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VCE Mathematical Methods ¾
Application of Differentiation Exam Skills [0.13]
Workshop

Error Logbook:

New Ideas/Concepts	Didn't Read Question
Pg / Q #: ____________________ Notes:	Pg / Q #: ____________________ Notes:
Algebraic/Arithmetic/ Calculator Input Mistake	**Working Out Not Detailed Enough**
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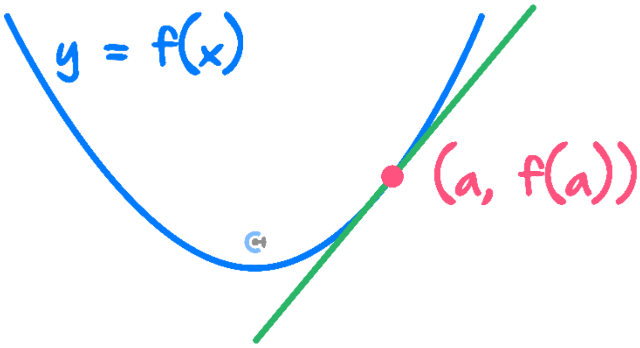
Section A: Recap

Tangents

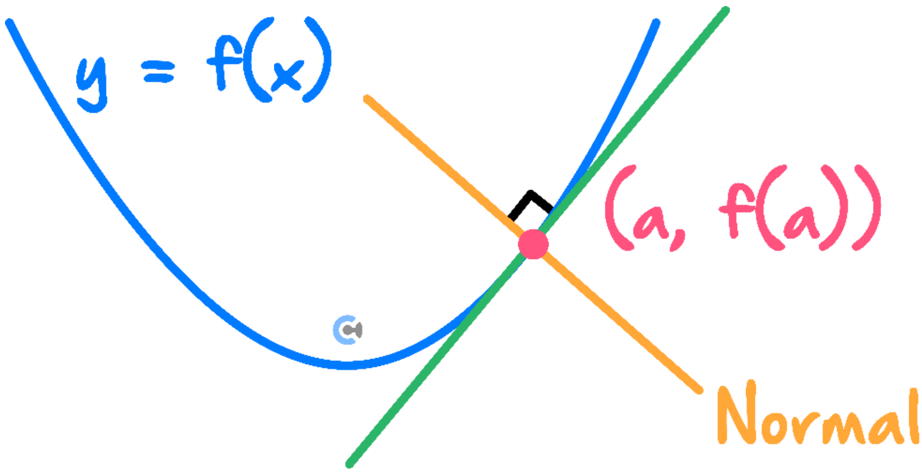
- A **tangent** is a linear line which **just touches** the curve.
- The gradient of a tangent line has to be equal to the gradient of the curve at the intersection.



$$m_{tangent} = f'(a)$$

Normals

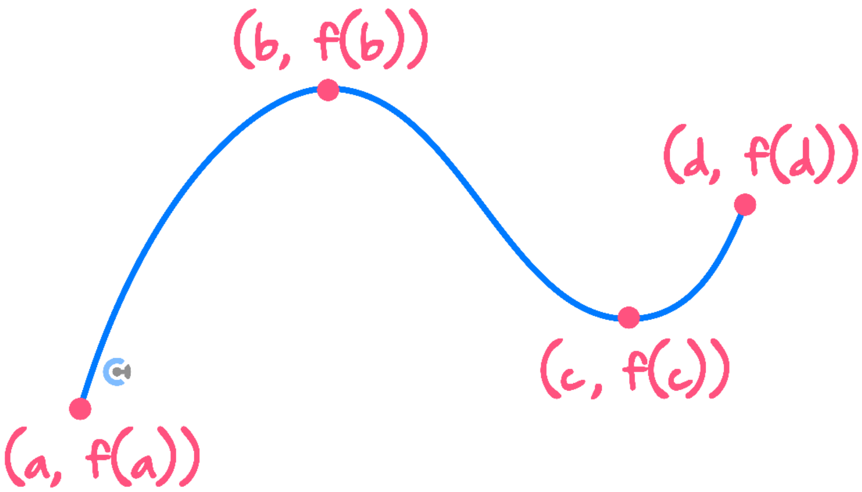
- A **normal** is a linear line which is **perpendicular** to the tangent.
- The gradient of a normal line has to be equal to the **negative reciprocal** of the gradient of the curve at the intersection.



$$m_{normal} = -\frac{1}{f'(a)}$$

Absolute Maximum and Minimum

- **Absolute Maxima/Minima** are the overall **largest/smallest** y-values for the given domain.
- **They occur at either an endpoint** or a **turning point.**



Absolute Min: $f(a)$

Absolute Max: $f(b)$

- **Steps:**

1. Find stationary points and endpoints.
2. Find the largest/lowest y-value for max/min.

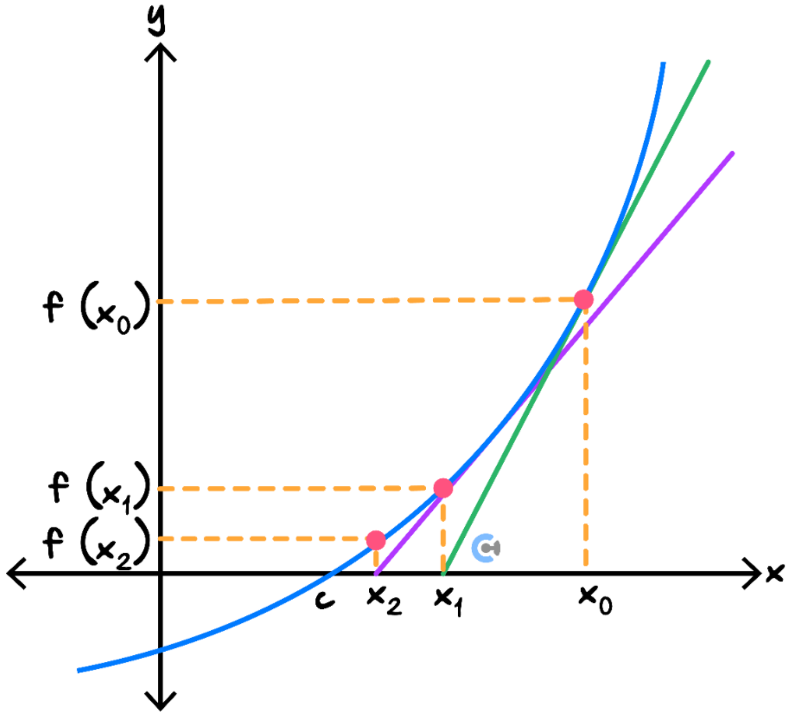
Optimisation Problems

- Applying absolute maxima and minima in a real-world setting.
- **Steps:**

1. Construct a function for the subject you want to find the maximum or minimum of.
2. Find its domain if appropriate.
3. Find its endpoints and turning points.
4. Identify the maximum or minimum y-value.

Newton's Method

- It is a method of approximating the x-intercept using **tangents.**



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Steps:**

1. Find the tangent at the x-value given.
2. Find the x-intercept of the tangent using an iterative formula.
3. Find the next tangent at the $x = x$-intercept of the previous tangent.
4. Repeat until the value doesn't change by much.

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Definition

Tolerance

- **Definition**: The maximum difference between x_n and x_{n+1} we can have when we stop the iteration.

$$\textit{We stop when } |x_{n+1} - x_n| < \textit{Tolerance}.$$

- The question will give us the tolerance level.

Definition

Limitation of Newton's Method

- **Terminating Sequence**: Occurs when we hit a stationary point.
- **Approximating a Wrong Root**: Occurs when we start on the wrong side.
- **Oscillating Sequence**: Occurs when we oscillate between two values without getting closer to the real root.

Definition

Newton's Method: **Finding x_0 values which result in an oscillating sequence.**

- **Step 1:** Define $n(x) = x - \frac{f(x)}{f'(x)}$.
- **Step 2:** Solve $n(n(x)) = x$.
- **Step 3:** Reject the roots of the function.

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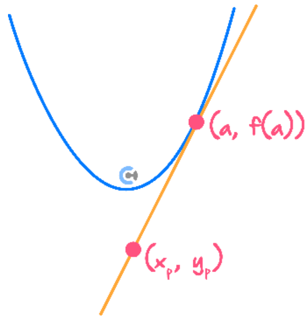
Definition

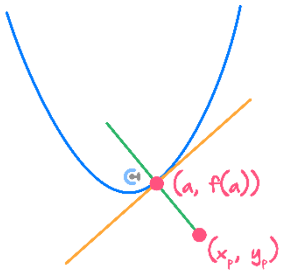
Finding Tangents/Normals to Functions, which also pass through a given point

- The tangent of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$f'(a) = \frac{f(a) - y_p}{a - x_p}$$

- Normal of $f(x)$ at $x = a$ passes through (x_p, y_p).



$$-\frac{1}{f'(a)} = \frac{f(a) - y_p}{a - x_p}$$

Definition

Finding Tangents/Normals with Coordinate Geometry

1. Using derivative = gradient of the tangent, find the point where the tangent is made.

2. Find the tangent.

Definition

Finding Maximum/Minimum Instantaneous Rate of Change

- Find the turning point of the derivative function.

Section B: Warm Up (8 Marks)

INSTRUCTION:

- Regular: 8 Marks. 8 Minutes Writing.
- Extension: Skip

Question 1 (8 marks)

a. Consider the function $f(x) = x^2 - 3x + 4$. Find the equation of the tangent to f, with a negative gradient that passes through the point $(3,0)$. (3 marks)

b. Find the equation of the tangent to $f(x) = x^2 - 4x + 3$ that makes an angle of $45°$ with the positive x-axis. (3 marks)

c. Find the maximum instantaneous rate of change of $f(x) = -x^3 + 6x^2 + 4x - 2$. (2 marks)

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Section C: Exam 1 Questions (19 Marks)

INSTRUCTION:

- Regular: 19 Marks. 5 Minutes Reading. 26 Minutes Writing.
- Extension: 19 Marks. 5 Minutes Reading. 19 Minutes Writing.

Question 2 (4 marks)

The function $f(x) = 14x^2 - ax^4$, where $a > 0$, has a stationary point when $x = -\frac{\sqrt{7}}{2}$.

a. Show that $a = 4$. (2 marks)

b. Hence, find the coordinates for all stationary points of f. (2 marks)

Question 3 (5 marks)

Consider the function $f(x) = x^3 - 3x^2 + 5x$.

a. Find the value(s) of x where $f(x)$ is concave down. (2 marks)

b. Hence, state the point of inflection of $f(x)$. (1 mark)

c. Show that there are two lines tangent to $f(x)$ which pass through the origin $(0,0)$. (2 marks)

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Question 4 (4 marks)

Consider the function $f(x) = x^3 + 16$.

a. Find the equation of the line tangent to f that passes through the origin. (3 marks)

Let $g(x) = kx$, where $k \in \mathbb{R}$..

b. Hence, find the value(s) of k such that, $g(x)$ and $f(x)$ have three points of intersection. (1 mark)

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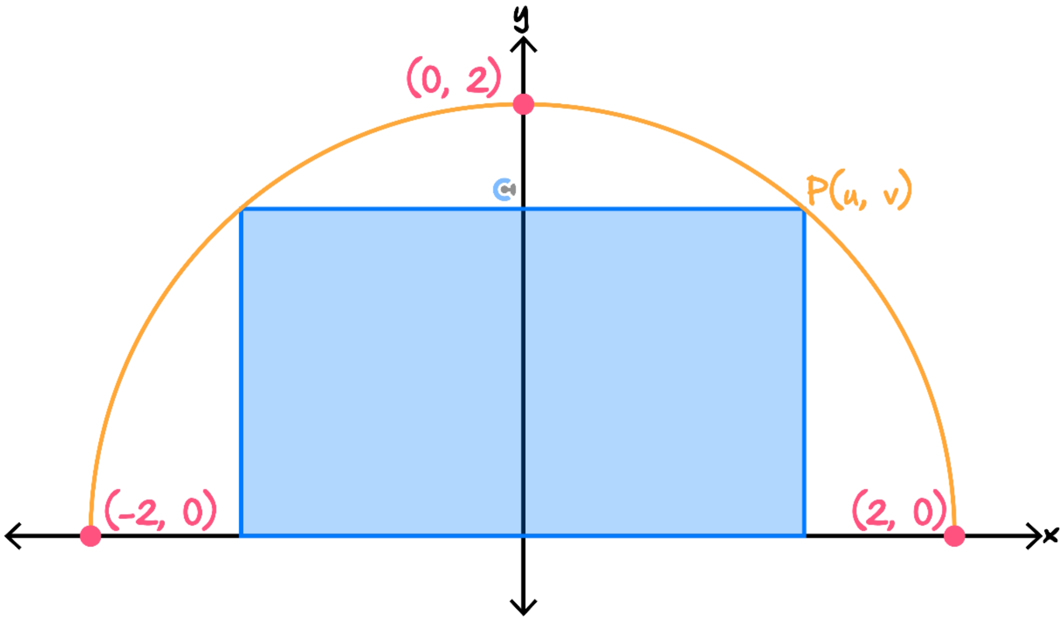
Question 5 (6 marks)

a. The tangent to the curve $y = \sqrt{a - x^2}$, where $a > 0$, at the point $x = \frac{3}{2}$ makes an angle of $150°$ with the positive x-axis. Determine the value of a. (3 marks)

b. A rectangle has two vertices on the graph of $y = \sqrt{4 - x^2}$, one at the point $P(u, v)$, where $u > 0$, and two on the x-axis, as shown in the diagram below.



i. Show that the area of the rectangle may be given by the expression $A = 2u\sqrt{4 - u^2}$. (1 mark)

ii. Hence, find the maximum area of the rectangle. (2 marks)

Section D: Tech Active Exam Skills

Calculator Commands: Finding Derivatives

Mathematica

```
f'[x]
```

```
D[f[x], x]
```

TI

- Shift Minus

$$\frac{d}{dx}(f(x))$$

Casio

- Math 2

$$\frac{d}{dx}(f(x))$$

Calculator Commands: Finding Second Derivatives

Mathematica

```
f''[x]
```

```
D[f[x], {x, 2}]
```

TI

- Shift Minus

$$\frac{d^2}{dx^2}(f(x))$$

Casio

- Math 2

$$\frac{d^2}{dx^2}(f(x))$$

Calculator Commands: Finding tangents on CAS

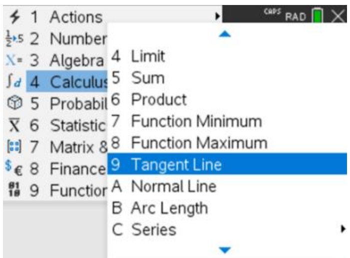
Mathematica

```
<< SuiteTools`

TangentLine[f[x], x, a]
```

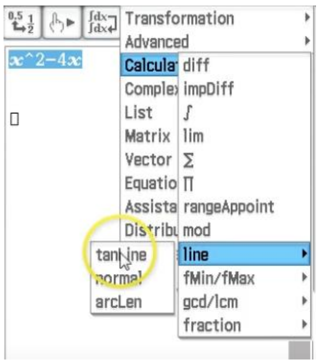
TI-Nspire

- Menu 4 9



tangentLine$(f(x),x,a)$

Casio Classpad



tangentLine$(f(x),x,a)$

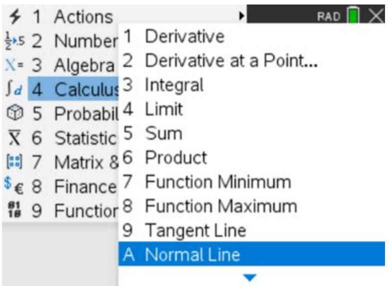
Calculator Commands: Finding normals on CAS

Mathematica

```
<< SuiteTools`
NormalLine[f[x], x, a]
```

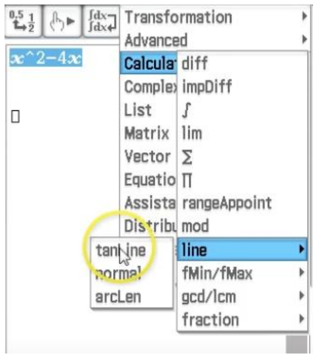
TI-Nspire

Menu 4 A



normalLine(f(x),x,a)

Casio Classpad



normalLine(f(x),x,a)

Calculator Commands: Finding Absolute max and min for $x \in [a, b]$

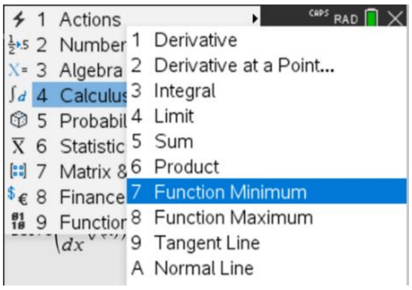
Mathematica

```
Maximize[{f[x], a ≤ x ≤ b}, x]

Minimize[{f[x], a ≤ x ≤ b}, x]
```

TI-Nspire

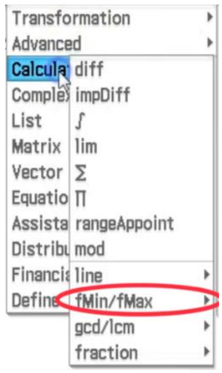
Menu 4 7 and Menu 4 8



fMax(f(x),x,a,b)

fMin(f(x),x,a,b)

Casio Classpad



fMax(f(x),x,a,b)

fMin(f(x),x,a,b)

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Calculator Commands: Newton's Method on Technology

- Consider finding a root to $f(x) = x^3 - 2$ with initial value $x_0 = 1$.

- **Mathematica**

```
In[531]:= f[x_] := x^3 - 2

In[533]:= n[x_] := x - f[x]/f'[x]

In[534]:= n[x] // Together
```

$$\text{Out[534]=} \ \frac{2\left(1 + x^3\right)}{3x^2}$$

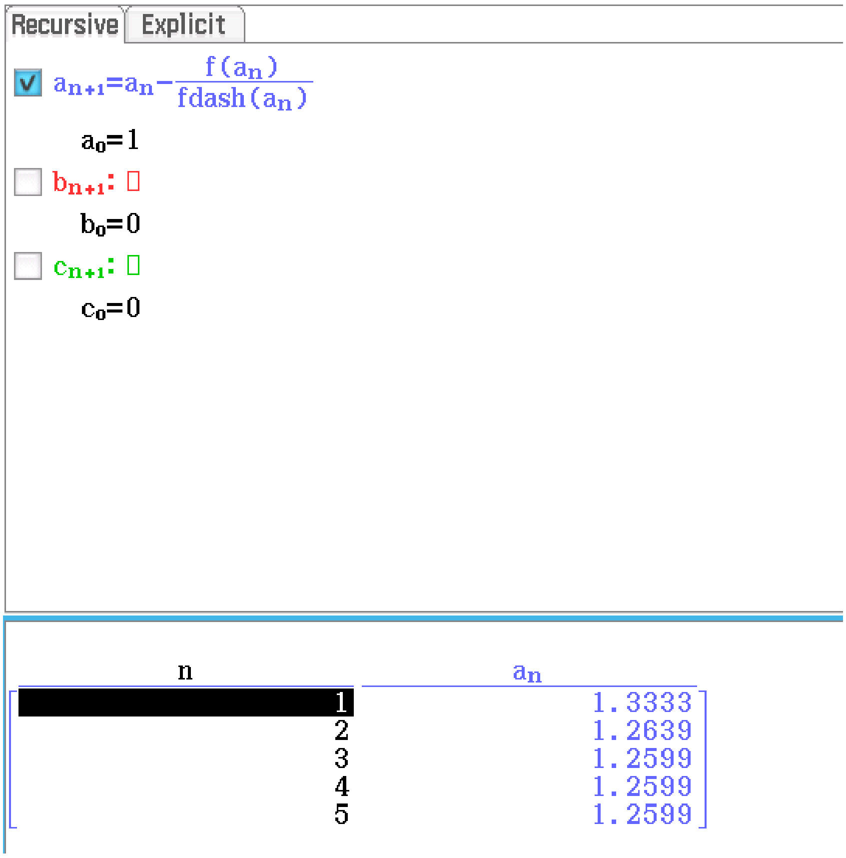
$$\text{In[537]:= For}\left[i = 1;\ x = 1,\ i < 5,\ i{+}{+},\ x = \frac{2\left(1.0 + x^3\right)}{3x^2};\ \text{Print}[x]\right]$$

```
1.33333
1.26389
1.25993
1.25992
```

- **TI.** Define the $n(x)$ function then keep iterating by putting your previous value back into $n(x)$.

Define $f(x)=x^3-2$	*Done*
$x-\dfrac{f(x)}{\frac{d}{dx}(f(x))}$	$\dfrac{2\cdot\left(x^3+1\right)}{3\cdot x^2}$
Define $n(x)=\dfrac{2\cdot\left(x^3+1\right)}{3\cdot x^2}$	*Done*
$n(1)$	1.33333
$n(1.3333333333333)$	1.26389
$n(1.2638888888889)$	1.25993

- **Classpad**
  - Under Sequences.



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Calculator Commands: Stationary Point

- ALWAYS sketch the graph first to get an idea of the nature of the stationary point.
- The turning points for a function $f(x)$ can be found by solving $f'(x) = 0$ and subbing the result into f.
- **Example:** Find the turning point for $f(x) = e^{-x^2+2x}$.
- **TI:**

Define $f(x)=e^{-x^2+2\cdot x}$	*Done*
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$x=1$
$f(1)$	e

- **Casio:**

define $f(x) = e^{-x^2+2x}$	done
$\text{solve}\left(\frac{d}{dx}(f(x))=0,x\right)$	$\{x=1\}$
$f(1)$	e

- **Mathematica:**

```
In[4]:= f[x_] := Exp[-x^2 + 2 x]
In[5]:= Solve[f'[x] == 0 && y == f[x], Reals]
Out[5]= {{x → 1, y → e}}
```

Calculator Commands: Finding tangents/normals which pass through a point.

- Suppose we want to find the equation of a tangent/normal to the graph of $f(x)$ that passes through the point $P(x_1, y_1)$.
- Steps:
  1. Find the equation of the tangent to $f(x)$ at arbitrary point $x = a$.
  2. Let this tangent line be $t(x)$.
  3. Solve the equation $t(x_1) = y_1$ to find possible value(s) of a.
  4. Find the equation of the tangent at $x = a$.
- A similar procedure for the normal line.
- **Example:** Find the equation of a tangent to $f(x) = x^3 - 2x$ that passes through the point $(0,2)$.

```
In[564]:= f[x_] := x^3 - 2 x

In[565]:= TangentLine[f[x], {x, a}]
Out[565]= -2 a^3 + (-2 + 3 a^2) x

In[566]:= t[x_] := -2 a^3 + (-2 + 3 a^2) x

In[568]:= Solve[t[0] == 2, a, Reals]
Out[568]= {{a → -1}}

In[570]:= t[x] /. a → -1
Out[570]= 2 + x

In[571]:= TangentLine[f[x], {x, -1}]
Out[571]= 2 + x
```

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Section E: Exam 2 Questions (22 Marks)

INSTRUCTION:

- Regular: 22 Marks. 5 Minutes Reading. 33 Minutes Writing.
- Extension: 22 Marks. 5 Minutes Reading. 22 Minutes Writing.

Question 6 (1 mark)

The equation of a tangent to $y = -x^2 + 3x + 4$ that makes an angle of $135°$ with the positive x-axis is:

A. $y = x + 8$

B. $y = -x + 8$

C. $y = -x + 6$

D. $y = x + 6$

Question 7 (1 mark)

The tangent to the graph of $y = x^2 - ax^2 + 5$ at $x = 1$ passes through the point $(1, 0)$. The value of a is:

A. 1

B. 2

C. 3

D. 4

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Question 8 (1 mark)

The maximum instantaneous rate of change of the function $f(x) = xe^{-x^2}$ is:

A. e

B. $\frac{1}{e}$

C. 1

D. $\frac{1}{2}$

Question 9 (1 mark)

The tangent to the graph of $y = -\log_e(ex)$ at the point $(a, -\log_e(ea))$ crosses the x-axis at the point $(b, 0)$, where $b > 0$. Which of the following statements is **false**?

A. The graph of any tangent to y has a negative y-intercept for all x within the domain.

B. $0 < a < 1$.

C. The gradient of all tangent lines is negative.

D. y is a strictly decreasing function.

Question 10 (1 mark)

Let $f(x) = -\log_e(x + e)$. A tangent to the graph of f has an x-intercept at $(-c, 0)$. The minimum value of c is:

A. $\sqrt{e} - 1$

B. $1 - e$

C. $e - 1$

D. $1 - \sqrt{e}$

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Question 11 (1 mark)

Consider the function $y = x(x-4)(x+4)$.

For what x_0 values, would Newton's method have an oscillating sequence?

A. $\frac{4\sqrt{5}}{5}$ or $-\frac{4\sqrt{5}}{5}$.

B. $\frac{\sqrt{5}}{5}$ or $-\frac{\sqrt{5}}{5}$.

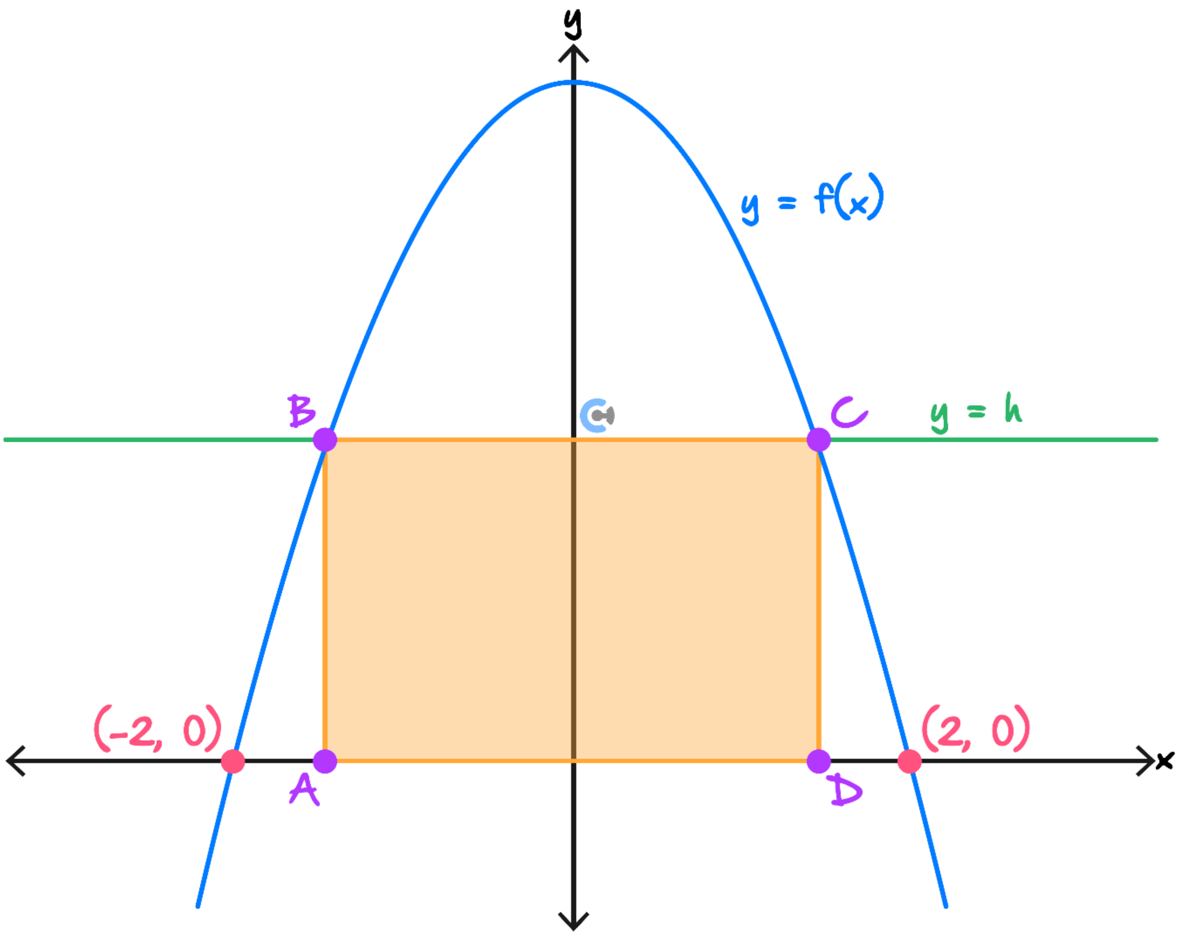
C. 4 or -4.

D. 2 or -2.

Question 12 (16 marks)

Let $f: \mathbb{R} \to \mathbb{R}, f(x) = (2-x)(x+2)$.

The line $y = h$ intersects f at the points B and C as shown in the graph below.



a.

i. Show that the coordinates of point B may be given as $\left(-\sqrt{4-h}, h\right)$. (2 marks)

ii. Hence, state the length of the line segment BC. (1 mark)

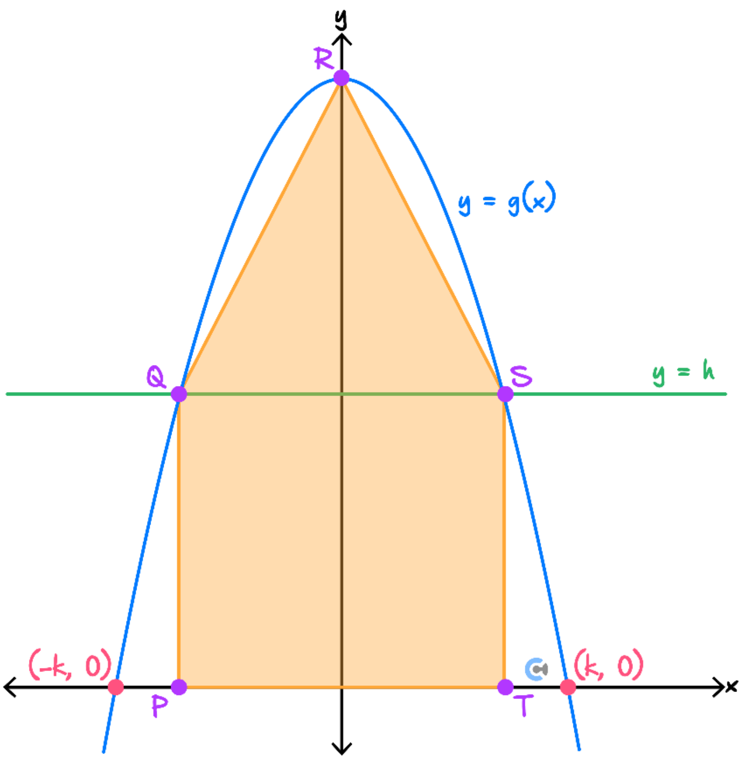
b. Show that the area of rectangle $ABCD$ is equal to $2h\sqrt{4-h}$. (1 mark)

c. Hence, find the maximum area of the rectangle $ABCD$. (3 marks)

Let $g: \mathbb{R} \to \mathbb{R},\ g(x) = (k - x)(x + k)$, where $k > 0$.

The line $y = h$, where $h > 0$, intersects g at the points Q and S as shown in the diagram below.

The point R is on the turning point of g and the points P and T lie directly below Q and S respectively.



d. Find the length QS in terms of h and k. (1 mark)

e. Find the area of the triangle QRS in terms of h and k. (1 mark)

f. Hence, find the maximum area of the polygon $PQRST$ in terms of k. (4 marks)

g. Determine the exact value of k for which the maximum area of polygon $PQRST$ occurs when the triangle QRS is equilateral. (3 marks)

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Section F: Extension Exam 1 (9 Marks)

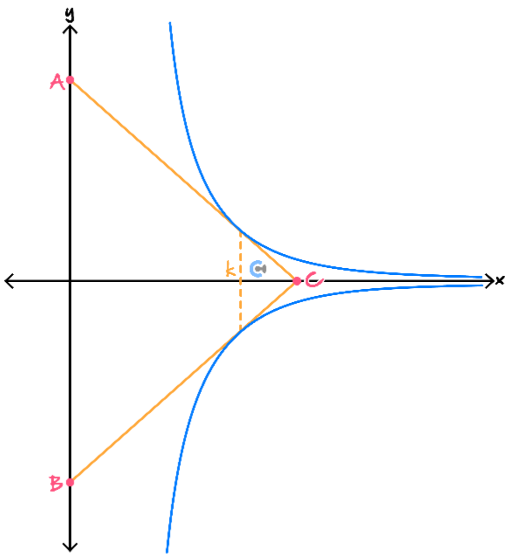
INSTRUCTION:

- Regular: Skip
- Extension: 9 Marks. 2 Minutes Reading. 13 Minutes Writing.

Question 13 (9 marks)

Let $k \in [1,3]$ be a constant. Consider the functions $f: (0, \infty) \to \mathbb{R}, f(x) = \frac{k}{x^3}$ and $g(x) = -\frac{k}{x^3}$.

A tangent is drawn to both f and g when $x = k$, and a triangle ABC, with a vertex on the x-axis and two vertices on the y-axis is formed, as shown in the diagram below.



a. Find the equation of the tangent to f at the point where $x = k$. (2 marks)

b. Find an expression, in terms of k, for the area of the triangle ABC. (2 marks)

c. Find the maximum and minimum area for the triangle ABC. (2 marks)

d. Find the area of the triangle ABC, if it is an equilateral triangle. (3 marks)

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Section G: Extension Exam 2 (17 Marks)

INSTRUCTION:

- Regular: Skip
- 17 Marks. 5 Minutes Reading. 24 Minutes Writing.

Question 14 (17 marks)

Consider the function $f: D \to \mathbb{R}, f(x) = 4\sqrt{x} - 3$.

a. The function f can be obtained from the graph of $y = \sqrt{x}$ under a transformation:

$$T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + h, y - k)$$

Find the values of a, h, and k. (2 marks)

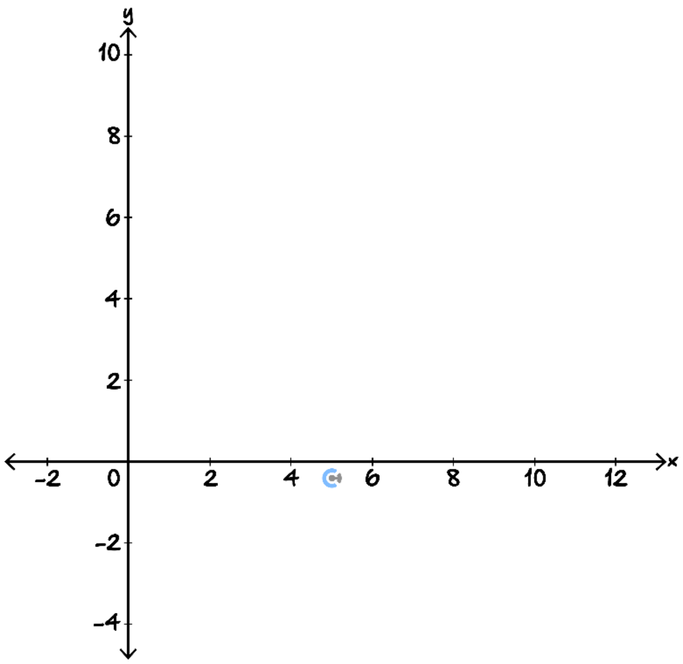
b. Find the equation of the line tangent to f that passes through the point $(0,1)$. (2 marks)

c. Define f^{-1}, the inverse function of f. (2 marks)

d. Let u be a real number such that, $f(u) = f^{-1}(u)$.

i. Find all possible values of u. (1 mark)

ii. Sketch the graphs of f and f^{-1} on the axes below. Label all axes intercepts and points of intersection with coordinates. (2 marks)



iii. Find the smallest acute angle between the tangents to the curves $y = f(x)$ and $y = f^{-1}(x)$ when they intersect. Give your answer in degrees correct to one decimal place. (2 marks)

Now, consider the family of functions $g: [0, \infty) \to \mathbb{R}, g(x) = k\sqrt{x} - k$, where $k \neq 0$.

e. Find the value(s) of k for which the graphs of $y = g(x)$ and $y = g^{-1}(x)$ do not intersect. (2 marks)

f. Find the value of $k > 0$ for which the angle made by tangents to g and g^{-1} at the **first** point where they intersect is $45°$. (2 marks)

g. Find the values of k such that, g and g^{-1} intersect three times. (2 marks)

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