left\{\frac{9}{4}\right\}$

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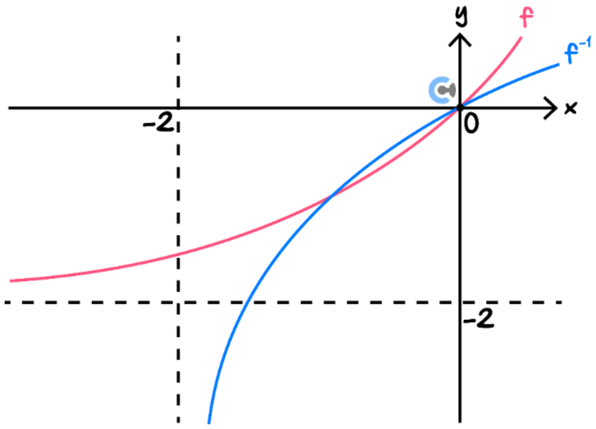
Question 15 (10 marks)

Consider a function $f(x) = 2e^x - 2$.

Part of the graphs of f and f^{-1} are shown below.



a. Find the gradient of f and the gradient of f^{-1} at $x = 0$. (2 marks)

The functions of g_k, where $k \in R^+$, are defined with domain R such that $g_k(x) = 2e^{kx} - 2$.

b. Find the value of k such that $g_k(x) = f(x)$. (1 mark)

c. Find the rule for the inverse functions g_k^{-1} of g_k, where $k \in R^+$. (1 mark)

d. Describe the transformation that maps the graph of g_1 onto the graph of g_k. (1 mark)

e. Describe the transformation that maps the graph of g_1^{-1} onto the graph of g_k^{-1}. (1 mark)

f. The lines L_1 and L_2 are the tangents at the origin to the graphs of g_k and g_k^{-1} respectively. Find the value(s) of k for which the angle between L_1 and L_2 is $30°$. (2 marks)

g. Let p be the value of k for which $g_k(x) = g_k^{-1}(x)$ has only one solution. Find p. (2 marks)

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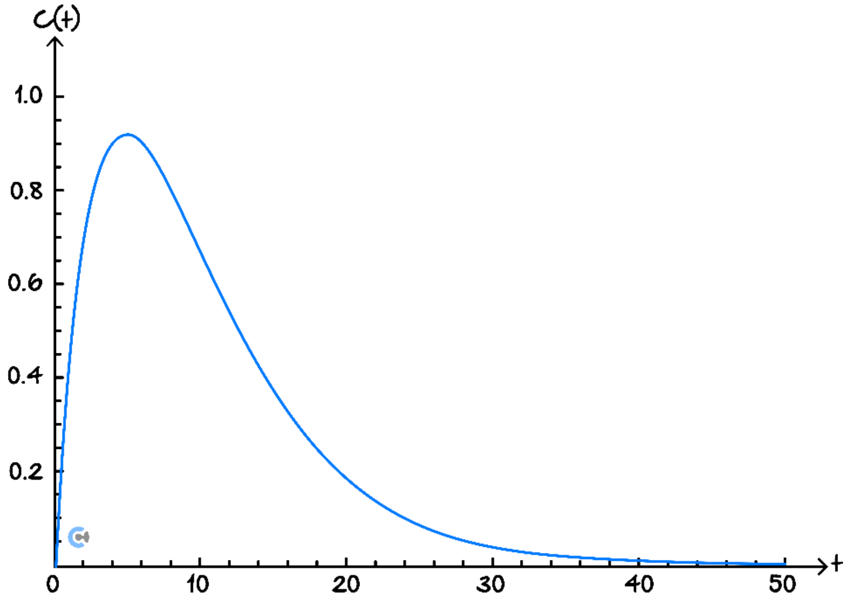
Question 16 (11 marks)

The function $C(t) = 0.5te^{-0.2t}$ is a reasonable model of the measured blood cyanide concentrations in $\mu g/mL$ after t minutes, which is shown in the figure below.



a. Find the value of t for which the concentration is maximum. (2 marks)

b. Find the inflection point of $C(t)$. Give your answer correct to two decimal places. (2 marks)

Another function $C_1(t) = ate^{-bt}$ is used to measure the concentration of another substance in the blood in $\mu g/mL$ after t minutes.

c. Find the values of a and b if the maximum amount of this substance in the blood was 120 $\mu g/mL$ after 2 hours. (2 marks)

It is known that $C_1(t)$ has a maximum value of 100 $\mu g/mL$.

d. Solve for the value of a in terms of b. (2 marks)

e. Solve for the value(s) of b such that $C_1(t)$ is above 80 $\mu g/mL$ for less than 10 minutes.

Give your answer correct to two decimal places. (3 marks)

Section F: Extension Exam 1 (11 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 11 Marks. 2 Minutes Reading. 15 Minutes Writing.

Question 17 (11 marks)

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = e^{-kx} + 2x$, where k is a real number.

a.

i. Find in terms of k, the coordinates for the stationary point of the graph of $y = f(x)$ (when it exists) and specify the range of values of k for which this stationary point exists. (3 marks)

ii. Find the value of k for which the stationary point of f occurs on the x-axis. (1 mark)

iii. Find the value(s) of k such that the x-coordinate of the stationary point of f is a positive number. (1 mark)

b. For a particular value of k, the tangent to f at $x = -4$ passes through the origin. Find this value of k and the equation of this tangent. (3 marks)

Consider the function $g(x) = e^{-\frac{x}{2}} + 2x$.

c. State how many real solutions the equation $g(x) = 0$ has. (1 mark)

It is known that the tangent to $g(x)$ when $x = -4$ has equation $y = \left(\frac{4-e^2}{2}\right)x - e^2$.

d. Newton's method is used to approximate a root of g, with $x_0 = -4$. Find the value of x_1 without performing any differentiation. (2 marks)

Space for Personal Notes

Section G: Extension Exam 2 (13 Marks)

INSTRUCTION:

- Regular: Skip
- Extension: 13 Marks. 3 Minutes Reading. 18 Minutes Writing.

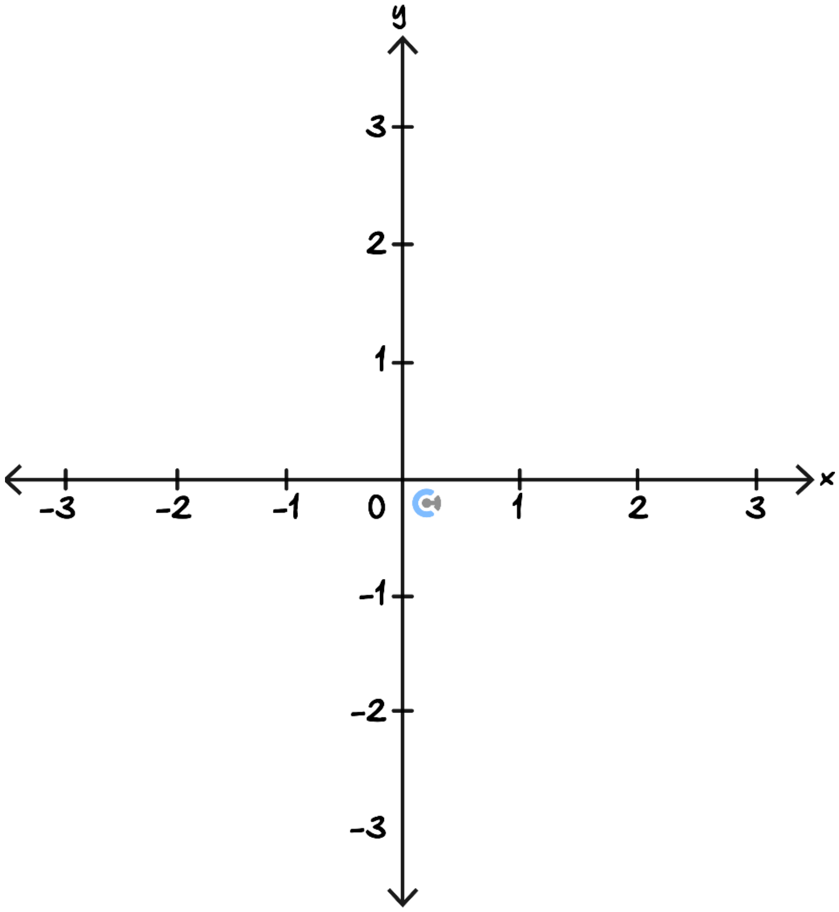
Question 18 (13 marks)

Let f be the hyperbolic tangent function, that is $f(x) = \tanh(x)$.

The function f may be defined, $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{e^{2x}-1}{e^{2x}+1}$.

a. Define the inverse function of f. Write down the rule for $f^{-1}(x)$ in the form $f^{-1}(x) = \log_e\big(g(x)\big)$, for some function g. (2 marks)

b. Sketch the graphs of f and f^{-1} on the axes below. Label all asymptotes with equations and points of intersection with coordinates. (3 marks)



Now consider the family of functions $f : \mathbb{R} \to \mathbb{R},\ g_k(x) = \frac{e^{kx}-1}{e^{kx}+1}$. The function g_1 specifies the function g_k when $k = 1$.

c. Find the acute angle between the tangents to g_1 and g_1^{-1}, at their point of intersection. Give your answer in degrees correct to one decimal place. (2 marks)

d. Find the value(s) of k such that g_k and g_k^{-1} intersect exactly once. (2 marks)

e. Find the value(s) of k such that g_k and g_k^{-1} intersect once and make an acute angle of $45°$ at this point of intersection. (2 marks)

f. Find the value(s) of k such that g_k and g_k^{-1} intersect at a point when $x = -\frac{1}{2}$. (2 marks)

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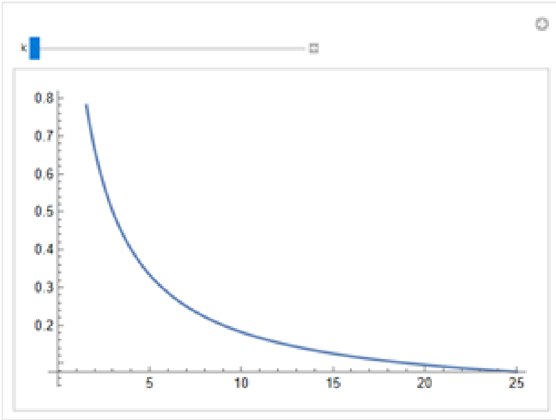
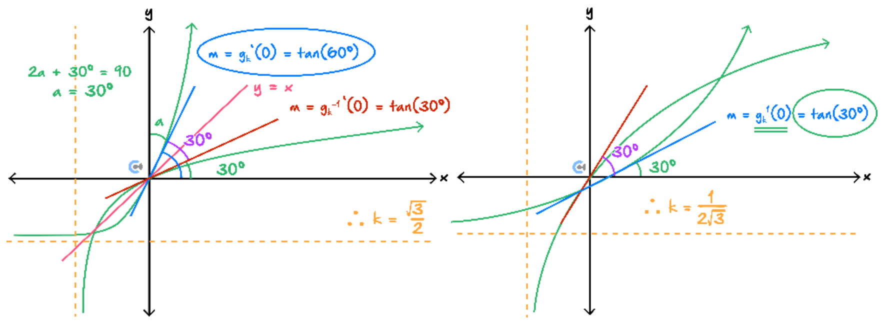
Section H: Tech-Active Solutions

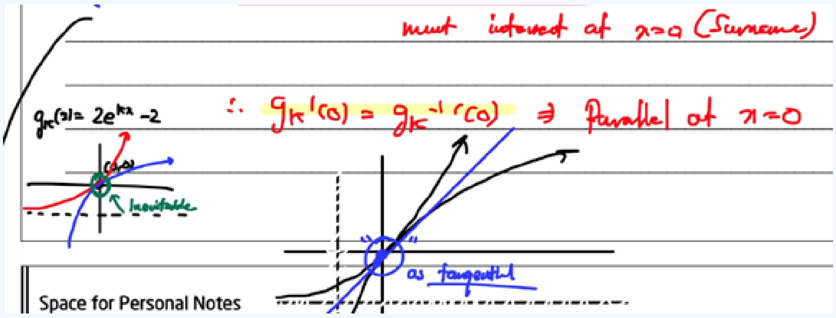
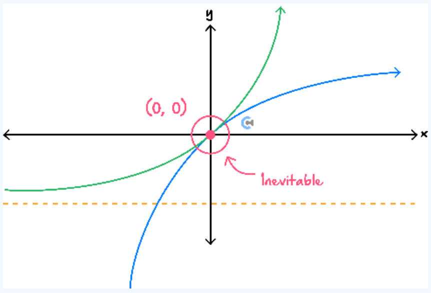
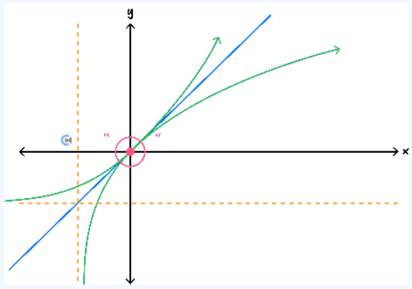
Question Number	Solutions
9	$k > 6 + 2\sqrt{2}$ or $k < 6 - 2\sqrt{2}$ In[27]:= `Solve[k x - 2 == x^2 + 6 x, x]` Out[27]= $\left\{\left\{x \to \frac{1}{2}\left(-6 + k - \sqrt{28 - 12k + k^2}\right)\right\}, \left\{x \to \frac{1}{2}\left(-6 + k + \sqrt{28 - 12k + k^2}\right)\right\}\right\}$ In[29]:= `Solve[28 - 12 k + k^2 == 0, k]` Out[29]= $\{\{k \to 2(3 - \sqrt{2})\}, \{k \to 2(3 + \sqrt{2})\}\}$

10	$k = 0, \frac{1}{4}$
	In[40]:= `f[x_] := Exp[x^2 - 4 k x + k]`
	In[41]:= `Solve[f'[x] == 0, x] // Quiet`
	Out[41]= $\{\{x \to 2k\}\}$
	In[42]:= `Solve[f[2 k] == 1, k, Reals]`
	Out[42]= $\left\{\{k \to 0\}, \left\{k \to \frac{1}{4}\right\}\right\}$

11	$-1 \le k < 2$ In[45]:= `f[x_] := -6 - 7 x + x^2 + 2 x^3` In[46]:= `Solve[f[x] == 0, x]` Out[46]= $\left\{\left\{x \to -\frac{3}{2}\right\}, \{x \to -1\}, \{x \to 2\}\right\}$
12	$c < -6$ or $c > -2$ In[50]:= `f[x_] := x^3 - 3 x + 4` In[51]:= `Solve[f'[x] == 0, x]` Out[51]= $\{\{x \to -1\}, \{x \to 1\}\}$ In[53]:= `f[-1]` Out[53]= 6 In[54]:= `f[1]` Out[54]= 2

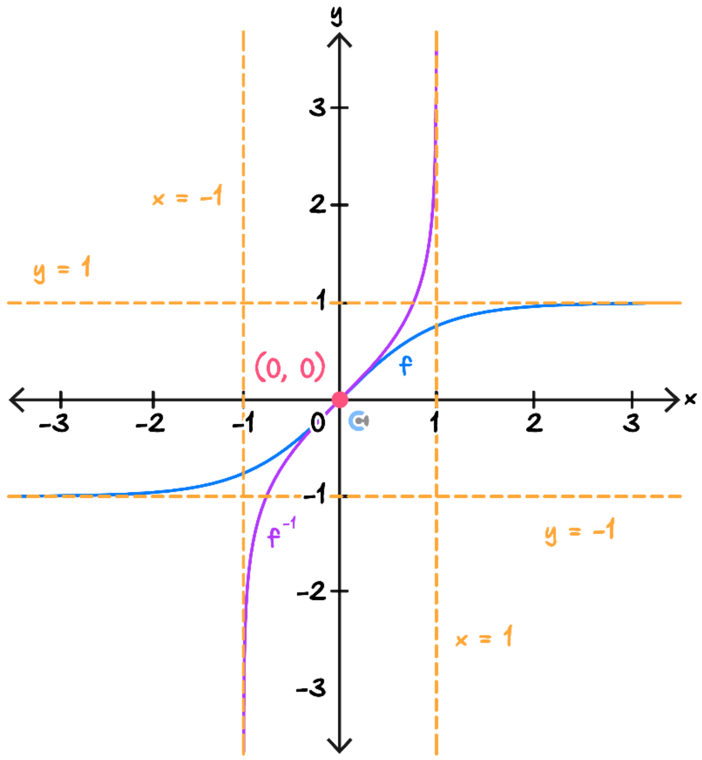
13	$(0, \frac{25}{338}]$ In[1]:= `f[x_] := 2/(x + 1) + k/(30 - x)` In[2]:= `Manipulate[Plot[2/(x + 1) + k/(30 - x), {x, 0, 25}], {k, 0, 10}]` Out[2]=  In[7]:= `Solve[f'[25] == 0, k]` Out[7]= $\{\{k \to \frac{25}{338}\}\}$
14	$(-\infty, 2) \cup \left\{\frac{9}{4}\right\}$ In[8]:= `f[x_] := (x - 2)^2 + k` In[10]:= `Solve[f[x] == x, x]` Out[10]= $\{\{x \to \frac{1}{2}(5 - \sqrt{9 - 4k})\}, \{x \to \frac{1}{2}(5 + \sqrt{9 - 4k})\}\}$ In[11]:= `Solve[9 - 4 k == 0, k]` Out[11]= $\{\{k \to \frac{9}{4}\}\}$
15 (a)	In[19]:= `f[x_] := 2 Exp[x] - 2` In[20]:= `Solve[f[y] == x, y, Reals]` Out[20]= $\{\{y \to \text{Log}[\frac{2 + x}{2}] \text{ if } x > -2\}\}$ In[21]:= `f1[x_] := Log[(2 + x)/2]` In[22]:= `f'[0]` Out[22]= 2 In[23]:= `f1'[0]` Out[23]= $\frac{1}{2}$

15 (b)	In[24]:= `g[x_] := 2 Exp[k x] - 2` In[26]:= `Solve[f[x] == g[x], k, Reals]` Out[26]= $\{\{k \to 1\}\}$
15 (c)	In[27]:= `Solve[g[y] == x, y, Reals]` Out[27]= $\left\{\left\{y \to \boxed{\frac{\text{Log}\left[\frac{2+x}{2}\right]}{k} \text{ if } x > -2}\right\}\right\}$
15 (d)	Dilation by factor $\frac{1}{k}$ from the y-axis.
15 (e)	Dilation by factor $\frac{1}{k}$ from the x-axis.
15 (f)	 In[32]:= `g'[0]` Out[32]= 2 k In[34]:= $\text{Solve}\left[\text{Abs}\left[\frac{2k - \frac{1}{2k}}{1 + 2k * \frac{1}{2k}}\right] == \text{Tan}[30\,\text{Degree}] \;\&\&\; k > 0,\; k\right]$ Out[34]= $\left\{\left\{k \to \frac{1}{2\sqrt{3}}\right\}, \left\{k \to \frac{\sqrt{3}}{2}\right\}\right\}$

15 (**g**)	   In[41]:= Solve[g[x] == x && g'[x] == 1, Reals] Out[41]= $\left\{\left\{k \to \frac{1}{2}, x \to 0\right\}\right\}$
16 (**a**)	In[12]:= c[t_] := 1 / 2 t Exp[-1 / 5 t] In[14]:= c'[t] Out[14]= $\frac{e^{-t/5}}{2} - \frac{1}{10} e^{-t/5} t$ In[13]:= Solve[c'[t] == 0, t] Out[13]= {{t → 5}}

16 (**b**)	In[15]:= c''[t]
	Out[15]= $-\frac{1}{5} e^{-t/5} + \frac{1}{50} e^{-t/5} t$
	In[16]:= Solve[c''[t] == 0 , t]
	Out[16]= {{t → 10}}
	In[18]:= {10, c[10]} // N
	Out[18]= {10., 0.676676}
	(10.00,0.68)

16 (c)	= Solve[f'[120] == 0 && f[120] == 120] 풀이 함수 … Solve: Inconsistent or redundant transcendenta … Solve: Inverse functions are being used by Solv = $\{\{a \to e, b \to \frac{1}{120}\}\}$
16 (d)	Solve[f'[t] == 0, t] 풀이 함수 $\{\{t \to \frac{1}{b}\}\}$ Solve[f[1/b] == 100, a] 풀이 함수 {{a → 100 b e}}
16 (e)	f[t_] := 100 b $e^{1-b\,t}$ t NSolve[f[t] == f[t + 10] && f[t] == 80, {t, b}, Reals] 수치 해석 실수 영역 {{t → 3.48685, b → 0.135272}, {t → -13.4868, b → -0.135272}} (* b>0.14: Check using sliders *)
18 (a)	$f^{-1}:(-1,1)\to\mathbb{R},\ f^{-1}(x)=\log_e\left(\sqrt{\frac{1+x}{1-x}}\right)$ (1M for domain, 1M for rule in correct form) In[140]:= f[x_] := $\frac{\text{Exp}[2x]-1}{\text{Exp}[2x]+1}$ In[141]:= Solve[f[y] == x, y, Reals] Out[141]= $\{\{y \to \frac{1}{2}\text{Log}\left[\frac{-1-x}{-1+x}\right] \text{ if } -1 < x < 1\}\}$

18 (b)	 (1M asymptotes, 1M shape of f, 1M shape of f^{-1})

18 (c)	Intersect at $(0,0)$. $g_1'(0) = \frac{1}{2} \implies (g^{-1})'(0) = 2$ (1M).
	Therefore angle is $\lvert\arctan(2) - \arctan(1/2)\rvert = 36.9°$ (1M)
	In[142]:= `g[x_] := (Exp[k x] - 1)/(Exp[k x] + 1)`
	In[143]:= `g'[0] /. k → 1`
	Out[143]= 1/2
	In[144]:= `Abs[ArcTan[2] - ArcTan[1/2]] / Degree // N`
	Out[144]= 36.8699

18 (d)	Always intersect at the origin. Use sliders/our graph from part b. $-2 < k < 2$ (1M) In[153]:= `FindInstance[g[x] == x && g'[x] == 1, {x, k}, Reals]` Out[153]= `{{x → 0, k → 2}}` In[156]:= `Solve[g'[0] == 1]` Out[156]= `{{k → 2}}`

18 (**e**)	Intersect at the origin and $-2 < k < 2. g_k'(0) = \frac{k}{2}$ $So\ we\ solve\ \left\lvert \frac{\frac{k}{2} - \frac{2}{k}}{1+1} \right\rvert = \tan(45°)\ and -2 < k < 2$ $\Rightarrow k = 2 + 2\sqrt{2}, 2 - 2\sqrt{2}$

18 (**f**)	Note that for $k > 2$ intersections occur along the line $y = x$ but for $k < -2$ intersections occur along the line $y = -x$. (1M)
	$k = \pm 2\log_e(3)$ (1M)
	`In[163]:= Solve[g[-1/2] == 1/2, Reals]`
	`Out[163]= {{k → -2 Log[3]}}`
	`In[164]:= Solve[g[-1/2] == -1/2, Reals]`
	`Out[164]= {{k → 2 Log[3]}}`
	`In[165]:= Solve[g[y] == x, y, Reals]`
	`Out[165]= {{y → Log[(-1-x)/(-1+x)]/k if -1 < x < 1}}`
	`In[166]:= g1[x_] := 1/k Log[(-1 - x)/(x - 1)]`
	`In[168]:= Solve[g[-1/2] == g1[-1/2], {x, k}, Reals]`
	`Solve: Equations may not give solutions for all "solve" variables.`
	`Out[168]= {{k → -2.20...}, {k → 2.20...}}`
	`In[172]:= Solve[g[x] == g1[x] /. k → 2 Log[3], x, Reals]`
	`Out[172]= {{x → -1/2}, {x → 0}, {x → 1/2}}`
	`In[173]:= Solve[g[x] == g1[x] /. k → -2 Log[3], x, Reals]`
	`Out[173]= {{x → -1/2}, {x → 0}, {x → 1/2}}`

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